Concerning the Ladyzhenskaya-Smagorinsky turbulence model of the Navier-Stokes equations

Hugo Beirão da Veiga,

Dipartimento di Matematica Applicata "U.Dini", Via Buonarrotti, 1/C, 56127 Pisa, Italy

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Abstract

In some recent papers we have been pursuing regularity results *up to the boundary*, in $W^{2,l}(\Omega)$ spaces for the velocity, and in $W^{1,l}(\Omega)$ spaces for the pressure, for fluid flows with shear dependent viscosity. To fix ideas, we assume the classical non-slip boundary condition. From the mathematical point of view it is appropriate to distinguish between the shear thickening case, $p > 2$, and the shear thinning case, $p < 2$, and between flat-boundaries and smooth, arbitrary, boundaries. The $p < 2$, non flat boundary case, is still open. The aim of this work is to extend to smooth boundaries the results proved in reference [23]. This is done here by appealing to a quite general method, introduced in reference [21], suitable for considering non-flat boundaries.

Résumé

Sur le modèle de turbulence de Ladyzhenskaya-Smagorinsky des équations de Navier-Stokes. Dans des articles récents (voir ci-après) nous avons démontré des résultats de régularité dans des espaces L^q pour les dérivées secondes de la vitesse et les dérivées premières de la presssion, pour des systèmes de Stokes et Navier-Stokes avec des viscosités qui dépendent de la partie symétrique du gradient des vitesses. Nous considérons seulement des résultats de régularité valables *jusqu'au bord*. Dans ces articles nous avons considéré des frontières plates. Tous récemment nous avons généraliser ces résultats aux cas des frontières arbitraires. Le but de cette note est de décrire ces résultats, avec des commentaires adéquats. *Pour citer cet article : H. Beirão da Veiga, C. R. Acad. Sci. Paris, Ser. I 345 (2007).*

1. Introduction

The Navier-Stokes system of equations with shear dependent viscosity has been studied in the last forty years by a great number of researchers, not only in pure and applied mathematics, but also in engineering, physics and biology. A typical model of generalized stationary Navier-Stokes system of equations with shear dependent viscosity is the well known model (general

Email address: bveiga@dma.unipi.it (Hugo Beirão da Veiga).

$$\begin{cases} -\nabla\cdot T(u,\pi)+\,u\,\cdot\,\nabla\,u\,=\,f, \\ \\ \nabla\cdot u=\,0\,, \end{cases} \tag{1}$$

where T denotes the stress tensor olg

$$T=-\pi\,I+\nu_T(u)\,\mathcal{D}\,u \tag{2}$$

and

$$\mathcal{D}\,u=\,\frac{1}{2}\,(\nabla\,u+\,\nabla\,u^T)\,.$$

The first mathematical studies on the above class of equations go back to O.A. Ladyzhenskaya in a series of remarkable contributions. See [45], [46], [47] and [48]. In references [49] and [50], Chap.2, n.5, J.-L. Lions considers the case in which $\mathcal{D}\,u$ is replaced by ∇u. However in this case the Stokes principle, see [66] and [65] page 231, is not satisfied. Such models, an instance of which is (3), were intensively studied in the eighties and nineties by J.Nečas and his school.

Nonlinear shear dependent viscosities are used, in particular, to model properties of materials. The cases $p>\,2$ and $p<\,2$, see equation (6), capture shear thickening and shear thinning phenomena, respectively. The case $p=\,3$ was introduced by Smagorinsky, see [15], as a turbulence model. In the sequel, we concentrate on, and assume that, $1<\,p\leq\,2$ and (for convenience) that $n=\,3$.

For comments and references, both to modeling and theory, we refer the reader to [44], [53], [54] and [59].

In a series of recent papers we introduced a general scheme suitable to solving the problem of the regularity up to the boundary of p-fluid flows, under typical boundary conditions. We began this series of papers by considering the half space case, see [18], and the case of a cubic domain, see [19] and [20]. In this last case the interesting boundary condition is given on two opposite faces, and space-periodicity in assumed in the other two directions. These two frameworks avoid the need of appealing to localization techniques and to changes of variables (in order to flatten the boundary). It is worth noting that when $p\neq 2$ the extension of regularity results from flat boundaries to arbitrary, regular, boundaries presents many new unusual obstacles, compared to the (still non trivial) classical case $p=2$.

Since we have dedicated a certain number of papers to the above subject, the following overview could help the interested reader. In [18], we establish the main lines to treat problems in the flat boundary case. More precisely, we consider the slip and the non-slip boundary value problem in the half space $\mathbb{R}^n{}_+$, and $p>\,2$. In reference [19] we replace, for simplicity, the half-space $\mathbb{R}^n_+$ by the above three dimensional cube and consider the non-slip boundary condition. Further, we introduce the convective term and the evolution problem.

In reference [20] we consider the $p<\,2$ case. Here, an idea borrowed from the Lemma 6 in reference [38], is crucial (see the Lemma 3.2, [20]). Further, by introducing a new device (see [19], Remark 5.1), we drop the $-\,\Delta\,u$ term from the equations.

It is worth noting that the addition of a $-\,\Delta\,u$ term on the left hand side of the equations simplify the proofs. Actually, it allows much stronger regularity results, specially in the $p<\,2$. This case, much easier to handle, is more in accordance with the physical problems. Actually, in [22], it is shown that weak solutions belong to $W^{1,\,q}(\Omega)$, for any finite q, provided that an x-dependent growth condition holds, $p=p(x)\leq\,2$. Convexity-type assumptions are not assumed. Under more classic assumptions, see for instance [30], one shows that $u\in\,W^{2,\,2}$.

To finish this overview on our recent contributions, we refer to [24], where the previous results on the shear thickening case are improved.

A main open problem remains the extension of the above types of results to non-flat boundaries (in this context, see also [52]). This requires really new ideas, since the presence of the $\mathcal{D}\,u$ term together with $p\neq\,2$, makes the boundary value problem particularly difficult. We solve this problem in reference [21], where $p>\,2$.

In the mean-time, in references [28] and [29], F. Crispo has extended the $p<\,2$ results to cylindrical domains, by appealing to cylindrical coordinates. This change of coordinates requires particular care, due to the non linear p-term. Further, L.C. Berselli, see [25], improves the argument followed in [19], by replacing the classical (isotropic) Sobolev embedding theorems by anisotropic embedding theorems. This very fruitful idea is used by us below (in section **??**). Next, in reference [23], we improve previous results shown for the shear thinning case. In particular, we obtain a better value for the parameter p_0 in the Navier-Stokes problem (see bellow) by replacing the device borrowed from [38], by a different idea. It remains the open problem of the extension, from flat to regular boundaries, of the sharp results proved in [23]. This is the aim of this paper.

For convenience, we call "the Stokes problem" the problem without the convective term $(u \cdot \nabla)\, u$, and "the Navier-Stokes problem" the problem with the above term included. Concerning our approach to $W^{2,\,l}(\Omega)$ regularity results up to the boundary, the really new points mostly concern the stationary Stokes problem. In fact, in our proofs, the inclusion of the convective term, and the consideration of the evolution problem, are reduced in a very simple way to the stationary Stokes problem. It goes without saying that we do not claim that it is not possible to obtain better results by different methods. In our approach, a) The Stokes evolution problem can be easily reduced to the stationary Stokes problem, with the same range of admissible values of p. b) In the stationary case, the presence of the convective term requires an assumption of the type $p > p_0$ for some $p_0 < 2$, see Theorem 2.3. Under this assumption the regularity results for the Stokes and the Navier-Stokes stationary problems, coincide. c) For the Navier-Stokes evolution problem we need a condition $p > p_1$, for some $p_1 > 2$. Hence the shear thinning case is excluded, except for sufficiently small initial data. In this last case we believe that it should be not difficult to prove the existence of a global, regular, solution.

————————————————————————-Comptes

In the sixties Olga Ladyzhenskaya proposed the following system of equations

$$\partial_t u + (u \cdot \nabla)\, u - \nabla \cdot T(u, \pi) = f\,, \qquad \nabla \cdot u = 0\,, \tag{3}$$

in $\Omega \times]0, T]$, as a model for turbulence phenomena. In this model the stress tensor $\mathbb{T} = -\pi\, I + \nu_T(u)\, \mathcal{D}\, u$ depends on the symmetric part of the gradient of the velocity $\mathcal{D}\, u = \frac{1}{2}\, (\nabla\, u + \nabla\, u^T)$, in a polynomial way, with p-rate of growth, for $p > 2$. To avoid additional technical difficulties, we consider the typical model of stress tensor $\mathbb{T}$ given by $\nu_T(u) = \nu_0 + \nu_1 |\mathcal{D}\, u|^{p-2}$, where ν_0 and ν_1 are strictly positive constants. In this case the Stokes Principle (see [66], and [65] page 231) is satisfied.

The first mathematical studies on the above class of equations go back to O.A. Ladyzhenskaya. See [45], [46], [47] and [48]. The case $p = 3$ was introduced by Smagorinsky, see [15], as a turbulence model. In reference [50], Chap.2, n.5, J.-L. Lions considers the case in which $\mathcal{D}\, u$ is replaced by ∇u. However in this case the Stokes principle is not satisfied.

Such kind of models were intensively studied in the eighties and nineties by J.Nečas and his school, see [51], in order to study certain particular kinds of fluids, and in particular to describe shear thickening ($p > 2$) and shear thinning ($p < 2$) phenomena. In particular, L^q regularity results, up to the boundary, for $p > 2$ and under the non-slip boundary condition

$$u_{|\Gamma} = 0\,, \tag{4}$$

are stated in [52]. The mathematical analysis concerning such kind of models is far from being trivial, and usually involves a large amount of delicate arguments of both technical and substantial nature.

In recent papers we have considered the question of regularity up the boundary of solutions to the above kind of modified Navier-Stokes equations. In particular, in [18] and [19], the case $p > 2$ is studied. In these references the very basic results are those proved for the generalized *Stokes stationary problem*

$$\begin{cases} -\dfrac{\nu_0}{2}\, \Delta\, u - \nu_1\, \nabla \cdot (|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, u) + \nabla\, \pi = f, \\ \nabla \cdot u = 0\,, \end{cases} \tag{5}$$

since the results remain to be true in the presence of the convective term (for $p > 2$, its role is not so crucial), and suitable extensions to the evolution problem can be proved in a quite simple way.

The main aim of the article [21] is to extend our previous results to the case of curvilinear boundaries. This is a quite difficult technical problem, especially for equations containing viscosity depending on the module of the symmetric part of the gradient. For it, the known scheme developed for the case in which coefficients in the equations depends on the module of the gradient (Lions model), does not work (see the Remark 1 below). Our proof is done via a very careful analysis up to boundary, and a suitable application of a modified difference quotient method (this is the relevant novelty of the paper) overcoming the simultaneous appearance of three difficulties: boundary regularity (that is, how to recover the vertical derivatives of $\mathcal{D}\,u$ from the tangential ones), the divergence constraint to be met at each choice of the test functions, and the fact that the system actually depends on the symmetric part of the gradient, rather than on the gradient itself. This leads to the introduction of a certain number of interesting new tricks. The results are anyway proved by first arguing locally, via a suitable flattening of the boundary, and then by a covering argument to recover the final global estimate.

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2. Main results

In the sequel we consider the following very basic model of generalized Stokes stationary problem, where $\nu_T(u) = (1 + |\mathcal{D}\,u|)^{p-2}$: (particular)

$$\begin{cases} -\nabla \cdot (\,(1+\,|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,u\,) + \nabla\,\pi = \,f, \\ \nabla \cdot u = \,0\,, \quad \text{in} \quad \Omega\,, \end{cases} \tag{6}$$

under the non-slip boundary condition (dirich)

$$u_{|\Gamma} = \,0\,. \tag{7}$$

The domain Ω is a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a manifold of class C^2.

In the sequel we use the following exponents (quesgr)

$$r(q) = \frac{2\,q}{2\,(2-p)+\,q}\,, \quad \lambda(q) = \frac{2\,q}{2-\,p+\,q}\,, \quad \mathcal{Q}(q) = \frac{6\,q}{8-\,4\,p+\,q}\,, \tag{8}$$

and also (essses)

$$\overline{q} = \,4\,p-\,2\,, \quad l = \frac{4\,p-\,2}{p+\,1}\,. \tag{9}$$

Theorem 2.1 *Assume that $f \in L^{p'}(\Omega)$ and let $u \in V_p$ be a solution to the problem (6), (7), where $\frac{3}{2} < p < 2$. Assume that (osdes)*

$$\mathcal{D}\,u \in L^q(\Omega)\,, \tag{10}$$

for some q satisfying

$$p \le \,q \le \,6\,.$$

Then (ertence)

$$u \in \,W^{1,\,\mathcal{Q}(r)}(\Omega) \cap W^{2,\,r(q)}(\Omega)\,, \quad \nabla\,\pi \in \,L^{r(q)}(\Omega)\,. \tag{11}$$

Further,(qmesmo)

$$\|\nabla\,u\|_{\mathcal{Q}(q),\,\Omega} \le C\,(1+\,\|f\|_{p'}\,)\,\left(1+\,\|\,\nabla\,u\,\|_{q,\,\Omega}^{\frac{2\,(2-\,p)}{3}}\right)\,, \tag{12}$$

and (sego)

$$\|D^2\,u\|_{r(q),\,\Omega} + \,\|\nabla\,\pi\|_{r(q),\,\Omega} \le C\,(1+\,\|f\|_{p'}\,)\,\left(1+\,\|\,\nabla\,u\,\|_{q,\,\Omega}^{\frac{2-\,p}{2}}\right)\,. \tag{13}$$

Note that the assumption (10) holds for $q = p$. This furnishes a first regularity theorem (statement left to the reader). Furthermore, the Theorem 2.1 allows a bootstrap argument, similar to that introduced in references [18] and [19]. This leads to the following improvement, and extension to general boundaries, of the Theorem 1.4 in [20].

Theorem 2.2 *Assume that* $f \in L^{p'}(\Omega)$ *and let* $u \in V_p$*, see* (19)*, be a solution to the problem* (6), (7)*, where* $\frac{3}{2} < p < 2$*. Then (see* (9)*)*

$$u \in W^{2,l}(\Omega) \cap W^{1,\overline{q}}(\Omega), \qquad \nabla \pi \in L^l(\Omega). \tag{14}$$

Moreover, (rei)

$$\|u\|_{1,\overline{q}} \leq C\,(1 + \|f\|_{p'}^{\frac{3}{2p-1}}). \tag{15}$$

and reiz

$$\|u\|_{2,l} \leq C\,(\|f\|_{p'} + |f\|_{p'}^{\frac{5-p}{2p-1}}). \tag{16}$$

Concerning the full Navier-Stokes system (generil) one has the following result.
Theorem 2.3 *Let* u *be a solution to the full Navier-Stokes equations* (**??**) *under the boundary condition* (7)*. Set (pzo)*

$$p_0 = \frac{20}{11} = 1.8181... \tag{17}$$

Then, under the assumption $p > p_0$, (14) *holds.*
Besides the extension to general boundaries, the above result improves the lower bound $p_0 == \frac{15}{8}$ obtained in [20] and the lower bound $p_0 = \frac{7+\sqrt{35}}{7}$, obtained in [25]. It coincides with the value p_0 that was reached in reference [23].

It is worth noting that the boundedness of Ω is not essential here. In fact, our proof is done locally, i.e., in "small" neighborhoods of each point $x_0 \in \Gamma$. Consequently, the results hold, in particular, in any bounded subset of Ω, since boundedness is used only in order to work with a compact boundary (just to guarantee that local parameters associated with the boundary Γ have uniform bounds).

3. Notation. Weak Solutions

In general we set

$$T_{sym} = \frac{1}{2}\,(T + T^T), \tag{18}$$

where T is a generic tensor field and T^T is it transpose. In particular, $\mathcal{D}\,u = (\nabla\,u)_{sym}$.

The symbol $\|\,.\,\|_p$ denotes the canonical norm in $L^p(\Omega)$. Further, $\|\,.\,\| = \|\,.\,\|_2$. We denote by $W^{k,p}(\Omega), k$ a positive integer and $1 < p < \infty$, the usual Sobolev space of order k, by $W_0^{1,p}(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $C_0^\infty(\Omega)$ and by $W^{-1,p'}(\Omega)$ the strong dual of $W_0^{1,p}(\Omega)$, where $p' = p/(p-1)$. The canonical norms in these spaces are denoted by $\|\,.\,\|_{k,p}$. $L^p_\#(\Omega)$ denotes the subspace of L^p consisting of functions with vanishing mean value.

In notation concerning duality pairings, functional spaces and norms, we will not distinguish between scalar and vector fields. Very often we also omit from the notation the symbols indicating the domains Ω or Γ, provided that the meaning remains clear.

————————————————————————————-puns for any open subset Ω_0 of $\mathbb{R}^3$.

We set

$$V_p = \left\{ v \in W^{1,p}(\Omega) : (\nabla \cdot v)_{|\Omega} = 0\,;\ v_{|\Gamma} = 0 \right\}. \tag{19}$$

Note that, by appealing to inequalities of Korn's type, one shows that there is a positive constant c such that

$$\|\nabla v\|_p + \|v\|_p \leq c\,\|\mathcal{D}v\|_p \tag{20}$$

for each $v \in V_p$. Hence the two quantities above are equivalent norms in V_p. Actually, $\|\mathcal{D}\,v\|_p$ is a norm in $W_0^{1,p}$.

We denote by c, $\overline{c}$, c_1, c_2, etc., positive constants that depend, at most, on Ω, and p. The dependence of the constants c on p is not crucial provided that $1 < p_0 \leq p \leq p_1 < \infty$. The same symbol c may denote different constants, even in the same equation.

————————————————————————————- Below Ω is a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary $\Gamma = \partial\,\Omega$, a manifold of class C^2. For saving space, in this note we do not write estimates. Concerning the stationary case, we prove three main theorems. Theorem 3.1, which can be considered as a "starting point", ensures the global higher differentiability of the solution (u, π). Rather explicit a priori estimates are also proved.

Theorem 3.1 *Let be* $2 \leq p \leq 3$. *Assume that* $f \in L^2(\Omega)$ *and let* (u, π) *be the weak solution to problem* (6) *under the boundary condition* (7). *Assume, in addition, that* $\mathcal{D}\, u \in L^q(\Omega)$ *for some* $p \leq q \leq 6$. *Then*

$$u \in W^{2,r}(\Omega), \quad \nabla \pi \in L^{\overline{p}}(\Omega), \quad \textit{and} \quad \pi \in L^r(\Omega), \tag{21}$$

where $1/r = (p-2)/2\,q + 1/2$ *and* $1/\overline{p} = (p-2)/q + 1/2$.

Note that at this stage the assumption $\mathcal{D}\, u \in L^q$ is only satisfied with q = p, since a solution, by standard monotonicity methods, can be initially found in $W^{1,p}$. Therefore the application of this result with $q = p$ leads to the following regularity result:

Theorem 3.2 *Let be* $2 \leq p \leq 3$. *Assume that* $f \in L^2(\Omega)$ *and let* (u, π) *be the weak solution to problem* (6) *under the boundary condition* (7). *Then*

$$u \in W^{2,p'}(\Omega) \quad \nabla \pi \in L^{p_0}(\Omega), \quad \textit{and} \quad \pi \in L^{p'}(\Omega), \tag{22}$$

where $p_0 = 2\,p/(3p-4)$.

The next step, which leads to the main regularity result of the paper, is to iterate Theorem 3.1 starting from Theorem 3.2 as follows: Theorem 3.2 allows to get higher integrability of $\mathcal{D}\, u$ via standard Sobolev embedding, let's say $\mathcal{D}\, u \in L^q$; then Theorem 3.1 allows to get higher integrability of second derivatives; in turn this implies higher integrability of $\mathcal{D}\, u$ and so on. Note that this kind of iterations usually lead to establish higher integrability for every exponent strictly less than a certain limiting one, say l. Actually, the exponent l can be reached since in Theorems 3.1 and 3.2 explicit boundary estimates are provided, in turn allowing for a precise control of the constants in the iteration procedure. We have therefore the following result:

Theorem 3.3 *Let be* $2 \leq p \leq 3$, *and let* f, u *and* π *be as in Theorem 3.2. Then*

$$u \in W^{2,l}(\Omega) \quad \textit{and} \quad \nabla \pi \in L^m(\Omega), \tag{23}$$

where $l = 3\,(4-p)/(5-p)$ *and* $m = 6\,(4-p)\,/(8-p)$.

Denote by s^* the immersion Sobolev exponent for which $W^{1,s} \subset L^{s^*}$. In Theorem 3.3 the exponent l turns out to be just the exponent for which Theorem 3.1, with $q = l^*$, yields $u \in W^{2,l}$. Then, by a Sobolev embedding Theorem, $u \in W^{1,l^*}$. In other words, $q = l^*$ is the fixed point of the map $q \to r \to r^*$. So, further regularity cannot be obtained by appealing to Theorem 3.1.

The extension to solutions to the stationary generalized Navier-Stokes system (obtained by addition of $(u \cdot \nabla)\, u$ to the left hand side of (6)) is straightforward.

Theorem 3.4 *All the regularity results stated in the Theorems 3.1, 3.2 and 3.3 hold for the generalized Navier-Stokes equations.*

Finally, we turn to examine the situation in the evolution case, where similar results can be obtained provided the exponent p is large enough to avoid the appearance of super-critical non-linearities in the convective term.

Remark 1 **On the regularity up to the boundary.** In proving interior regularity by the classical translation method, the translations are admissible in all the n independent directions. This allows suitable L^2-estimates for $\nabla\, \mathcal{D}\, u$, where the full gradient ∇ is obtained thanks to the possibility of appealing to translations in all the directions. On the other hand, $c\,|\nabla\,\nabla\, u| \leq |\nabla\, \mathcal{D}\, u| \leq C\,|\nabla\,\nabla\, u|$. These two facts together lead to a small distinction if we replace $\mathcal{D}u$ by ∇u in the expression of the stress tensor. However, in proving regularity up to the boundary, the two cases are completely distinct. In fact, solutions to the J.-L. Lions model belong to $W^{2,2}$ up to the boundary. It seems not accidental that there is a very extensive literature on interior regularity for the above class of problems but, as far as we know, scant literature concerning regularity up to the boundary, in the $3-D$ case.

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