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TURBULENCE MODELS, p–FLUID FLOWS, AND $W^{2,L}$ REGULARITY OF SOLUTIONS

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Abstract. In this article we prove some sharp regularity results for the stationary and the evolution Navier-Stokes equations with shear dependent viscosity, under the no-slip boundary condition. This is a classical turbulence model, considered by von Neumann and Richtmeyer in the 50's, and by Smagorinski in the beginning of the 60's (for $p = 3$). The model was extended to other physical situations, and deeply studied from a mathematical point of view, by Ladyzhenskaya in the second half of the 60's. In the sequel we consider the case $p > 2$. We are interested in regularity results in Sobolev spaces, *up to the boundary*, in dimension $n = 3$, for the second order derivatives of the velocity and the first order derivatives of the pressure. In spite of the very rich literature on this subject, sharp regularity results up to the boundary are quite new. In the sequel we improve in a very substantial way all the known results in the literature. In order to emphasize the very new ideas, we consider a flat boundary (the so called "cubic-domain" case). However, all the regularity results stated here hold in the presence of smooth boundaries, by following [3].

1. **Introduction.** Throughout this work u and π denote, respectively, the velocity and the pressure of a viscous incompressible fluid. We are interested on regularity results for solutions to the Navier-Stokes equations for flows with shear dependent viscosity, namely

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nabla \cdot T(u, \pi) = f, \qquad \nabla \cdot u = 0, \tag{1}$$

where $T = -\pi I + \nu_T(u)\mathcal{D}u$ denotes the Cauchy stress tensor, and $\frac{1}{2}\mathcal{D}u$ denotes the symmetric gradient, $\mathcal{D}u = \nabla u + \nabla u^T$.

The system of equations (1), for $p = 3$ (for the meaning of p, see below), was introduced by J.S. Smagorinsky in [23], as a turbulence model. From a mathematical point of view (and for arbitrary p) the system was studied by O.A. Ladyzenskaya, already as a turbulence model, in references [10], [11], [12] and [13]. J.-L. Lions considered similar models, in which $\mathcal{D}u$ is replaced by ∇u. See [14] and [15], Chap.2, n.5. It is worth noting that (3) satisfies the Stokes Principle, see [24]. A clear and rigorous discussion on this subject is given by J. Serrin in reference [22], page 231. Nonlinear shear dependent viscosity also models properties of certain materials. The cases $p > 2$ and $p < 2$ captures shear thickening and shear thinning phenomena,

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respectively. See, for instance, [8], [17], [18], and [21]. We assume here that $2 < p \leq 3$. However this restriction is not at all necessary, in the sense that, basically, the same argument gives similar results for $p \leq 4$.

Often, the exact form of $\nu_T(u)$ is not essential. For instance, our proofs apply to wider classes of $\nu_T(u)$ generalized viscosities than that considered below, depending essentially on convexity type properties, and behavior near zero and infinity. A typical example (considered in references [1] and [2]) is given by $\nu_T(u) = \nu_0 + \nu_1 |\mathcal{D}u|^{p-2}$, where ν_0 and ν_1 are strictly positive constants and $p \geq 2$. The results and proofs shown in the sequel apply to this case. However we consider here the more difficult case

$$\nu_T(u) = (1 + |\mathcal{D}u|)^{p-2}. \tag{2}$$

The lack of the $-\nu_0 \Delta u$ in the left hand side of (3) below is overcome here by appealing to a new device. See also the Remark 5.1 in [2].

Under the assumption (2) the stationary Stokes system reads

$$\begin{cases} -\nabla \cdot \Big((1 + |\mathcal{D}u|)^{p-2} \mathcal{D}u \Big) + \nabla \pi = f, \\ \nabla \cdot u = 0. \end{cases} \tag{3}$$

Below, we consider solutions under the no-slip boundary condition

$$u_{|\partial\Omega} = 0. \tag{4}$$

As already shown in previous papers (see in particular [2], to which this paper is strongly related), the proofs of our regularity results in the presence of the convective term $(u \cdot \nabla) u$, and for the evolution problem, are easily obtained from the results proved for the stationary Stokes equations. So, we state these results below, and leave the straightforward proofs to the interested reader. The main lines are exactly that followed in [2]. Hence, our main concern is the study of the stationary generalized Stokes system (3). Roughly speaking, our basic result states that weak solutions to (3), under the no-slip boundary condition, satisfy equation (31) below. Concerning previous results, related to (31), in [2] it is shown that $u \in W^{2, \frac{3(4-p)}{5-p}}(\Omega)$, and in [7] that $u \in W^{2, \frac{8-p}{3}}(\Omega)$.

Higher order regularity results up to the boundary in regular bounded open sets $\Omega \subset \mathbb{R}^3$, under the no-slip boundary condition, and $p > 2$, are studied for the first time in reference [16]. For $n = 2$, see [9]. In reference [1] and [2], sharper results are obtained in the case of a flat boundary. In particular, in reference [2], as below, we consider the cubic-domain case, see subsection 1.1. Further, in reference [3], we introduce a general method that, essentially, provides the extension of results from flat to non flat boundaries. Actually, in [3], we have extended to smooth boundaries the results proved in reference [2]. By now, having in hands the results shown here, we can easily modified a few points in the proofs in [3], and extend in this way all the results stated below to non-flat boundaries. Actually, a similar extension for the case $p < 2$ has already been done by us, see [4].

It is worth noting that the presentation of some arguments and proofs below could be reduced by appealing to some of our previous papers. However, the reading of the paper would became unclear and unpleasant to really interested readers.

1.1. **The "cubic-domain" framework, and related notation.** In the following we consider a 3-dimensional cubic domain $\Omega = (]0,1[)^3$, and impose our boundary condition (4) only on the two opposite faces $x_3 = 0$ and $x_3 = 1$. On the x_1 and x_2 directions we assume periodicity, as a device to avoid unessential technical difficulties. This choice is made so that we work in a bounded domain Ω and, at the same time, with a flat boundary. By working in this simple context, we concentrate on the basic ideas of proofs. More precisely, the boundary condition (4) will be imposed only on

$$\Gamma = \Gamma_- \cup \Gamma_+.$$

where

$$\Gamma_- = \{x : |x_1|, |x_2| < 1, x_3 = 0\}, \quad \Gamma_+ = \{x : |x_1|, |x_2| < 1, x_3 = 1\}.$$

The problem is assumed to be periodic, with period equal to 1, both in the x_1 and the x_2 directions. Obviously, the “significant” boundary is Γ. We set

$$x' = (x_1, x_2).$$

By x'-periodic we mean periodic of period 1 both in x_1 and x_2. A similar convention is assumed for expressions like x'-periodicity and so on.

Hence our “boundary condition” reads

$$u_{|\Gamma} = 0, \qquad \text{and} \quad u(x) \quad \text{is} \quad x' - \text{periodic}. \tag{5}$$

We denote by D^2u the set of all the second derivatives of u and by D^2_*u the second order derivatives $\partial^2 u_j/\partial x_i \partial x_k$ with the exclusion of the normal derivatives $\partial^2 u_j/\partial x_3^2$, for $j = 1, 2$. Further,

$$|D^2_*u|^2 := \left|\frac{\partial^2 u_3}{\partial x_3^2}\right|^2 + \sum_{\substack{i,j,k=1\\(i,k)\neq(3,3)}}^{3} \left|\frac{\partial^2 u_j}{\partial x_i \partial x_k}\right|^2. \tag{6}$$

Similarly, ∇^* denotes first order partial derivatives, except for $\partial\ /\partial x_3$.

1.2. **Weak solutions.** In the sequel the symbol $\|\cdot\|_p$ denotes the canonical norm in $L^p(\Omega)$, and $\|\cdot\| = \|\cdot\|_2$. By $W^{k,q}(\Omega)$ we denote the usual Sobolev spaces. We use the same notation for functional spaces and norms for both scalar and vector fields. We set

$$V_p = \left\{v \in W^{1,p}(\Omega) : (\nabla \cdot v)_{|\Omega} = 0; v_{|\Gamma} = 0; v \text{ is } x' - \text{periodic}\right\}. \tag{7}$$

It is well known that there is a positive constant c such that the estimate

$$\|\nabla v\|_p + \|v\|_p \le c\|\mathcal{D}v\|_p \tag{8}$$

holds, for each $v \in V_p$. Hence the two above quantities are equivalent norms in V_p. See, for instance, [20], Proposition 1.1.

Assume that $f \in (V_p)'$, the strong dual of V^p. We say that u is a *weak solution* to problem (3), (4) if $u \in V_p$ satisfies

$$\frac{1}{2}\int_\Omega (1 + |\mathcal{D}u|)^{p-2}\mathcal{D}u \cdot \mathcal{D}v dx = \int_\Omega f \cdot v dx \tag{9}$$

for all $v \in V_p$. It is well known that existence and uniqueness of the weak solution follow by appealing to the method described in the Chap.2, Sect.2 of [15].

By replacing v by u in equation (9) one gets

$$\|\mathcal{D}u\|^2 + \|\mathcal{D}u\|_p^p \le c| < f, u > |, \tag{10}$$

where the symbols $< \cdot, \cdot >$ denote a duality pairing.

From (10) and (8) there readily follows the basic estimates

$$\|\nabla u\| \le c\|f\| \quad \text{and} \quad \|\nabla u\|_p \le c\|f\|_{p'}^{\frac{1}{p-1}}. \tag{11}$$

Note that the second estimate is stronger than the first one.

Well known devices show the existence of a distribution π (determined up to a constant) such that

$$\nabla \pi = -\nabla \cdot \left[(1 + |\mathcal{D}u|)^{p-2} \mathcal{D}u \right] + f. \tag{12}$$

Hence the first equation (3) holds in the distributions sense. Actually, by appealing to (12) and (11), one may show that $\pi \in L^{p'}(\Omega)$ and that

$$\|\pi\|_{L^{p'}_{\#}} \leq c(\|\mathcal{D}u\|_{p'} + \|\mathcal{D}u\|_p^{p-1}) \leq c(\|f\| + \|f\|_{p'}),$$

where, in general, $L^\alpha_{\#} = L^\alpha / \mathbb{R}$.

1.3. **Main integrability exponents.** Integrability exponents play a crucial role in our proofs. For the reader's convenience, we introduced all these exponents together.

In the sequel p denotes an exponent that lies in the interval

$$2 \leq p \leq 3 \tag{13}$$

and q an exponent that, in the sequel (but not necessarily), lies in the interval

$$p \leq q \leq p + 4.$$

We denote by p' the dual exponent

$$p' = \frac{p}{p-1}. \tag{14}$$

In general, for $1 < a < 3$ we define the Sobolev embedding exponent a^* by the equation

$$\frac{1}{a^*} = \frac{1}{a} - \frac{1}{3}. \tag{15}$$

Moreover we define $r = r(q)$ by

$$\frac{1}{r(q)} = \frac{p-2}{2q} + \frac{1}{2}, \tag{16}$$

$\mathcal{Q} = \mathcal{Q}(q)$ by

$$\frac{1}{\mathcal{Q}(q)} = \frac{5(p-2)}{6(p-1)q} + \frac{1}{6(p-1)}, \tag{17}$$

and $\overline{q} = \overline{q}(q)$ by

$$\frac{1}{\overline{q}(q)} = \frac{p-2}{\mathcal{Q}(q)} + \frac{1}{2}. \tag{18}$$

Note that

$$\mathcal{Q}(q) > q, \tag{19}$$

since $q < p + 4$.

We set

$$\widetilde{q}(q) = \min\{\overline{q}(q), r(q)\}. \tag{20}$$

Note that $r(q) \geq \overline{q}(r)$ is equivalent to $q \geq 7 - 2p$.

2. **The stationary Stokes problem. Main results.** In the sequel we denote by c a generic positive constant that may change from equation to equation. The positive constants c do not depend on the parameters p and q, in the usual sense (i.e., they are bounded from above for p and q varying in the ranges considered here).

Our first statement concerns the regularity of the tangential derivatives.

Theorem 2.1. *Assume that*

$$f \in L^2(\Omega) \tag{21}$$

and let u, π be the weak solution to problem (3) under the boundary condition (5) (problem (9)).

Then the derivatives $D_*^2 u$ belong to $L^2(\Omega)$, moreover

$$\|D_*^2 u\|^2 + \sum_{k=1}^{2} \left\| (1+|\mathcal{D}u|)^{\frac{p-2}{2}} \mathcal{D} \frac{\partial u}{\partial x_k} \right\|^2 \leq c\|f\|^2. \tag{22}$$

Concerning the regularity of the remaining derivatives, we start from the following conditional result.

Theorem 2.2. *Let f, u and π be as in Theorem 2.1 and assume, in addition, that*

$$\mathcal{D}u \in L^q(\Omega) \tag{23}$$

for some $p \leq q \leq 6$. Then, in addition to (22), one has

$$\|\nabla^* \pi\|_{r(q)} + \|D^2 u\|_{r(q)} + \|(1+|\mathcal{D}u|)^{p-2} \nabla^* \mathcal{D}u\|_{r(q)} \leq \mathcal{K}_q, \tag{24}$$

where $r = r(q)$ is given by (16) and $\mathcal{K}_q$ satisfies the estimate

$$\mathcal{K}_q \leq c\|f\| + c\|\mathcal{D}u\|_q^{\frac{p-2}{2}} \|f\|. \tag{25}$$

Furthermore,

$$\|u\|_{1,\mathcal{Q}(q)} \leq A_q =: c_0 \|f\|^{\frac{1}{p-1}} + c_0 \|\nabla u\|_q^{\frac{5(p-2)}{6(p-1)}} \|f\|^{\frac{1}{p-1}} + c\|\nabla u\|_p, \tag{26}$$

where $\mathcal{Q}(q)$ is given by (17).

Since (23) holds for $q = p$, and $r(p) = p'$, one has the following result.

Corollary 1. *Let f, u and π be as in Theorem 2.1. Then*

$$\|\nabla^* \pi\|_{p'} + \|D^2 u\|_{p'} + \|(1+|\mathcal{D}u|)^{p-2} \nabla^* \mathcal{D}u\|_{p'} \leq c\|f\| + c\|1+|\mathcal{D}u|\|_p^{\frac{p-2}{2}} \|f\| \tag{27}$$

and

$$\|u\|_{1,\mathcal{Q}(p)} \leq c_0 \|f\|^{\frac{1}{p-1}} + c_0 \|\nabla u\|_p^{\frac{5(p-2)}{6(p-1)}} \|f\|^{\frac{1}{p-1}} + c\|\nabla u\|_p, \tag{28}$$

where $\mathcal{Q}(p) = \frac{3p-5}{3p(p-1)}$.

Concerning the regularity of the derivative $\frac{\partial \pi}{\partial x_3}$ one has the following result.

Proposition 1. *Under the assumptions of Theorem 2.2 one has*

$$\left\| \frac{\partial \pi}{\partial x_3} \right\|_{\widetilde{q}} \leq c\left[1 + A_q^{p-2}\right] \|f\| + c\mathcal{K}_q \tag{29}$$

where $\widetilde{q}$ is defined in (20) and A_q is the right hand side of (26). In particular, by (24),

$$\|\nabla \pi\|_{\widetilde{q}} \leq c\left[1 + A_q^{p-2}\right] \|f\| + c\mathcal{K}_q. \tag{30}$$

The reason that leads us to separate Proposition 1 from Theorem 2.2 is to emphasize that the regularity of $\frac{\partial \pi}{\partial x_3}$ is simply obtained as a final by-product of the regularity of all other derivatives.

The next is our main result.

Theorem 2.3. *Let f, u and π be as in Theorem 2.1. Then, in addition to (22),*

$$u \in W^{1,p+4}(\Omega) \cap W^{2,\frac{p+4}{p+1}}(\Omega). \tag{31}$$

More precisely,

$$\|u\|_{1,p+4} \leq c\|f\|^{\frac{1}{p-1}} + c\|f\|^{\frac{6}{p+4}} + c\|\nabla u\|_p. \tag{32}$$

Furthermore,

$$\|\nabla^* \pi\|_l + \|D^2 u\|_l + \|(1+|\mathcal{D}u|)^{p-2}\nabla^* \mathcal{D}u\|_l \leq c(1+\|\nabla u\|_p^{\frac{p-2}{2}})\|f\| + c\|f\|^{\frac{2(p-2)}{p+4}}, \tag{33}$$

where

$$l = \frac{p+4}{p+1}. \tag{34}$$

Finally,

$$\frac{\partial \pi}{\partial x_3} \in L^m(\Omega), \tag{35}$$

where $m = \overline{q}(p+4)$. In particular,

$$\nabla \pi \in L^m(\Omega),$$

and

$$\|\nabla \pi\|_m \leq c\left[1 + A_{p+4}^{p-2}\right]\|f\| + c\mathcal{K}_{p+4}. \tag{36}$$

Note that, by (32), $\mathcal{K}_{p+4}$ and A_{p+4} are bounded (in terms of $\|f\|$ and $\|\nabla u\|_p$). Further, $m = 2$ if $p = 2$.

Remark 1. Note that, by a Sobolev embedding theorem, it follows from $u \in W^{2,l}$ that $u \in W^{1,l^*}$, where $l^* = \frac{3(p+4)}{2p-1}$. Since $l < 2$ one has $l^* < 2^* = 6$. Nevertheless, we prove here that $u \in W^{1,p+4}$, where $6 < p+4$.

Note that, for $p = 2$, all the inequalities written in this remark turn into equalities.

Remark 2. The second order "tangential derivatives", see (22), belong to $L^2(\Omega)$. Hence, they are more regular than the remaining (purely normal) second order derivatives, see (24). Consequently, instead of appealing to classical Sobolev embedding theorems (as done in [2], to show (3.8)), we appeal here to anisotropic Sobolev-type embedding theorems, see [25], in order to show (26) (which is here the counterpart of (3.8) in [2]). This fruitful idea was introduced in reference [6]. Clearly, by appealing to the estimate (26), we get better integrability exponents than that in [2], obtained by appealing to the weaker estimate (3.8).

Remark 3. As already remarked, all the regularity results stated here hold in the presence of smooth boundaries. This can be shown, without particular difficulties, by readers already acquainted with the approach introduced in [3]. The significant changes must be made only in a very small part of the proof in [3], concerning a couple of estimates proved in the context of the "flat" system of coordinates y used in the above reference. Roughly speaking, in this last reference, we extend to the y variables some basic estimates proved in [2] in the simpler context of the original

x variables. In order to extend to non-flat boundaries the results proved below (instead of that in [2]), it is sufficient to appeal to the equivalent basic estimates proved in the following. For instance, the fundamental estimate (10.19) in reference [3] corresponds to the estimate (3.8) in reference [2], and to the estimate (26) below. Roughly speaking, in the original proof of the estimate (10.19) in [3], we extend some manipulations made in the proof of the estimate (3.8) in reference [2], from the x to the y variables. In the context of the new results claimed above, we extend, in a quite similar way, the corresponding manipulations made in proving the estimate (26) below instead of that concerning the estimate (3.8) in [2].

Finally, in the forthcoming paper [5] all the above results are improved. In [5] stationary and evolution (Stokes and Navier-Stokes) problems are studied in the presence of non-flat boundaries, for all $p \geq 2$ and all $n \geq 2$.

Remark 4. In our previous papers [1], [2], [3] and [4] (as well as in our papers concerning the case $p < 2$) we obtain the largest integrability exponents (like l in (34)) by appealing to a boot-strap argument. Actually, in the above papers we may avoid the boot-strap, by arguing just as shown in section 6 below.

3. **Stationary and evolution Navier-Stokes equations. Results.** The proofs of the following extensions of the above results to solutions of the Navier-Stokes stationary and evolution equations are left to the interested reader, since they are done by straightforward modifications of the corresponding proofs shown in reference [2]. We note that, in our papers, the main novelties concern the generalized Stokes system (3). In fact, in our opinion, the new obstacles related to the boundary value problems already appear in this particular case. Regularity results for solutions to the stationary and evolution generalized Navier-Stokes equations are proved by us as more or less straightforward consequences of the results obtained for the generalized Stokes system. Actually, we realize that a more stringent use of the estimates proved for the system (3) is possible. However, we did not push in this direction.

Theorem 3.1. *The regularity results stated in the Theorems 2.1, 2.2 and 2.3, and in the Lemma 1, hold for the generalized Navier-Stokes equations*

$$\begin{cases} -\nabla \cdot \Big((1+|\mathcal{D}u|)^{p-2}\mathcal{D}u \Big) + (u\cdot\nabla)u + \nabla\pi = f, \\ \nabla \cdot u = 0. \end{cases} \tag{37}$$

Consider now the evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - \nabla \cdot \Big((1+|\mathcal{D}u|)^{p-2}\mathcal{D}u \Big) + (u\cdot\nabla)u + \nabla\pi = f, \\ \nabla \cdot u = 0, \\ u(0) = u_0(x). \end{cases} \tag{38}$$

One has the following results.

Theorem 3.2. *Let u be a weak solution to problem (38) under the boundary condition (4) plus x'-periodicity, where $u_0 \in V_p$ and $f \in L^2(0,T;L^2)$. Assume that $p \geq 2 + \frac{2}{5}$. Then*

$$\begin{cases} u \in L^2(0,T;W^{2,p'}) \cap L^\infty(0,T;W^{1,p}), \\ \dfrac{\partial u}{\partial t} \in L^2(0,T;L^2). \end{cases} \tag{39}$$

Theorem 3.3. *Under the assumptions of Theorem 3.2 one has*

$$u \in L^{\frac{p+4}{p-2}}(0,T;W^{2,l}) \cap L^{\infty}(0,T;W^{1,p}). \tag{40}$$

The assumption $p \geq 2+\frac{2}{5}$ is superfluous if the convective term is not present. Corresponding results for the pressure are easily obtained, as well as estimates for the norms that appear in the above theorems.

4. **Proof of Theorem 2.1.** In this section we prove the Theorem 2.1. By assumption $u \in V_p$ satisfies (9) for each $v \in V_p$. For arbitrary scalar or vector fields v set $v^h(x) = v(x_1+h, x_2, x_3)$ or $v^h(x) = v(x_1, x_2+h, x_3)$ where $h \in \mathbb{R}$. We also set

$$\Delta_h v = \frac{v - v^{-h}}{h}.$$

By writing (9) with v replaced by v^h and by replacing, in the integrals on the left hand side, the variable x_k by $x_k - h$, $k = 1,2$, one easily shows that

$$\frac{1}{2}\int (1+|\mathcal{D}u^{-h}|)^{p-2}\mathcal{D}u^{-h}\cdot \mathcal{D}v dx = \int f \cdot v^h dx. \tag{41}$$

By taking the difference between equations (9) and (41), respecting the left and right sides, and by dividing by h one gets

$$\frac{1}{2h}\int \left((1+|\mathcal{D}u|)^{p-2}\mathcal{D}u-(1+|\mathcal{D}u^{-h}|)^{p-2}\mathcal{D}u^{-h}\right)\cdot \mathcal{D}v dx = \frac{1}{h},\int f\cdot(v-v^h)dx. \tag{42}$$

By setting $v = \Delta_h u$ in equation (42), by appealing to the identity

$$\|\mathcal{D}\Delta_h u\|^2 = 2\|\nabla(\Delta_h u)\|^2 \tag{43}$$

and by using the estimate

$$\left|\frac{1}{h}\int f\cdot(v-v^h)dx\right| \leq \|f\| \left\|\frac{v-v^h}{h}\right\| \leq \|f\|\|\nabla v\|, \tag{44}$$

it follows that

$$\frac{1}{2h}\int \left((1+|\mathcal{D}u|)^{p-2}\mathcal{D}u-(1+|\mathcal{D}u^{-h}|)^{p-2}\mathcal{D}u^{-h}\right)\cdot (\mathcal{D}\Delta_h u)dx \leq c\|f\|\|\mathcal{D}(\Delta_h u)\|. \tag{45}$$

Next, by a well known convex analysis estimate (set $U = \mathcal{D}u$ and $V = \mathcal{D}u^{-h}$ in equation (5.1), [2]), it follows that

$$\int \left(1+|\mathcal{D}u|+|\mathcal{D}u^{-h}|)\right)^{p-2}|\mathcal{D}\Delta_h u|^2 \leq c\|f\|\|\mathcal{D}(\Delta_h u)\|. \tag{46}$$

In particular,

$$\|D_*^2 u\|^2 + \int \left(1+|\mathcal{D}u|+|\mathcal{D}u^{-h}|)\right)^{p-2}|\mathcal{D}\Delta_h u|^2 \leq c\|f\|^2. \tag{47}$$

Note that, as a first steep, we obtain the above equation with $\|D_*^2 u\|^2$ replaced by $\|\mathcal{D}\Delta_h u\|^2$, hence by $\|\nabla\Delta_h u\|^2$ (apply (43)). Further, the uniform bound of this last quantity with respect to h allows us to replace it by $\|\nabla\nabla_* u\|^2$. Finally, differentiation with respect to x_3 of the equation $\nabla\cdot u = 0$ allows the inclusion of the derivative $\frac{\partial^2 u_3}{\partial x_3^2}$ in the above estimate, hence to replace $\|\nabla\nabla_* u\|$ by $\|D_*^2 u\|$.

Next, (as in [2]), by passing to the limit in (47), as $h \to 0$, one proves (22).

We note that it is not strictly necessary to appeal to (43). See the remark 5.1 in reference [2].

5. Proof of the Theorem 2.2.

5.1. Proof of the estimate (24). We start this section by recalling the following result. Let $g(x)$ be a scalar field in Ω such that $g = \nabla \cdot w_0$ and $\nabla g = \nabla \cdot W$, where $w_0 \in L^\beta(\Omega)$ and $W \in L^\alpha(\Omega)$, for some $\alpha \geq \beta > 1$. Then

$$\|g\|_{L^\alpha(\Omega)} \leq c\left(\|w_0\|_{L^\beta(\Omega)} + \|W\|_{L^\alpha(\Omega)}\right). \tag{48}$$

For $\beta = \alpha$ this result is proved in reference [19]. The above extension is straightforward.

It is worth noting that our constants c are independent of p, q, r since the constants that appear in the embedding theorems used in the sequel, as well as in (48), are uniformly bounded from above. This follows, since the exponents lie away from the critical values. Note that $2 \leq p \leq 3$, $p \leq q \leq 6$ and $\frac{4}{3} \leq r \leq 2$.

Lemma 5.1. *Assume (23). For $k = 1, 2$, the terms $(1 + |\mathcal{D}u|)^{p-2}\mathcal{D}\frac{\partial u}{\partial x_k}$ and the derivatives $\frac{\partial \pi}{\partial x_k}$ satisfy the estimate (24), i.e.,*

$$\|\nabla^* \pi\|_{r(q)} + \|(1 + |\mathcal{D}u|)^{p-2}\nabla^*\mathcal{D}u\|_{r(q)} \leq \mathcal{K}_q.$$

Proof. The proof is a straightforward copy of the proof of the Lemma 6.2 in reference [2]. We present it just for the reader's convenience.

By Hőlder's inequality and assumption (23), one has

$$\left\|(1 + |\mathcal{D}u|)^{p-2}\mathcal{D}\frac{\partial u}{\partial x_k}\right\|_{r(q)} \leq \|1 + |\mathcal{D}u|\|_q^{\frac{p-2}{2}} \left\|(1 + |\mathcal{D}u|)^{\frac{p-2}{2}}\mathcal{D}\frac{\partial u}{\partial x_k}\right\|. \tag{49}$$

Hence, by (22), it follows that

$$\left\|(1 + |\mathcal{D}u|)^{p-2}\mathcal{D}\frac{\partial u}{\partial x_k}\right\|_{r(q)} \leq c\|1 + |\mathcal{D}u|\|_q^{\frac{p-2}{2}} \|f\|. \tag{50}$$

This proves the second statement in the Lemma.

On the other hand, straightforward calculations show that

$$\begin{aligned}&\frac{\partial}{\partial x_k}\left((1 + |\mathcal{D}u|)^{p-2}\mathcal{D}u\right)\\ =&(1 + |\mathcal{D}u|)^{p-2}\mathcal{D}\frac{\partial u}{\partial x_k} + (p-2)(1 + |\mathcal{D}u|)^{p-3}|\mathcal{D}u|^{-1}\left(\mathcal{D}u \cdot \mathcal{D}\frac{\partial u}{\partial x_k}\right)\mathcal{D}u.\end{aligned} \tag{51}$$

Hence

$$|\frac{\partial}{\partial x_k}\left((1 + |\mathcal{D}u|)^{p-2}\mathcal{D}u\right)| \leq c(1 + |\mathcal{D}u|)^{p-2}\left|\mathcal{D}\frac{\partial u}{\partial x_k}\right|, \tag{52}$$

almost everywhere in Ω. Hence,

$$\left\|\frac{\partial}{\partial x_k}\left((1 + |\mathcal{D}u|)^{p-2}\mathcal{D}u\right)\right\|_{r(q)} \leq c\|1 + |\mathcal{D}u|\|_q^{\frac{p-2}{2}} \|f\|. \tag{53}$$

Further, by differentiation of equation (3) with respect to x_k, $k = 1, 2$, it follows that

$$\nabla \frac{\partial \pi}{\partial x_k} = \nabla \cdot \left[-\frac{\partial}{\partial x_k}\left((1 + |\mathcal{D}u|)^{p-2}\mathcal{D}u\right)\right] + \frac{\partial f}{\partial x_k}. \tag{54}$$

By appealing to (48), with $g = \frac{\partial \pi}{\partial x_k}$, $\alpha = r$ and $\beta = p'$, and by (53) and (54), (recall also (12) and (11)) it follows that

$$\left\| \frac{\partial \pi}{\partial x_k} \right\|_{r(q)} \leq c \left(\|f\| + \|f\|_{p'} + \|1 + |\mathcal{D}u|\|_q^{\frac{p-2}{2}} \|f\| \right). \tag{55}$$

Hence,

$$\left\| \frac{\partial \pi}{\partial x_k} \right\|_{r(q)} \leq \mathcal{K}_q. \tag{56}$$

□

Note that from equations (50) and (56) we get the estimate (24) for the first and the last term on the left hand side. The missing term is the subject of the following lemma.

Lemma 5.2. *The derivatives* $\frac{\partial^2 u_j}{\partial x_3^2}$, $j = 1, 2$ *satisfy the estimate*

$$\sum_{l=1}^{2} \left\| \frac{\partial^2 u_l}{\partial x_3^2} \right\|_{r(q)} \leq \mathcal{K}_q. \tag{57}$$

Proof. It is worth noting that the proof is a pedestrian copy of the proof of the Lemma 6.3 in reference [2]. Nevertheless, we believe that it is pleasant to the interested reader to have it hereby.

By using (51), the $j.^{th}$ equation (3) may be written in the form

$$\begin{aligned} &-(1 + |\mathcal{D}u|)^{p-2} \sum_{k=1}^{3} \frac{\partial^2 u_j}{\partial x_k^2} \\ &-(p-2)(1 + |\mathcal{D}u|)^{p-3} |\mathcal{D}u|^{-1} \sum_{l,m,k=1}^{3} \mathcal{D}_{lm} \mathcal{D}_{jk} \left(\frac{\partial^2 u_l}{\partial x_m \partial x_k} + \frac{\partial^2 u_m}{\partial x_l \partial x_k} \right) + \frac{\partial \pi}{\partial x_j} = f_j, \end{aligned} \tag{58}$$

where $\mathcal{D}_{ij} = (\mathcal{D}u)_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$ and $1 \leq j \leq 3$. Let us write the first two equations (58), $j = 1, 2$, as follows:

$$(1+|\mathcal{D}u|)^{p-2} \frac{\partial^2 u_j}{\partial x_3^2} + 2(p-2)(1+|\mathcal{D}u|)^{p-3} |\mathcal{D}u|^{-1} \mathcal{D}_{j3} \sum_{l=1}^{2} \mathcal{D}_{l3} \frac{\partial^2 u_l}{\partial x_3^2} = F_j(x) + \frac{\partial \pi}{\partial x_j} - f_j, \tag{59}$$

where the $F_j(x), j \neq 3$, are given by

$$\begin{aligned} F_j(x) &:= -(1 + |\mathcal{D}u|)^{p-2} \sum_{k=1}^{2} \frac{\partial^2 u_j}{\partial x_k^2} \\ &- 2(p-2)(1 + |\mathcal{D}u|)^{p-3} |\mathcal{D}u|^{-1} \left\{ \mathcal{D}_{33} \mathcal{D}_{j3} \frac{\partial^2 u_3}{\partial x_3^2} + \sum_{\substack{l,m,k=1 \\ (m,k) \neq (3,3)}}^{3} \mathcal{D}_{lm} \mathcal{D}_{jk} \frac{\partial^2 u_l}{\partial x_m \partial x_k} \right\}. \end{aligned} \tag{60}$$

In the sequel, the equations (59), $j = 1, 2$, will be treated as a 2×2 linear system in the unknowns $\frac{\partial^2 u_j}{\partial x_3^2}, j \neq 3$. Note that, with an obviously simplified notation, the measurable functions F_j satisfy

$$|F_j(x)| \leq c(1 + |\mathcal{D}u(x)|)^{p-2} |D_*^2 u(x)|, \tag{61}$$

a.e. in Ω.

We denote by $\widetilde{F}_j$ the right hand sides

$$\widetilde{F}_j(x) := F_j(x) + \frac{\partial \pi}{\partial x_j} - f_j, \tag{62}$$

that appear in the above 2×2 system (59).

Let us show that the 2×2 system (59) can be solved for the unknowns $\frac{\partial^2 u_j}{\partial x_3^2}$, $j = 1, 2$, for almost all $x \in \Omega$.

The elements a_{jl} of the matrix system A are given by

$$a_{jl} = (1 + |\mathcal{D}u|)^{p-2}\delta_{jl} + 2(p-2)(1 + |\mathcal{D}u|)^{p-3}|\mathcal{D}u|^{-1}\mathcal{D}_{l3}\mathcal{D}_{j3},$$

for $j, l \neq 3$. Note that $a_{jl} = a_{lj}$. One easily shows that

$$\sum_{j,l=1}^{2} a_{jl}\xi_j\xi_l = (1 + |\mathcal{D}u|)^{p-2}|\xi|^2 + 2(p-2)(1 + |\mathcal{D}u|)^{p-3}|\mathcal{D}u|^{-1}\left[(\mathcal{D}u)\cdot\xi\right]_3^2.$$

Hence the matrix A is symmetric and positive definite. Moreover, the above identity shows that all the eigenvalues are larger than or equal to $(1 + |\mathcal{D}u|)^{p-2}$. Hence,

$$detA \geq \left((1 + |\mathcal{D}u|)^{p-2}\right)^2.$$

Next, by setting $\xi_l = \frac{\partial^2 u_l}{\partial x_3^2}$, we get from (59), i.e. from

$$\sum_{l=1}^{2} a_{jl}\xi_l = \widetilde{F}_j, \tag{63}$$

that

$$\sum_{l,j=1}^{2} a_{jl}\xi_l\xi_j = \sum_{j=1}^{2} \widetilde{F}_j\xi_j. \tag{64}$$

Consequently $(1 + |\mathcal{D}u|)^{p-2}|\xi|^2 \leq |\widetilde{F}||\xi|$, which shows that

$$(1 + |\mathcal{D}u|)^{p-2}\sum_{l=1}^{2}\left|\frac{\partial^2 u_l}{\partial x_3^2}\right| \leq |\widetilde{F}| := \left(\sum_{j=1}^{2}|\widetilde{F}_j|^2\right)^{1/2}, \tag{65}$$

almost everywhere in Ω. By appealing to (61) and (62) one shows that

$$(1 + |\mathcal{D}u|)^{p-2}\sum_{l=1}^{2}\left|\frac{\partial^2 u_l}{\partial x_3^2}\right| \leq c(1 + |\mathcal{D}u|)^{p-2}|D_*^2 u(x)| + c\left(|\nabla^*\pi| + |f|\right). \tag{66}$$

In particular,

$$\sum_{l=1}^{2}\left|\frac{\partial^2 u_l}{\partial x_3^2}\right| \leq c|D_*^2 u(x)| + c\left(|\nabla^*\pi| + |f|\right), \tag{67}$$

almost everywhere in Ω. There readily follows, by appealing to (56), that (57) holds. The proof of the estimate (24) is accomplished. □

5.2. **Proof of the estimate** (26)**.** The following anisotropic, Sobolev type, embedding theorem is a particular case of more general results proved by Troisi in reference [25]. It is a crucial tool for proving the Theorem 2.3.

Proposition 2. *Let* Ω *be as above, and let* $v \in W^{1,1}(\Omega)$*. Assume that*

$$\partial_k v \in L^{p_k}, \textit{for} \quad k = 1, 2, 3, \tag{68}$$

where

$$\frac{1}{\overline{p}} := \frac{1}{3} \sum_{k=1}^{3} \frac{1}{p_k} - \frac{1}{3}. \tag{69}$$

Then $v \in L^{\overline{p}}(\Omega)$ *and*

$$\|v\|_{\overline{p}} \leq c \prod_{k=1}^{3} \|\partial_k v\|_{p_k}^{\frac{1}{3}} + c\|v\|_p. \tag{70}$$

Obviously, we may replace $\|v\|_p$ by any other L^s norm, $s \geq 1$.

An essential point in order to get the limit exponent l in the proof of Theorem 2.3 is that the constant c on the right hand side of (70) does not depend on the values of the exponents p_k used in the sequel. This property holds provided that $\overline{p}$ lies bounded away from 3. This follows essentially from the equation (1.15) in the above reference (nevertheless, note that each of the values p_k used in our proof lie bounded away from 3).

We start by noting that

$$|\partial_{x_k} |\mathcal{D}u|^{p-1}| \leq (p-1)|\mathcal{D}u|^{p-2} |\partial_{x_k} \mathcal{D}u|. \tag{71}$$

Lemma 5.3. *Assume that the hypotheses in the Theorem 2.2 hold. Then*

$$\|\nabla^* |\mathcal{D}u|^{p-1}\|_r \leq \mathcal{K}_q \tag{72}$$

and

$$\|\partial_{x_3} |\mathcal{D}u|^{p-1}\|_s \leq c\|\mathcal{D}u\|_q^{p-2} \mathcal{K}_q, \tag{73}$$

where $s = s(q)$ *is given by*

$$\frac{1}{s(q)} := \frac{p-2}{q} + \frac{1}{r(q)} = \frac{3p+q-6}{2q}. \tag{74}$$

Proof. The estimate (72) follows from (71), for $k = 1, 2$, together with (24).

On the other hand, from (71) and Hőlder's inequality, one gets

$$\|\partial_{x_3} |\mathcal{D}u|^{p-1}\|_s \leq c\|\mathcal{D}u\|_q^{p-2} \|\partial_{x_3} \mathcal{D}u\|_r. \tag{75}$$

The estimate (73) follows by appealing to (24). ◻

Define $\alpha = \alpha(q)$ by

$$\frac{1}{\alpha(q)} = \frac{1}{3} \left(\frac{2}{r(q)} + \frac{1}{s(q)} \right) - \frac{1}{3}, \tag{76}$$

Note that

$$\frac{1}{\alpha(q)} = \frac{1}{r(q)} + \frac{p-2}{3q} - \frac{1}{3}, \tag{77}$$

moreover, recall (17),

$$\mathcal{Q}(q) = (p-1)\alpha(q).$$

Lemma 5.4. *Assume that the hypotheses in the Theorem 2.2 hold. Then* $\nabla u \in L^{\mathcal{Q}(q)}$*, moreover,*

$$\|\nabla u\|_{\mathcal{Q}(q)} \leq c\|f\|^{\frac{1}{p-1}} + c\|\mathcal{D}u\|_q^{\frac{5(p-2)}{6(p-1)}}\|f\|^{\frac{1}{p-1}} + c\|\nabla u\|_p. \tag{78}$$

In particular, if $q < p+4$*,*

$$\|\nabla u\|_{\mathcal{Q}(q)} \leq c(\|f\|^{\frac{1}{p-1}} + \|\nabla u\|_p) + c\|f\|^{\frac{6}{p+4}}. \tag{79}$$

Proof. From (72), (73) and Proposition 2, it follows that

$$\||\nabla u|^{p-1}\|_\alpha \leq c\mathcal{K}_q\|\mathcal{D}u\|_q^{\frac{p-2}{3}} + c\|\nabla u\|_p^{p-1}. \tag{80}$$

Equation (78) follows from (80) and (25). Next, since $q < \mathcal{Q}(q)$ for $q < p+4$, we may write (78) with $\|\mathcal{D}\,u\|_q$ replaced by $\|\mathcal{D}\,u\|_{\mathcal{Q}(q)}$. The estimate (79) follows, since $\frac{5\,(p-2)}{6\,(p-1)} < 1$. ☐

6. Proof of Theorem 2.3.

Define the increasing sequence

$$q_1 = p, \qquad q_{n+1} = \mathcal{Q}(q_n). \tag{81}$$

Since $q_n < p+4$ for each n, and $\nabla u \in L^{q_1}$, an induction argument shows (79), for each $q = q_n$. Since

$$\lim_{n\to\infty} \mathcal{Q}(q_n) = p+4, \tag{82}$$

equation (32) holds.

Finally, the estimates (33) follows by applying once more the Theorem 2.2, now with $q = p+4$. In this case the equation (16) shows that $r(p+4) = l$, with l given by (34). Hence, from (24), it follows that

$$\|\nabla^*\pi\|_l + \|D^2u\|_l + \|(1+|\mathcal{D}u|)^{p-2}\nabla^*\mathcal{D}u\|_l \leq \mathcal{K}_{p+4} \leq c\|f\| + c\|1+|\mathcal{D}u|\|_{p+4}^{\frac{p-2}{2}}\|f\|. \tag{83}$$

Finally, by appealing to (32) we show (33).

Regularity and estimates for $\frac{\partial\pi}{x_3}$ (hence, for $\nabla\pi$), see (36), follows immediately by appealing to the Lemma 1 with $q = p+4$. Note that $m = \widetilde{q}(p+4) = \overline{q}(p+4)$ since $p+4 > 7-2p$ (see the remark after (20)).

7. Proof of proposition 1.

Proof. From equation (58) written for $j = 3$, we get

$$\left|\frac{\partial\pi}{\partial x_3}\right| \leq c(1+|\mathcal{D}u(x)|)^{p-2}|D_*^2u(x)| + c(p-2)(1+|\mathcal{D}u(x)|)^{p-2}\sum_{l=1}^{2}\left|\frac{\partial^2 u_l}{\partial x_3^2}\right| + |f_3(x)|, \tag{84}$$

almost everywhere in Ω. Hence, by (66), one has

$$\left|\frac{\partial\pi}{\partial x_3}\right| \leq c(1+|\mathcal{D}u|)^{p-2}|D_*^2u| + c(|\nabla^*\pi| + |f|). \tag{85}$$

On the other hand, by Hőlder's inequality,

$$\|(1+|\mathcal{D}u|)^{p-2}D_*^2u\|_{\overline{q}(q)} \leq \|1+|\mathcal{D}u|\|_{\mathcal{Q}(q)}^{p-2}\|D_*^2u\|. \tag{86}$$

By (22) and (26) one gets

$$\|(1+|\mathcal{D}u|)^{p-2}D_*^2u\|_{\overline{q}} \leq c\left[1+A_q^{p-2}\right]\|f\|. \tag{87}$$

Finally, (29) follows by appealing to (85), (87) and (24). □

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