Model theory

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Why am I here?

I was asked to speak about model theory at the 2014 meeting of the italian logic association (AILA). This is a huge task and I am not sure I am able to do it. I limited myself to some pointers to a very personal selection of topics. The result is rather biased. I apology for the many omissions even within the italian panorama.

What is model theory?

Model theory is the study of structures with particular emphasis on definability/interpretability issues. A **structure** is a set equipped with some operations or relations. Groups, rings, modules, ordered sets, boolean algebras, etc., are examples of structures. Some general references are [40, 46, 34, 38, 48]

The integers

- A first basic distinction to be made is whether a given structure **interprets** the ring of integers.
- Gödel proved that in $(\mathbb{N}, +, \cdot)$, or equivalently in the ring $\mathbb{Z}$, one can give **first order** definitions of every computable function. It follows that the first order theory of the integers $Th(\mathbb{Z})$ is undecidable.
- Through the work of Davis, Putnam, J. Robinson and Matiyasevich this led to the negative solution of Hilbert's 10th problem: there is no algorithm to test the solvability of a diophantine equation (Matiyasevich 1970).
- Any structure which interprets the ring of integers $(\mathbb{Z}, +, \cdot)$ (or a model of a few of its axioms) has an undecidable theory. So in particular Zermelo-Fraenkel set theory is undecidable.

$\mathbb{C}, \mathbb{R}$ and $\mathbb{Q}$

- Tarski: The ordered field of real numbers $\mathbb{R}$ is decidable and admits **elimination of quantifiers** (EQ). Any definable subset of $\mathbb{R}^n$ is semialgebraic, so in particular it has finitely many connected components (hence $\mathbb{Z}$ is not definable).
- Similarly the theory of the complex field $\mathbb{C}$ is decidable and admits elimination of quantifiers. Any definable subset of $\mathbb{C}^n$ is a boolean combinations of algebraic varieties. In particular $\mathbb{Z}$ is not definable.
- By contrast, Julia Robinson proved that the subring $\mathbb{Z}$ is first order definable in the field $\mathbb{Q}$. So $\mathbb{Q}$ has an undecidable theory. It is still open whether $\mathbb{Z}$ is existentially definable in $\mathbb{Q}$. This would yield the undecidability of Hilbert's 10th problem over $\mathbb{Q}$.
- Recently Koenigsmann proved that $\mathbb{Z}$ is universally definable in $\mathbb{Q}$ [29].

Model completeness

- It follows from EQ that the theories of $\mathbb{C}$ and $\mathbb{R}$ are **model complete**: every substructure is an **elementary substructure**.
- Early applications of model completeness include Abraham Robinson's alternative approach to Hilbert's Nullstellensatz and Hilbert's 17th problem (every positive definite polinomial $p(x) \in \mathbb{R}[x]$ is a sum of squares of rational functions).
- By the work of Ax, Kochen (1965) and Ershov, two Henselian valued fields or residual characteristic zero are **elementary equivalent** iff so are their residue fields and value groups. This has applications on diophantine problems over p-adic numbers. A correct version of a conjecture of Artin was thus established.
- Elementary substructures also play a crucial role in Robinson's **non-standard** analysis, represented in Pisa by Di Nasso, Benci, Forti and their students (Luperi, Lupini, etc.).

Decidability

Initially the emphasis was on decidability/undecidability. Let us consider two old problems by Tarski:

- It is still not known whether the theory of the real exponential field $(\mathbb{R}, \exp)$ is decidable (we will return to this).
- Any two free groups with at least two generators have the same first order theory [41]. Moreover the theory is decidable [28].

Categoricity

A natural question is whether one can classify the models of a given first order theory T by attaching them some cardinal invariants.

- A vector space V over $\mathbb{Q}$ is determined, up to isomorphism, by its dimension. It follows that the theory of $\mathbb{Q}$-vector spaces is **κ-categorical** for every uncountable κ, namely it has a unique model of cardinality κ up to isomorphism.
- An algebrically closed field is determined, up to isomorphism, by its transcendence degree over the prime subfield. As above, κ-categoricity for every uncountable κ follows.
- Morley (1965): If a first-order theory in a countable language is categorical in some uncountable cardinality, then it is categorical in all uncountable cardinalities.
- The proof uses the model theoretic notion of **Morley rank**. In the case of $\mathbb{C}$ one recovers a classical notion: The dimension of a complex algebraic variety coincides with its Morley rank.

Stability

- The work of Morley was the source of ispiration for Shelah's classification program [43]: find cardinal invariants to classify the models of any given theory T or show that a classification is impossible.
- The class of **stable** theories (those which do not have formulas that define a total ordering on an infinite set) lies at the core of the classification program. An unstable theory has too many models, so they cannot be classified in Shelah's sense. On the other hand, any uncountably categorical theory is stable.
- The simplest kind of stable structures are the **strongly minimal** ones, those of Morley rank and Morley degree equal to one. The complex field $\mathbb{C}$ is strongly minimal.

Geometric stability

The possibility of obtaining geometrically significant notions (such as the Morley rank) out of model theoretic assumptions was an outcome of the work of Morley and Shelah.

- The geometric approach was pushed further by Hrushovski and Zilber with the investigation of the so called **"Zariski geometries"**, a synthetic approach to Zariski topology not mentioning fields [26].
- It was shown that "Zilber's trichotomy conjecture", while false in general [21], holds within the class of Zariski structures: a Zariski geometry is either "trivial" or comes essentially from a vector space or it is biinterpretable with an algebrically closed field.
- Hodges coined the slogan "Model theory = algebraic geometry minus fields".
- In 1996 Hrushovski contributed to substantiate the slogan with his proof of the Mordell-Lang conjecture for function fields [22].

O-minimality I

- While $\mathbb{C}$ strongly minimal (every definable subset if finite or cofinite), the real field $\mathbb{R}$ is **o-minimal**: every definable subset of $\mathbb{R}$ is a finite union of intervals.
- O-minimal structures do not need to be based on the reals, but they are always expansions of a total order (so they are not stable). It can be proved that in any o-minimal structure every definable set is a finite union of "cells" [50].
- Wilkie proved that the real exponential field $(\mathbb{R}, \exp)$ is model complete and o-minimal [51]. In particular every subset of $\mathbb{R}^n$ definable in $(\mathbb{R}, \exp)$ has finitely many connected components (since cells in $\mathbb{R}$ are connected). Thus $\mathbb{Z}$ is not definable.
- Another example of o-minimal structures is given by the expansion $\mathbb{R}_{an}$ of the field $\mathbb{R}$ by all the analytic functions restricted to $[-1, 1]^n$ [15]. Subanalytic geometry can be accommodated in this setting. The structure $(\mathbb{R}_{an}, \exp)$ is also o-minimal [49][16].

O-minimality II

- Macintyre and Wilkie proved that the decidability of $(\mathbb{R}, \exp)$ follows from "Schanuel's conjecture" in transcendental number theory [32].
- Motivated by the decidability problem, we proved with T. Servi [11] an effective version of Wilkie's o-minimality result: there are effective (even primitive recursive) upper bounds on the number of connected components of a definable set in $(\mathbb{R}, \exp)$ (and other expansions of $\mathbb{R}$ by "Pfaffian functions").
- Servi and Jones proved the decidability of the real field with a generic power function [27].
- Like stability, also o-minimality found applications to diophantine problems. In (2006) Pila and Wilkie [35] studied the asymptotics of rational points on a definable set in an o-minimal structure.
- Pila and Zannier [36] used these results for a new proof of the Manin-Munford conjecture.

Tame but not o-minimal

- For structures based on the real line, the undefinability of $\mathbb{Z}$ is a weaker condition than o-minimality, but it still implies some form of "tameness" for the structure of definable sets (Fornasiero, Hieronymi, Miller [20]).
- More generally one can prove a wild/tame dychotomy for the class of "definably complete" structures [19].
- Such class was also investigated by Fornasiero and Servi in connection to the model theory of Pfaffian functions [42]

Dependent theories

- The class of **NIP** (or "dependent") theories, introduced by Shelah, includes the stable and the o-minimal theories and is currently the object of intense investigation [45].

$$\text{Stable} \cup \text{O-minimal} \subseteq \text{NIP}$$

- A basic example of a NIP structure which is neither stable nor o-minimal is the ordered group $(\mathbb{Z}, +, <)$ (Presburger arithmetic). Important examples of NIP structures arise from the study of valued fields.
- A trivial example of a NIP structure is given by the disjoint union $\mathbb{R} \sqcup (\mathbb{Z}, +)$ of the o-minimal structure $\mathbb{R}$ and the stable structure $(\mathbb{Z}, +)$.
- The model theoretic study of group extensions leads naturally to $\mathbb{R} \sqcup (\mathbb{Z}, +)$ [24][7, 10].

Simple theories

- The class of **simple theories**, characterized by the existence of a good "independence relation", is another generalization of the stable theories which was proved relevant for the study of profinite fields (see [46]).
- If T is simple and NIP it is stable.

$$\text{Stable} \cup \text{O-minimal} \subseteq \text{NIP}$$

$$\text{NIP} \cap \text{Simple} = \text{Stable}$$

Complex exponentiation

$\mathbb{Z}$ is definable in the complex exponential field $(\mathbb{C}, \exp)$, so the theory is undecidable.

- It has been conjectured that $(\mathbb{C}, \exp)$ is "quasi-minimal" (every infinite definable subset is countable or co-countable).
- Zilber proposed some axioms for $(\mathbb{C}, \exp)$ in an infinitary logic and proved that his axioms have a unique model $\mathbb{B}$ of cardinality $2^{\aleph_0}$. He conjectured that $\mathbb{B} \cong (\mathbb{C}, \exp)$. This would have important number theoretic implications including the Schanuel conjecture.
- If Zilber's conjecture is true, then $\mathbb{B}$ would have an involution. Mantova proved that this is indeed the case [33].
- Terzo, D'Aquino and Macintyre investigated the consequences of Zilber's and Schanuel's conjectures [47, 13, 14]
- Exponentiations on Lie groups and algebras have been investigated by L'Innocente, Macintyre, Point, Toffalori [30, 31].

Definable groups

- The investigation of definable groups plays an important role in the investigation of stable and o-minimal structures.
- In the stable case, the Cherlin-Zilber conjecture (or "algebraicity conjecture") states that every infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field. The sheer size of results in this direction is collected in various monographs [12].
- In the o-minimal case there is a connection with Lie groups, rather than algebraic groups.
- If G is a group definable in an o-minimal structure M, then G is a Lie group "over M". So if M is an expansion $\mathbb{R}$ we obtain a real Lie group [37].
- A second connection comes from "PIllay's conjectures" in [39] (see below).

Hyperdefinability

- In [39] Pillay showed that one can associate to a definable group G in an o-minimal structure M a compact topological group G^*/G^{*00} by first going to an elementary extension $G^* \succ G$ (with the whole structure induced by M) and then taking a quotient by the so called "infinitesimal subgroup" G^{*00} (a "**type-definable**" group).
- G^*/G^{*00} is **hyperdefinable**. It does not depend on the model.
- In collaboration with Otero, Peterzil and Pillay we showed [9] that G^*/G^{*00} admits the structure of a real Lie group.
- Shelah proved that G^{*00} exists even in the NIP context [44], although the compact group G^*/G^{*00} does not need to be a Lie group in this case.
- In general G^*/G^{*00} can be seen as a kind of completion of G. For instance starting from $\mathbb{Z}$ one obtains its classical completion $\widehat{\mathbb{Z}}$.

Transfer modulo G^{00}

Determining up to what extent we can trasfer properties from G^*/G^{*00} to G is a difficult task.

- If G is a definably compact group in an o-minimal structure, the dimension of G^*/G^{*00} as a Lie group coincides with the o-minimal dimension of G [23]
- Moreover G is elementary equivalent to G^*/G^{*00} [24] and is "compactly dominated" [25].
- In a series of papers in collaboration [4, 8, 2, 3] we investigated the topology of G^*/G^{*00} (with G definably compact) through the study of homology and homotopy functors.
- It was finally proved that the isomorphism type of G^*/G^{*00} in the Lie category determines the homotopy type of G in the definable category[6, 1].
- In dimension $\neq 4$ the homeomorphism type is also determined [2] (unknown in dimension 4).

Local definability

- Another important category is the category of "locally definable" sets and functions, which includes certain inductive limits of definable objects.
- The commutator subgroup of a definable group is in general only locally definable, namely is a countable union of definable sets.
- The universal cover of a definable group G in an o-minimal structure is locally definable [17].
- Under a connectedness assumption it is reasonable to conjecture [18] that locally definable groups still reflect many properties of Lie groups.
- For instance in [5] it is proved that in the abelian case the n-torsion subgroup is finite and every "discrete" subgroup has finite rank. It is still open whether every discrete subgroup is finitely generated.

Omissions

- This is slightly self-referencial: I am afraid that if I mention the omissions they do not count as omissions any more.
- In any case I should at least mention the work of Toffalori and L'Innocente on the model theory of modules and representations, in particular the representations of the universal enveloping algebra of the Lie algebra $sl(2, \mathbb{C})$ and the decidability of modules over various domains.
- Unfortunately I have no time to do it, ... or did I?
- Thank you anyway for your attention!

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