Differential and Integral Equations Volume 15, Number 3, March 2002, Pages 345–356

ON THE REGULARIZING EFFECT OF THE VORTICITY DIRECTION IN INCOMPRESSIBLE VISCOUS FLOWS

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(Submitted by: Haim Brezis)

Abstract. We improve results in reference [6] concerning the effect of the direction of the vorticity on the regularity of weak solutions to the 3D Navier–Stokes equations. In particular, we prove that, if the direction of the vorticity belongs to suitable Sobolev spaces, then there exists a unique smooth solution of the Cauchy problem for the Navier–Stokes equations.

1. Introduction

In this paper we study how the knowledge of some conditions on the direction of vorticity field can be used to prove the smoothness of solutions to the 3D Navier–Stokes equations. We improve results obtained by Constantin and Fefferman in reference [6], by following essentially their approach. Here we relax the assumptions on the direction of vorticity

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|},$$

needed to ensure the smoothness of the solutions.

We consider the Navier–Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)\,u - \nu\Delta u + \nabla p = f \quad \text{in } \mathbb{R}^3\times[0,T], \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3\times[0,T], \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^3. \end{cases} \tag{1.1}$$

For simplicity we suppose that the external force f vanishes and that the initial datum is smooth. A classical result, that dates back to Leray [12],

Accepted for publication: January 2001.
AMS Subject Classifications: 35Q30, 35B65, 76C99.

states that for any fixed $T > 0$ there exists at least a *weak solution* of system (1.1) in $(0,T)$, i.e., a divergence-free vector field u such that

$$u \in C_w(0,T;L^2) \cap L^2(0,T;H^1)$$

and that

$$\int_0^T \int_{\mathbb{R}^3} \left[u\frac{\partial \phi}{\partial t} - \nu \nabla u \cdot \nabla \phi - (u \cdot \nabla) u \phi \right] dx\, dt = \int_{\mathbb{R}^3} u(T)\phi(T) - u_0 \phi(0)\, dx,$$

for all divergence-free $\phi \in C^1(0,T;H^1)$. In the sequel we denote by $L^p := L^p(\mathbb{R}^3)$, for $1 \leq p \leq \infty$ and equipped with norm $|\,.\,|_p$, the usual Lebesgue spaces, while $H^s := H^s(\mathbb{R}^3)$, for $s \geq 0$, are the classical Sobolev spaces. We indicate with the same symbol both scalar and vector function spaces. We also recall that $C_w(0,T;L^2)$ is the space of weakly continuous functions on $(0,T)$ with values in L^2.

By a *strong solution* we mean a weak solution u such that

$$u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2).$$

It is well-known that strong solutions are regular (say, classical) and unique in the class of weak solutions. Unfortunately, it is not known whether a strong solution exists or not in arbitrarily large time-intervals $(0,T)$. A main problem is then to prove that for smooth, divergence-free initial data there exists a strong solution for all times. There are some estimates on the Hausdorff dimension (in space-time) of the possible singular set and, as remarked in Caffarelli, Kohn, and Nirenberg [3], where the best results in this direction are obtained, ... *the theory remains fundamentally incomplete. In particular it is not known whether or not the velocity u develops singularities even if all the data are C^∞.*

It is important to note that for the problem in $\mathbb{R}^2$ the situation is completely different, since it is possible to prove the global existence in time of strong solutions; see Ladyženskaya [11]—see also J.L. Lions [13] or Constantin and Foiaş [7]. Another way to understand this result is to study the vorticity field

$$\omega(x,t) := \nabla \times u(x,t).$$

In two dimensions ω is a vector always perpendicular to the plane of motion. Furthermore, by taking the curl of the first equation in (1.1) we have

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\, \omega - \nu \Delta \omega = 0,$$

and then the vorticity satisfies a linear evolution equation. In particular, the modulus of the vorticity cannot increase.

In three dimensions the situation changes drastically: if we take the curl of the first equation in (1.1) we obtain

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\,\omega - \nu \Delta \omega = \omega \nabla u, \tag{1.2}$$

and the right-hand side is the so-called *stretching term*. The vorticity can increase its modulus and can change its direction.

A simple condition on ω that ensures regularity is that considered in reference [2]: the solution is regular if the vorticity ω belongs to $L^p(0,T;L^s)$, or equivalently if u belongs to $L^p(0,T:W^{1,s})$, for

$$\frac{2}{p} + \frac{3}{s} = 2 \quad \text{with} \quad 1 \le p \le 2 \tag{1.3}$$

(hence if $3 \le s \le \infty$). This condition is a very natural extension to the values $1 \le p \le 2$ of the classical criterion that establishes the regularity of the weak solutions that belong to $L^p(0,T;L^q)$ for

$$\frac{2}{p} + \frac{3}{q} = 1 \quad \text{with} \quad 2 \le p < \infty \tag{1.4}$$

(hence if $3 < q \le \infty$); see Galdi and Maremonti [8] and references therein. Let us show a heuristic, but significant, explanation of (1.3) in the light of (1.4). It is well-known that for $1 \le s < 3$ one has $W^{1,s} \subset L^q$, where

$$\frac{1}{q} = \frac{1}{s} - \frac{1}{3}. \tag{1.5}$$

Suppose now that for $s \ge 3$ this embedding theorem and the above classical regularity criterion are "still true." Clearly, the "space" L^q is now just a symbol, to be replaced in the true theorem by $W^{1,s}$, s given by (1.5). Assume, moreover, that u belongs to some $L^p(0,T;W^{1,s})$. Then, by the "heuristic embedding theorem," $u \in L^p(0,T;L^q)$, where

$$\frac{2}{p} + \frac{3}{q} = \frac{2}{p} + \big(\frac{3}{s} - 1\big). \tag{1.6}$$

Hence by the "heuristic classical criterion," u is regular if $\frac{2}{p} + \frac{3}{q} = 1$. Hence, by (1.6), u is regular if condition (1.3) holds. This is just the result proved in reference [2].

Note that for $p = 2$ condition (1.3) gives $u \in L^2(0,T;W^{1,3})$, and this space is not included in $L^2(0,T;L^\infty)$, the corresponding space in the above classical condition (1.4). Recently Kozono and Taniuchi [9] have shown that $u \in L^2(0,T;BMO)$ is a regularity class, where $BMO \supset L^\infty \cup W^{1,n}$ is the classical space of functions with "bounded mean oscillation."

It is interesting to note that the limit case $p = 1$ of condition (1.3) gives the Beale–Kato–Majda condition $u \in L^1(0,T;W^{1,\infty})$; see Beale, Kato, and Majda [1]. This condition has been relaxed by Kozono and Taniuchi in reference [10].

Finally, note that in the whole space $\mathbb{R}^3$ it is sufficient to assume the above condition (1.3) just for two of the three components of the vorticity; see Chae and Choe [4].

In the sequel, instead of regularity criteria involving the length of $\omega(x,t)$, we consider criteria involving its direction. To our knowledge, the only known condition involving the direction of vorticity is that considered in Constantin and Fefferman [6]. In this last reference, instead of giving conditions on the modulus (or on some norm) of the vorticity, some estimates involving the direction of vorticity are found. It is proved that if $\theta(x,x+y,t)$, the angle between the vorticities ω at points x and $x+y$ at time t, satisfies

$$\sin\theta(x,x+y,t) \leq \frac{|y|}{\rho(t)} \qquad \text{for} \quad \rho^{-12} \in L^1(0,T),$$

then the solution is smooth in $(0,T)$. Note that

$$\sin\theta(x,x+y,t) = |P^{\perp}_{\xi(x,t)}\xi(x+y,t)|,$$

where $P^{\perp}_{\xi(x,t)}\xi(x+y,t)$ denotes the projection of $\xi(x+y,t)$ on the plane orthogonal to $\xi(x,t)$.

In this paper we improve this result by proving a regularity criterion involving the following assumption.

Assumption A. *There exist* $\alpha \in [1/2,1]$, *a positive constant* k, *and* $g \in L^a(0,T;L^b)$, *where*

$$\frac{2}{a} + \frac{3}{b} = \alpha - \frac{1}{2} \quad \textit{with} \quad a \in \Big[\frac{4}{2\alpha-1}, \infty\Big]$$

such that

$$\sin\theta(x,x+y,t) \leq g(t,x)|y|^{\alpha} \tag{1.7}$$

holds in the region where the vorticity at both x *and* $x+y$ *is larger than* k.

Remark 1.1. Note that (1.7) holds, in particular, if

$$\sin\theta(x,x+y,t) \leq c\,|y|^{1/2}.$$

Assumption A will be used to prove the main result of this paper, namely the following criterion for the regularity of weak solutions.

Theorem 1.2. *Suppose that u is a weak solution of* (1.1) *in* $(0,T)$, *with* $u_0 \in H^1$ *and* $\nabla \cdot u_0 = 0$. *Suppose that Assumption A is satisfied. Then the solution is strong in* $(0,T)$, *hence regular.*

Remark 1.3. It is clear that it is possible to relax Assumption A by assuming that (1.7) is satisfied just for

$$|y| \leq \delta,$$

for an arbitrary positive constant δ.

From Theorem 1.2 one can derive regularity criteria that depend on suitable norms of ξ, instead of depending on pointwise relations, like (1.7). In this regard, see Corollaries 4.2 and 4.3.

2. Some kinematic estimates

In this section we recall some important results, proved in Constantin [5]. They are referred as kinematic (or "frozen time") estimates since some spatial properties of the velocity and vorticity are obtained by the Biot–Savart law, at each time t. Recall that the Biot–Savart law reads

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\Big(\nabla\frac{1}{|y|}\Big)\times\omega(x+y)\,dy. \tag{2.1}$$

By differentiating (2.1) we obtain the following expression for the *strain matrix:*

$$S[\omega](x) := \frac{1}{2}\left[\nabla u(x) + (\nabla u(x))^*\right] = \frac{3}{4\pi}\ P.V.\int_{\mathbb{R}^3} M(\hat{y},\omega(x+y))\frac{dy}{|y|^3}.$$

For convenience we set $S(x) = S[\omega](x)$. In the last formula $\hat{y}$ is the unit vector in the direction of y, while

$$M(\hat{y},\omega) := \frac{1}{2}\left[\hat{y}\otimes(\hat{y}\times\omega) + (\hat{y}\times\omega)\otimes\hat{y}\right]$$

is a symmetric traceless matrix. It defines a proper singular operator, since its mean value on the unit sphere is zero, when ω is held fixed. With this formula we can give a representation for the term

$$\alpha(x) := S(x)\xi(x)\cdot\xi(x), \tag{2.2}$$

the *stretching rate*, that is defined in the set $\{x \in \mathbb{R}^3 :\ |\omega(x)| > 0\}$. The importance of the formula representing α is made clear by recalling the equation satisfied by the vorticity magnitude:

$$\frac{1}{2}\Big(\frac{\partial}{\partial t} + u(x,t)\cdot\nabla - \nu\Delta\Big)|\omega(x,t)|^2 + \nu|\nabla\omega(x,t)|^2 = \alpha(x,t)|\omega(x,t)|^2.$$

From the representation formula for $S(x)$ one can deduce the following elegant formula:

$$\alpha(x) := \frac{3}{4\pi}\ P.V. \int_{\mathbb{R}^3} D(\hat{y}, \xi(x+y), \xi(x))|\omega(x+y)| \frac{dy}{|y|^3}, \tag{2.3}$$

where

$$D(v_1, v_2, v_3) := (v_1 \cdot v_3)\, \mathrm{Determinant}(v_1, v_2, v_3).$$

As remarked in [5], it is interesting to note that in formula (2.3) the stretching term vanishes if $\xi(x+y) = \pm\xi(x)$, because the determinant of $(\hat{y}, \xi(x+y), \xi(x))$ equals the volume of the prism whose edges are $\hat{y}, \xi(x+y)$ and $\xi(x)$. One can easily infer that the local (anti) alignment of the vorticity depletes the nonlinearity.

3. A priori estimates on weak solutions

When dealing with weak solutions of the Navier–Stokes equations, the strongest quantitative result, known up to now, is the energy estimate. If the external force vanishes, the kinetic energy of the fluid dissipates since

$$\frac{1}{2}|u(t)|_2^2 + \nu \int_0^t \int_{\mathbb{R}^3} |\nabla u(x,\sigma)|^2 \, dx \, d\sigma \leq \frac{1}{2}|u_0|_2^2, \quad \forall\, t \geq 0. \tag{3.1}$$

This inequality, stated in the seminal paper by Leray [12], is a fundamental tool in the proof of existence of weak solutions.

A more recent estimate is that obtained in Constantin and Fefferman [6], equation (20), Section 3. By multiplying (1.2) by the vorticity direction ξ (and with a smoothing process), and by integrating in the space-time variables it is possible to prove that

$$|\omega(t)|_1 + \nu \int_0^t \int_{\{x : |\omega(x,\sigma)|>0\}} |\omega(x,\sigma)||\nabla\xi(x,\sigma)|^2 \, dx \, d\sigma \leq |\omega_0|_1 + \frac{2}{\nu}|u_0|_2^2, \tag{3.2}$$

provided that the initial vorticity belongs to L^1.

As remarked in [5], this inequality shows that in region of high vorticity the direction of vorticity is regular, at least in an average sense. In particular, in the region where the modulus of vorticity is large the vortex lines can not bend too much.

Remark 3.1. In Constantin and Fefferman [6] estimate (3.2) is used to prove their regularity criterion. In particular, the uniform boundedness of the L^1 norm of the vorticity is used. Our proof is independent of this estimate and, consequently, we do not need an L^1 bound for the initial vorticity.

4. Proof of Theorem 1.2

In the sequel the symbol c denotes a generic positive constant independent of u, ν, and k. Distinct constants will be denoted by the same symbol c.

In this section we prove the main theorem of this paper. By using the continuation principle for strong solutions it is enough to prove the following lemma.

Lemma 4.1. *Let u, a weak solution of system* (1.1) *on* $(\tau-\delta,\tau)$, *be a strong solution in* $(\tau-\delta,\tau')$ *for each* $\tau'<\tau$. *Assume moreover that Assumption A is satisfied. Under these hypotheses, u is a strong solution in* $(\tau-\delta,\tau)$.

Since we work with solutions that are strong (hence regular) in the interval $(\tau-\delta,\tau')$, it is possible to handle the estimates without introducing smoothing processes to justify the formal calculations.

Proof of Lemma 4.1. By multiplying equation (1.2) by ω, and by some integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}|\omega|_2^2+\nu|\nabla\omega|_2^2=\int_{\mathbb{R}^3} S(x)\,\omega(x)\cdot\omega(x)\,dx. \tag{4.1}$$

Let k be the positive constant in Assumption A and split $\omega(x)$ as $\omega(x)=\omega_1(x)+\omega_2(x)$, where

$$\omega_1(x)=\begin{cases}\omega(x), & \text{if} \quad |\omega(x)|\le k,\\ 0, & \text{if} \quad |\omega(x)|>k,\end{cases}$$

and

$$\omega_2(x)=\begin{cases}0, & \text{if} \quad |\omega(x)|\le k,\\ \omega(x), & \text{if} \quad |\omega(x)|>k.\end{cases}$$

This induces the natural decomposition $S[\omega](x)=S[\omega_1](x)+S[\omega_2](x)$, or, with the simplified notation, $S(x)=S_1(x)+S_2(x)$. Note that, by the Calderón–Zygmund inequality,

$$|S_i|_q\le c\,|\omega_i|_q, \tag{4.2}$$

where $1<q<\infty$ and $i=1,2$.

The integrand $S\,\omega\cdot\omega$ in the right-hand side of equation (4.1) splits into a sum of eight terms of the form $S_i\,\omega_j\cdot\omega_k$, for $i,j,k=1,2$. By using (4.2) with $q=2$ it is immediate to show that

$$\Big|\int_{\mathbb{R}^3} S_i(x)\,\omega_j(x)\cdot\omega_k(x)\,dx\Big|\le c\,k\,|\omega|_2^2, \tag{4.3}$$

when $(j,k)\ne(2,2)$. Note that $|\omega_i|_2\le|\omega|_2$, for $i=1,2$.

Next, we want to prove that

$$B := \left| \int_{\mathbb{R}^3} S_1(x)\, \omega_2(x) \cdot \omega_2(x)\, dx \right| \leq \frac{\nu}{4} |\nabla \omega|_2^2 + c\, \nu^{-3/5} k^{4/5} |\omega|_2^{4/5} |\omega|_2^2. \quad (4.4)$$

Let us recall that (see Ladyžhenskaya [11], Section 1.1, Lemma 2)

$$|\omega|_4 \leq 4\, |\omega|_2^{1/4} |\nabla \omega|_2^{3/4}. \quad (4.5)$$

By Hölder's inequality and by (4.2) with $q = 4$ one gets

$$B \leq c\, |\omega_1|_4\, |\omega|_4\, |\omega|_2.$$

Therefore, due to (4.5),

$$B \leq c\, |\omega_1|_4\, |\omega|_2^{1/4} |\nabla \omega|_2^{3/4} |\omega|_2,$$

and by Young's inequality

$$B \leq \frac{\nu}{4} |\nabla \omega|_2^2 + c\, \nu^{-3/5} |\omega_1|_4^{8/5} |\omega|_2^2.$$

Since $|\omega_1|_4^4 \leq k^2 |\omega|_2^2$, estimate (4.4) follows.

Finally, we consider the term regarding the values $i, j, k = 2$. This term is just the one that requires the additional Assumption A. By (2.2) one has $S_2\, \omega_2 \cdot \omega_2 = |\omega_2|^2\, S_2\, \xi_2 \cdot \xi_2$. Consequently, by using (2.3),

$$S_2\, \omega_2 \cdot \omega_2 = \frac{3}{4\pi} |\omega_2(x)|^2\, P.V. \int_{\mathbb{R}^3} D(\hat{y}, \xi_2(x+y), \xi_2(x)) |\omega_2(x+y)|\, \frac{dy}{|y|^3}.$$

Moreover, Assumption A, together with $\xi_2 = \xi$, shows that

$$|D(\hat{y}, \xi_2(x+y), \xi_2(x))| \leq g(t,x) |y|^{\alpha}.$$

It readily follows that

$$|S_2(x)\, \omega_2(x) \cdot \omega_2(x)| \leq \frac{3}{4\pi} |\omega_2(x)|^2\, g(t,x)\, I(x), \quad (4.6)$$

where

$$I(x) = \int_{\mathbb{R}^3} |\omega_2(x+y)|\, \frac{dy}{|y|^{3-\alpha}}.$$

Let us recall that the Hardy–Littlewood–Sobolev inequality in $\mathbb{R}^n$ (see, for instance, Stein [14], Chapter V) states that if $\alpha \in (0, n)$ and if $\omega_2 \in L^r$, for some $r \in (1, n)$, then I belongs to L^q, $1 < q < \infty$, for $1/q = 1/r - \alpha/n$. Furthermore, the linear map $\omega_2 \mapsto I$, between L^r and L^q, is continuous. Clearly, in the above statement, ω_2 denotes an arbitrary function of $L^r(\mathbb{R}^n)$, and in the definition of $I(x)$, the symbol 3 must be replaced by n.

By applying this inequality with $r = 2$ and $n = 3$ we show that

$$|I|_q \leq c\, |\omega_2|_2, \quad (4.7)$$

where $1/q = 1/2 - \alpha/3$. By (4.6) and by Hölder's inequality one gets

$$C := \Big| \int_{\mathbb{R}^3} S_2(x)\, \omega_2(x) \cdot \omega_2(x)\, dx \Big| \leq \frac{3}{4\pi} |\omega(x)|_{2p}^2\, |g(t,x)|_b\, |I(x)|_q, \tag{4.8}$$

where $1/b + 1/p + 1/2 - \alpha/3 = 1$; i.e.,

$$\frac{1}{b} + \frac{1}{p} = \frac{1}{2} + \frac{\alpha}{3}. \tag{4.9}$$

Note that $b \in [6/(2\alpha - 1), \infty]$ and that $p \geq 6/(3 + 2\alpha)$. In the sequel we restrict p to the values

$$\frac{6}{3+2\alpha} \leq p < 3,$$

since we need $2p < 6$. By (4.7) and (4.8) it follows that

$$C \leq c\, |g|_b\, |\omega|_{2p}^2\, |\omega|_2.$$

By using the classical interpolation results in L^p-spaces (note that $2 \leq 2p < 6$)

$$|\omega|_{2p}^2 \leq |\omega|_2^{2\beta}\, |\omega|_6^{2-2\beta}, \qquad \beta = \frac{1}{2}\Big[\frac{3}{p} - 1\Big],$$

we get

$$C \leq c\, |g|_b\, |\omega|_2^{\frac{3}{p}}\, |\nabla\omega|_2^{\frac{3(p-1)}{p}}$$

since, by a Sobolev's embedding theorem, $|\omega|_6 \leq c\, |\nabla\omega|_2$. Next, we apply Young's inequality with exponents $2p/(3-p)$ and $2p/3(p-1)$. This yields

$$C \leq c\, \nu^{-\frac{3(p-1)}{3-p}}\, |g|_b^{\frac{2p}{3-p}}\, |\omega|_2^{\frac{6}{3-p}} + \frac{\nu}{4} |\nabla\omega|_2^2. \tag{4.10}$$

Finally, by (4.3), (4.4), and (4.10)

$$\frac{d}{dt} |\omega|_2^2 + \nu |\nabla\omega|_2^2 \leq c\, G(t) |\omega|_2^2, \tag{4.11}$$

where

$$G(t) = k + \nu^{-3/5} k^{4/5} |\omega|_2^{4/5} + \nu^{-\frac{3(p-1)}{3-p}} \left(|g|_b\, |\omega|_2\right)^{\frac{2p}{3-p}}. \tag{4.12}$$

By using (4.11) and (4.12) we show that $\omega \in L^\infty(\tau-\delta, \tau; L^2) \cap L^2(\tau-\delta, \tau; H^1)$, hence that u is a strong solution in the time interval $(\tau - \delta, \tau)$, provided that $G(t) \in L^1(\tau - \delta, \tau)$, hence provided that

$$|g|_b^{\frac{2p}{3-p}}\, |\omega|_2^{\frac{2p}{3-p}} \in L^1(\tau - \delta, \tau). \tag{4.13}$$

Let us show that this holds under Assumption A.

We start by setting $p = 3/2$ in condition (4.13). Hence, by (4.9), we have that $b = 6/(2\alpha - 1)$ (note that $b = +\infty$ if $\alpha = 1/2$). For $p = 3/2$

condition (4.13) is simply $|g|^2_{6/(2\alpha-1)}|\omega|_2^2 \in L^1(\tau-\delta,\tau)$, which holds if $g \in L^\infty(\tau-\delta,\tau;L^{\frac{6}{2\alpha-1}})$. This covers the proof of Lemma 4.1 in the case in which $a=\infty$, for each $\alpha \in [1/2,1]$.

Assume now that $a<\infty$ (hence $1/2<\alpha\le 1$), and let p satisfy

$$\frac{6}{3+2\alpha} \le p < \frac{3}{2}. \tag{4.14}$$

By applying the Hölder inequality with exponents $\theta = \frac{3-p}{3-2p}$ and $\theta' = \frac{3-p}{p}$, and, by recalling the energy estimate (3.1), we get

$$\int_{\tau-\delta}^{\tau} |g|_b^{\frac{2p}{3-p}} |\omega|_2^{\frac{2p}{3-p}}\, dt \le \Big[\int_{\tau-\delta}^{\tau} |g|_b^{\frac{2p}{3-2p}}\Big]^{\frac{3-2p}{3-p}} \Big[\frac{1}{2\nu}|u_0|^2\Big]^{\frac{p}{3-p}}.$$

Hence, the coefficient of $|\omega|_2^2$ on the right hand side of (4.11) is integrable over $(\tau-\delta,\tau)$ if

$$g \in L^{\frac{2p}{3-2p}}(\tau-\delta,\tau;L^{\frac{6p}{(3+2\alpha)p-6}}), \tag{4.15}$$

for p satisfying (4.14). Consequently, u is a strong solution on $(\tau-\delta,\tau)$, provided that condition (4.15) is satisfied.

By setting $a=\frac{2p}{3-2p}$ and $b=\frac{6p}{(3+2\alpha)p-6}$ (note that the value of b is given by (4.9)), it follows that $\frac{2}{a}+\frac{3}{b}=\alpha-\frac{1}{2}$. The restriction on p imposed by condition (4.14) shows that $\frac{4}{2\alpha-1}\le a<\infty$, which corresponds, in the statement of Lemma 4.1, to all the cases in which $a<\infty$. □

After having proved Lemma 4.1, we can now prove the main result of this paper.

Proof of Theorem 1.2. The proof follows immediately from Lemma 4.1, by using a standard continuation argument, together with the existence of a (unique) local strong solution for arbitrary divergence-free initial data in H^1.

In fact, the local existence theorem for strong solutions (see for instance Leray [12]) implies that system (1.1) has a unique strong solution in some interval $[0,T')$, for some strictly positive T'. For convenience, we assume that this interval is the *maximal* interval of existence of the strong solution starting from u_0 at time $t=0$.

Let us suppose, by contradiction, that $T'<T$. Lemma 4.1 implies that the solution is strong up to T'. The local existence theorem ensures that there exists a unique strong solution for the Cauchy problem with initial datum $u(T')$ in some interval $[T',T'+\epsilon)$, for some strictly positive ϵ. This is absurd, since T' was the endpoint of the maximal interval of existence.

Observe that if the initial datum u_0 does not belong to H^1, but just to L^2, then the regularity result holds on (t', T) for each $t' > 0$. □

In the case in which $\alpha = 1$, condition (1.7) holds in particular if g belongs to $L^a(0, T; L^b)$, with $2/a + 3/b = 1/2$. We observe that

$$|D(\hat{y}, \xi(x+y,t), \xi(x,t))| = |D(\hat{y}, \xi(x+y,t) - \xi(x,t), \xi(x,t))| .$$

Since $|\xi(x,t)| = 1$, the condition (1.7) can be replaced by

$$|\xi(x+y,t) - \xi(x,t)| \leq g(t,x)|y|.$$

It is well-known that this condition is equivalent to requiring that

$$\nabla\xi \in L^a(0, T; L^b). \tag{4.16}$$

We have then the following corollary.

Corollary 4.2. *Suppose that a weak solution u in $(0,T)$ satisfies (4.16) for*

$$\frac{2}{a} + \frac{3}{b} = \frac{1}{2}.$$

Then the solution is strong, hence regular in $(0,T)$.

It follows that it is not necessary to resort to a pointwise estimate of the direction of the vorticity. In fact, the global estimate (4.16) can be related to some mean properties of the flow.

The condition with α noninteger can be related, as well, to some regularity of ξ. In this case the right framework is that one of the Nikol'skij spaces $N^{p,\lambda}$. We recall that the spaces $N^{p,\lambda}$, for $1 \leq p < \infty$ and $0 < \lambda < 1$, are defined by

$$N^{p,\lambda} := \Big\{ u \in L^p \quad s.t. \quad \sup_{0 \neq h \in \mathbb{R}^n} \frac{|u(x+h) - u(x)|_p}{|h|^\lambda} < \infty \Big\},$$

equipped with the norm

$$\|u\|_{N^{p,\lambda}} = |u|_p + \sup_{0 \neq h \in \mathbb{R}^n} \frac{|u(x+h) - u(x)|_p}{|h|^\lambda}.$$

Observe that $N^{p,\lambda}$ is isomorphic to the Besov space $B_p^{\lambda,\infty}$. Furthermore, $N^{p,1} = W^{1,p}$, for $p > 1$. Moreover, we define the following space $\mathcal{N}^{p,\lambda}$:

$$\mathcal{N}^{p,\lambda} := \Big\{ u \text{ measurable} \quad s.t. \quad \sup_{0 \neq h \in \mathbb{R}^n} \frac{|u(x+h) - u(x)|_p}{|h|^\lambda} < \infty \Big\}.$$

A comprehensive reference regarding these spaces is Triebel's book [15].

Hence, the following result derives immediately from Theorem 1.2.

Corollary 4.3. *Suppose that u is a weak solution of (1.1) in $(0,T)$ and suppose also that the following condition is satisfied:*

$$\xi \in L^a(0,T;\mathcal{N}^{b,\alpha}), \quad \textit{with} \quad \frac{2}{a}+\frac{3}{b}=\alpha-\frac{1}{2} \quad \textit{for} \quad \frac{1}{2}\le \alpha \le 1,$$

where $4/2\alpha-1 \le a \le \infty$. Then the solution is strong in $(0,T)$, hence regular.

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