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REMARKS ON THE FLOW OF HOLES AND ELECTRONS IN CRYSTALLINE SEMICONDUCTORS

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1. INTRODUCTION

In this paper we study the following system of nonlinear partial differential equations, that describes the transport of holes and electrons in a semiconductor device

$$\begin{cases} \dfrac{\partial p}{\partial t} - \nabla \cdot (D_1 \nabla p + \mu_1 p \nabla u) = R(p,n) \ , \\ \dfrac{\partial n}{\partial t} - \nabla \cdot (D_2 \nabla n - \mu_2 n \nabla u) = R(p,n) \ , \\ -\nabla \cdot (a \nabla u) = f + p - n \qquad \text{in } \ \mathbb{R}_+ \times \Omega \ , \end{cases} \tag{1.1}$$

with boundary conditions

$$\begin{cases} p = \phi(x) \ , \ n = \psi(x) \qquad \text{on } \ \mathbb{R}_+ \times \ D \ , \\ (D_1 \nabla p + \mu_1 p \nabla u) \cdot \nu = (D_2 \nabla n - \mu_2 n \nabla u) \cdot \nu = 0 \ \text{ on } \ \mathbb{R}_+ \times \ B \ , \end{cases} \tag{1.2}$$

$$\begin{cases} u = U(x) \ \text{ on } \ \mathbb{R}_+ \times D \ , \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{ on } \ \mathbb{R}_+ \times B \ , \end{cases} \tag{1.3}$$

and initial conditions

$$p(0,x) = p_0(x) \ , \ n(0,x) = n_0(x) \ \text{ in } \ \Omega \ . \tag{1.4}$$

Here Ω is a bounded Lipschitzian domain in $\mathbb{R}^n$ (see Nečas, 1967). We assume that the boundary Γ of Ω is the union of two disjoint sets D and B, where D is closed. For convenience we assume that D has non vanishing $(n-1)$- dimensional measure. We denote by ν the unit outward normal to Γ. We refer to van Roosbroeck (1950), Moll (1964), Markowich, Ringhofer and Schmeiser (1990) for more detailed descriptions of

Navier-Stokes Equations and Related Nonlinear Problems
Edited by A. Sequeira, Plenum Press, New York, 1995

the model. The unknowns u, p and n denote the electrostatic potential, the free hole carrier concentration and the free electron carrier concentration. The solutions $p(x,t)$ and $n(x,t)$ are required to be nonegative. Usually, the above equations are written in the form

$$\frac{\partial p}{\partial t} - \nabla \cdot j_p = R(p,n) \ , \ \frac{\partial n}{\partial t} - \nabla \cdot j_n = R(p,n)$$

where the hole and electron current densities are given by

$$j_p = D_1 \nabla p + \mu_1 p \nabla u \ , \ j_n = D_2 \nabla n - \mu_2 n \nabla u$$

respectively. We do not assume that the hole and electron diffusion coefficients D_1 and D_2 and the hole and electron mobilities μ_1 and μ_2 are (necessarily) connected by the Einstein relations $D_i = (k\vartheta_0)\mu_i$, $i = 1,2$, where k is the Boltzmann's constant and ϑ_0 the constant temperature. We assume that D_1, D_2, μ_1, μ_2 and a (the dielectric permittivity) are positive constants. This leads to the equations (1.1).

In the sequel we assume that $\phi, \psi \in H^1(\Omega) \cap L^\infty_+(\Omega)$ and $U \in H^1(\Omega) \cap L^\infty(\Omega)$. The symbol "+" means the cone of nonegative functions (we point out that we will not define standard notation). Finally, we assume that the net density of ionized impurities f satisfies

$$f \in L^s(0,+\infty;\ L^r(\Omega)) \tag{1.5}$$

for some fixed $s \in [4,+\infty]$ and some fixed $r \in]2N,+\infty]$. In particular, the case of an arbitrary bounded measurable $f(t,x)$ is included. The most important case in applications is that in which f is independent of time. However, it has mathematical interest considering the above more general case (a similar remark holds if $N > 3$). Concerning the initial data we assume that

$$p_0,\ n_0 \in L^\infty_+(\Omega)\ . \tag{1.6}$$

The recombination term $R(p,n)$ is assumed to be a locally Lipschitz continuous function, defined on $\mathbb{R}_+ \times \mathbb{R}_+$, such that

$$\lim_{p+n \to +\infty} \frac{R(p,n)^+}{p+n} = 0\ , \tag{1.7}$$

where $z^+ = \max\{z,0\}$, moreover

$$\begin{cases} R(p,0) \geq 0 \ , \ \forall\, p \geq 0\ , \\ R(0,n) \geq 0 \ , \ \forall\, n \geq 0\ . \end{cases} \tag{1.8}$$

We point out that (1.7), (1.8) hold for the Shockley-Read-Hall recombination term

$$R(p,n) = \frac{1-pn}{r_0 + r_1 p + r_2 n}\ , \tag{1.9}$$

in which r_0, r_1, and r_2 are positive constants. It is worth noting that the solutions p and n must be nonegative. Under suitable hypotheses (see below) Gajewski and Gröger (1986), show that the solution of the above problem satisfies

$$p, n \in L^\infty(0,T; L^\infty_+(\Omega)) \tag{1.10}$$

for each fixed T. However, the $L^\infty(\Omega)$-norm of $p(t)$ and $n(t)$ may blow up, at most exponentially, as t goes to $+\infty$. For different boundary conditions a similar result is proved by Seidman and Troianiello (1985). For previous, related results, see Mock (1974, 1975) and Gajewski (1985). A main open question, in order to approach the problem of the qualitative behaviour of solutions for large values of t, is to know whether the solutions are uniformly bounded in $[0,+\infty[\times\Omega$. This is our main concern here. We will prove, under no smallness or under other restrictive assumptions, that

$$p,n \in L^\infty(0,+\infty);\ L^\infty_+(\Omega)) \ . \tag{1.11}$$

A partial result in this direction was obtained by Gröger, 1986. This author exhibits a sufficient condition in order that (1.11) holds. For the Shokley-Read-Hall recombination term (1.9) Gröger's condition corresponds to the following smallness assumption on f : $-M_1 \leq f(x) \leq M_2$ a.e. in Ω, where M_1 and M_2 satisfy $M_i < a/D_i(r_1+r_2)$, $i=1,2$. Here $\mu_i = D_i$.

With respect to Gajewski and Gröger (1986), we do not assume that ϕ and ψ are bounded from below by a strictly positive constant and that ∇ $(\log\phi + U)$ and ∇ $(\log\psi - U)$ are bounded. On the other hand, in Gajewski and Gröger (1986), the authors assume a more general boundary condition on the Neumann boundary B.

Before stating our main theorem we introduce some notation. We set $Q_T = (0,T)\times\Omega$, $Q = Q_\infty$. We denote by $\|\cdot\|_r$, $r\in[1,+\infty]$, the canonical norm in $L^r = L^r(\Omega)$ and by $\|\cdot\|_{r,s;T}$, $r,s\in[1,+\infty]$ and $T\in]0,+\infty]$ that in $L^s(0,T;L^r)$. For convenience, we set $\|\cdot\| = \|\cdot\|_2$ and $\|\cdot\|_{r,s} = \|\cdot\|_{r,s;+\infty}$. We denote by $|E|$ the N-dimensional Lebesgue measure of a set E.

We denote by V the Hilbert space $V = \{v\in H^1 : v=0 \text{ on } D\}$ and by V' its dual space. In order to use here a standard notation, let us set $H = L^2(\Omega)$. By identifying H with its dual H' one has $V\hookrightarrow H\hookrightarrow V'$, where each space is dense in the next one. The spaces V,H, and V' are in a typical situation, often considered on studying weak solutions of partial differential equations. We denote by $(\cdot,\cdot)$ the scalar product in H (or in H^N). We use the same notations for scalar and for vector fields) and by $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V. If v belongs to $L^2_{\text{loc}}(0,+\infty;V)$ we denote by v' the derivative of v as a distribution in $]0,+\infty[$ with values in V. Since $V\hookrightarrow V'$ it could be that $v'\in L^2_{\text{loc}}(0,+\infty;V')$. For properties connected to this (already classical) setting up we refer the reader to Lions and Magenes (1968), Dautray-Lions (1992).

We set $\mu_0 = \sqrt{\mu_1\mu_2}$, $\mu_3 = \max\{\mu_1,\mu_2\}$, $\mu_4 = \min\{\mu_1,\mu_2\}$, $\rho = \min\{D_1/\mu_1, D_2/\mu_2\}$, $b = r_0^{-1}\mu_3$. Moreover

$$M_0 = \max\{\|p_0\|_\infty,\ \|n_0\|_\infty,\ \|\phi\|_\infty,\ \|\psi\|_\infty\} \ . \tag{1.12}$$

We denote by c_0 a positive constant such that the Poincaré's inequality

$$\int v^2 dx \leq c_0 \int |\nabla u|^2 dx, \quad \forall\, v\in V \ , \tag{1.13}$$

holds and by c_1 a positive constant such that the Sobolev's embedding theorem

$$\left(\int v^{2^*}dx\right)^{1/2^*} \leq c_1 \left(\int |\nabla v|^2 dx\right)^{1/2} , \quad \forall\, v\in V \ , \tag{1.14}$$

holds. If $N\geq 3$, we denote by 2^* the embedding Sobolev exponent $2^* = 2N/(N-2)$ and by $\hat{2}$ its dual exponent $\hat{2} = 2N/(N+2)$. If $N=2$ (hence $r\in]4,+\infty]$; recall (1.5)) we set $2^* = 4r/(r-4)$ and $\hat{2} = 4r/(4+3r)$. Note that (1.14) also holds for $N=2$. Moreover $1/2^* + 1/\hat{2} = 1$.

Next we set

$$\delta_0 = \frac{\rho\mu_0^2}{4\mu_3 c_0} \tag{1.15}$$

and we define N_0 as being a positive constant such that

$$R(p,n)^+ \leq \delta_0(p+n) \qquad \text{if } p+n \geq N_0 \ . \tag{1.16}$$

N_0 exists, by the assumption (1.7). We set $N_1 = 2^{-1}\delta_0 \sup\limits_{p+n \leq N_0} R(p,n)^+$ and we define

$$M = \max\{M_0, 1, N_0, N_1\} \ . \tag{1.17}$$

In the particular case of the Schokley-Read-Hall recombination term (1.9), we simply set

$$M = \max\{M_0, 1\} \ . \tag{1.18}$$

Under the above hypotheses there is a weak solution (p,n,u) of problem (1.1)-(1.4) in the following class: $p - \phi$ and $n - \psi$ belong to $L^2_{\text{loc}}(0,+\infty;V)$; $u - U$ belongs to $L^r_{\text{loc}}(0,+\infty;V)$; p and n are nonegative a.e. in Q and belong to $L^\infty_{\text{loc}}(0,+\infty;L^\infty)$. Moreover, the solution is unique in the above class. Functions p, n and u in the above class are said to be a weak solution (1.1)-(1.4) if, for each fixed $v \in V$, one has $a(\nabla u, \nabla v) = (f + p - n, v)$, and also, in the sense of $\mathcal{D}'(]0,+\infty[)$ (or equivalently, almost everywhere in $]0,+\infty[$)

$$\langle p', v\rangle + (D_1\nabla p, \nabla v) + \mu_1(p\nabla u, \nabla v) = (R(p,n), v) \ ,$$

and

$$\langle n', v\rangle + (D_2\nabla n, \nabla v) + \mu_2(-n\nabla u, \nabla v) = (R(p,n), v) \ .$$

Moreover, $p(0) = p_0$, $n(0) = n_0$. Note that p and n are continuous on $[0,+\infty[$ with values in $H = L^2(\Omega)$. We may also write the above equations in terms of $y = p - \phi$, $z = n - \psi$ and $w = u - U$.

Our main result is the following

Theorem 1.1. *The above solution (p,n) of problem (1.1-(1.4) is uniformly bounded in $Q = R_+ \times \Omega$. More precisely,*

$$\sup_Q \,(p(t,x) + n(t,x)) \leq CM\left(1 + \|f\|_{r,s}^{2(1+\frac{1}{\chi})}\right) \tag{1.19}$$

where

$$\chi = \frac{2}{N} - \frac{4}{r} \quad \text{if} \quad N \geq 3\,, \quad \chi = \frac{1}{2} - \frac{2}{r} \quad \text{if} \quad N = 2 \ . \tag{1.20}$$

The constant C depends only on $N, c_0, c_1 a,\ D_1, D_2, \mu_1, \mu_2,\ |\Omega|$ and r.

The reader is assumed to be well acquainted with the formulation of PDE's in weak form. We adopt here classical terminology and notations in order to bring out clearly the underlying ideas. The interpretation of some of the terminology and the justification of some of the calculations (in terms of weak solutions, distributional derivatives, duality pairing, and so on) is done by using well known standard devices. We refer the unexperienced reader to Dautray- Lions (1992), Ladyzhenskaya, Solonnikov and Ural'ceva (1968), Lions and Magenes (1968), Ladyzhenskaya (1973); see, in particular Dautray- Lions (1992) Chapt. XVIII, §§ 1 and 3.

Some words about the proof of the Theorem 1.1. are in order. The proof consists in four steps. The first one consists in giving (q, m) (see (3.2)) and in solving for (p, n, u) the linear problem (3.3)-(3.6); the definition of $\hat{q}$ and $\hat{m}$ is given in (3.1). For the time being, the value of the positive constant μ in the definition of $\hat{q}$ and $\hat{n}$ is arbitrary.

The second step consists in proving the existence of a weak solution (p, n, u) of problem (3.11)-(3.14). This is done by proving the existence of a fixed point $(p, n) = (q, m)$ for the map S which associates to each (q, m) the (unique) solution of the linear problem (3.3)-(3.6).

The third step consists in showing that the solutions p and n of problem (3.11)-(3.14) are nonegative. In particular, from this result it follows that $\hat{p} = \max\{p, \mu\}$ and $\hat{n} = \max\{n, \mu\}$.

The proof of the above three steps is done by following Gajewski and Gröger (1986), and will be postponed to Section 3.

The fourth step is the main point in our paper. Here, we show that the solution p and n of problem (3.11)-(3.14) constructed above (solutions that depend on the particular value of the parameter μ) are bounded from above by the right hand side of equation (1.19). Hence, by choising μ larger than the above right hand side it follows that $\hat{p} = p$ and $\hat{n} = n$. Consequently, p, n (and u) are a (weak) solution of problem (1.1)-(1.4).

The above steps prove the existence part together with the main estimate (1.19). In order to give a better understanding of the underlying ideas developed in the fourth step (proof of the main estimate (1.19)) we rather prefer to present the corresponding calculations in the form of an a priori estimate. This is done in Section 2. The proof of (1.19) for the solution (p, n) of problem (3.11)-(3.14) (the above step four) is done by making (quite obvious) minor changes on the argument developed in Section 2. The few modifications to be done are indicated in Section 3.

Finally, the proof of the uniqueness of the solution follows the usual devices and is presented (at the end of Section 3) just for the reader's convenience.

Before going on let us remark that our proof can be adapted to more general situations: dependence on time of the data $\phi \geq 0$, $\psi \geq 0$, provided that they belong to $L^\infty(0, +\infty; H^1 \cap L^\infty)$ and that ϕ', ψ' belong to $L^2_{\text{loc}}(0, +\infty; V')$ (this generalization requires only a few modifications in the proofs); other boundary conditions (for instance, unilateral contraints, see Beirão da Veiga and Dias, 1972; Beirão da Veiga, 1974); dependence of the coefficients $D_i, \mu_i (i = 1, 2)$, and a on the solution itself and on (x, t), under suitable assumptions.

We also note that many regularity results follow as straightforward applications of well-known theorems or techniques. For instance, $u \in C(0, +\infty; C^{0,\alpha}(\overline{\Omega}))$ for some $\alpha > 0$, since $p, n \in C(0, +\infty; L^r)$, for arbitrarily large r; see Stampacchia (1960), and also Beirão da Veiga, 1972. We conjecture that p and n are Hölder-continuous on Q if $N = 2$ or 3 (under slight regularity assumptions on the boundary of D as a subset of $\partial\Omega$) but that (in general) this result is false if $N \geq 4$. However, we did not investigative in this direction.

It is worth noting that obvious modifications (in fact, simplifications) in our proofs (set $\partial p/\partial t = \partial n/\partial t = 0$ everywhere ...) yield the following result for *stationary solutions.*

Theorem 1.2. *Let ϕ, ψ, U be as above and let $f \in L^r(\Omega)$, $r \in]2N, +\infty]$. Then, the problem (1.1)-(1.3) admits a time independent solution (p, n, u) such that $p - \phi$,*

$n-\psi,\ u-U \in V$. *Moreover* $p \geq 0,\ n \geq 0$ *and*

$$\sup_{\Omega}\,(p(x)+n(x)) \leq C\ M(1+\|f\|_r^{2(1+1/\chi)}) \ . \tag{1.21}$$

Here, χ *and* C *are as in Theorem 1.1.. In the definition of* M *drop* $\|p_0\|_\infty$ *and* $\|n_0\|_\infty$.

A second basic question in order to study the asymptotic behaviour of the set of solutions is that of the existence (or non existence) of a (significant) functional space X and of a bounded set $B_0 \subset X$ that atracts (uniformly) each bounded subset B of X. We prove that this property holds for $X = L^2(\Omega)$. In order to prove this result, the first step consists in showing the existence and the uniqueness of a solution

$$p, n \in C(0,+\infty; L^2(\Omega)) \tag{1.22}$$

in correspondence to each (arbitrary) pair of initial data $p_0, n_0 \in L^2(\Omega)$. We prove the existence of a weak solution (p,n) to our problem in the class $p-\phi,\ n-\psi \in L^2_{\text{loc}}(0,+\infty;V)$, $p_t, n_t \in L^2_{\text{loc}}(0,+\infty;V_N')$ is the dual space of $V_N = \{v \in V : \nabla v \in L^N(\Omega)\}$; if $N=2$ replace N by q, $q>2$. In order to prove the uniqueness of the solution and also that $p_t, n_t \in L^2_{\text{loc}}(0,+\infty;V')$ (hence that (1.22) holds) we assume the property described below. Consider the elliptic mixed boundary value problem

$$-\nabla u = g \ \text{ in } \Omega \ , \ u = U \ \text{ on } D \ , \ \partial u/\partial v = 0 \ \text{ on } B \ . \tag{1.23}$$

We assume that there is a functional space Y and a real q ($q>2$ if $N=2$; $q=N$ otherwise) such that if $g \in L^2(\Omega)$ and $U \in Y$ then the variational solution u of problem (1.23) satisfies

$$\|\nabla u\|_q \leq c\,(\|g\| + \|U\|_Y) \ . \tag{1.24}$$

Note that this is an assumption on $\{\Omega, B, D\}$. This assumption is out of place if $N>4$ since $H^2(\Omega)$ is not contained in $L^N(\Omega)$. If $N=2$, it holds if Ω is a bounded domain with a polygonal boundary (or a regular transformation of such a set). In this case $q>2$ can be arbitrarily fixed, moreover $Y = W^{1-\frac{1}{q},q}(B)$. This follows essentialy from results by Lorenzi (1975). Since it is sufficient to have (1.24) for some $q>2$, it seems possible to use Gröger's results (1989).

If $N=3$ and if Ω is bounded convex set with a polyhedral boundary (or a regular transformation of it) then the solution of problem (1.23) belongs to $H^{3/2}(\Omega)$. This follows from results of Grisvard (1992), at least if $U=0$. Note that $\nabla u \in L^3(\Omega)$ since $H^{3/2}(\Omega) \hookrightarrow W^{1,3}(\Omega)$. It is worth noting that in Grisvard (1992), the author considers only homogeneous boundary conditions, but that looks inessential there. We also note that $W^{1,3}$-regularity holds under weaker hypotheses on the angles between faces than that needed to get $H^{3/2}$-regularity. But we do not know about precise statements in the literature.

The following result is proved in a forthcoming paper. For brevity, we assume that $f \in L^\infty(Q)$ and that $R(p,n)$ is given by (1.9).

Theorem 1.3. *Let the assumption (1.24) hold and let* ϕ, ψ, U, f *and* $R(p,n)$ *be as in Theorem 1.1.; moreover* $U \in Y$. *Then, to each pair of initial data* $(p_0, n_0) \in L^2(\Omega)$ *it corresponds a unique solution* (p,n) *of problem (1.1)-(1.4) in the class* $p-\phi, n-\psi \in L^2_{\text{loc}}(0,+\infty;V)$; $p_t, n_t \in L^2_{\text{loc}}(0,+\infty;V')$. *Moreover,* $p,n \in C(0,+\infty;L^2(\Omega))$ *and there is a positive constant* C_0 *which depends on the norms* $\|\phi\|_\infty$, $\|\psi\|_\infty$, $|||f|||_\infty$ *but not on* p_0, n_0 *and* U) *such that*

$$\|p(t)\|^2 + \|n(t)\|^2 \leq C_0 + ce^{\nu t}\,(\|p_0\|^2 + \|n_0\|^2) \ . \tag{1.25}$$

for each $t \geq 0$. The positive constant c and ν are independent of the data ϕ, ψ, U, f, p_0 and n_0.

The above result shows that the set

$$B_0 = \{(\overline{p}, \overline{n}) \in L^2(\Omega) : \overline{p} \geq 0,\ \overline{n} \geq 0,\ \|\overline{p}\|^2 + \|\overline{n}\|^2 \leq C_0\}$$

is a global bounded atractor in the space $L^2(\Omega)$.

2. PROOF OF THE MAIN ESTIMATE

As explained in Section 1, the proof of (1.19) will be carried out here as an a priori estimate for solutions of problem (1.1)-(1.4). However, according to the above explanation, the proof should be applicable to the solution of problem (3.11)-(3.14). We assume here that the solution of problem (1.1)-(1.4) belongs to the existence class, described before the statement of Theorem 1.1., since the solution of problem (3.11)-(3.14) belongs to this class and since the justification of each single calculation is the same in both cases.

For $k \geq 0$ we set

$$\overline{w} = w^{(k)} = \max\{w - k, 0\}\ .$$

The notation $\overline{w}$ will be used when there is no danger of misunderstanding. In the sequel $k \geq M_0$. Hence $\overline{p}$ and $\overline{n}$ belong to $L^2_{\text{loc}}(0, +\infty; V)$, moreover $\overline{p}(0) = \overline{n}(0) = 0$. See, for instance, Dautray- Lions (1992) Vol. 2, Chap. IV, §7, Prop. 6. Next multiply the equation $(1.1)_1$ by $\overline{p}$, integrate over Ω and made suitable integrations by parts. This yields

$$\frac{1}{2}\frac{d}{dt}\|\overline{p}\|^2 + D_1\|\nabla\overline{p}\|^2 + \mu_1 \int \nabla u \cdot \left(\frac{1}{2}\nabla\overline{p}^2 + k\nabla\overline{p}\right) dx = \int R(p,n)\overline{p}\, dx \tag{2.1}$$

where integrals are over Ω. Again by suitable integrations by parts, and also by taking into account the equation $(1.1)_3$ and the boundary conditions, one gets

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\overline{p}\|^2 + D_1\|\nabla\overline{p}\|^2 + \frac{\mu_1}{2a}\int (f+p-n)\overline{p}^2 dx + \frac{\mu_1}{a}k\int (f+p-n)\overline{p}\, dx = \\ = \int R(p,n)\overline{p}\, dx\,.\end{aligned} \tag{2.2}$$

In a similar way, by starting from $(1.1)_2$, one gets

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\overline{n}\|^2 + D_2\|\nabla\overline{n}\|^2 - \frac{\mu_2}{2a}\int (f+p-n)\overline{n}^2 dx + \frac{\mu_2}{a}k\int (f+p-n)\overline{n}\, dx = \\ = \int R(p,n)\overline{n}\, dx\,.\end{aligned} \tag{2.2$'$}$$

Next, multiply equation (2.2) by μ_2, equation (2.2 $'$) by μ_1, and add both equations. This yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int (\mu_2\overline{p}^2 + \mu_1\overline{n}^2)\, dx + \rho\mu_0^2\int |\nabla(\overline{p},\overline{n})|^2\, dx + \frac{\mu_0^2}{2a}\int (f+p-n)\,(\overline{p}^2 - \overline{n}^2)\, dx + \\ + \frac{\mu_0^2}{a}\, k\int (f+p-n)\,(\overline{p} - \overline{n})\, dx = \int R(p,n)\,(\mu_2\overline{p} + \mu_1\overline{n})\, dx\ ,\end{aligned} \tag{2.3}$$

where, for convenience, we set

$$|\nabla(\overline{p},\overline{n})|^2 = |\nabla\overline{p}|^2 + |\nabla\overline{n}|^2 \ .$$

Since p and n are nonegative, one easily proves that $(\overline{p}-\overline{n})(p-n) \geq 0$ and that $|\overline{p}-\overline{n}| \leq |p-n|$. In particular $(\overline{p}-\overline{n})(p-n) \leq (p-n)^2$. Hence

$$(\overline{p}^2 - \overline{n}^2)\,(p-n) \geq (\overline{p}-\overline{n})^2\,(\overline{p}+\overline{n}) \ .$$

On the other hand

$$|f\,(\overline{p}^2-\overline{n}^2)| \leq \frac{1}{2}\,|f|^2\,(\overline{p}+\overline{n}) + \frac{1}{2}\,(\overline{p}-\overline{n})^2\,(\overline{p}+\overline{n}) \ .$$

It readily follows that

$$(f+p-n)\,(\overline{p}^2-\overline{n}^2) \geq \frac{1}{2}\,(\overline{p}-\overline{n})^2\,(\overline{p}+\overline{n}) - \frac{1}{2}\,|f|^2\,(\overline{p}+\overline{n}) \ . \tag{2.4}$$

Similarly, $(f+p-n)\,(\overline{p}-\overline{n}) \geq \frac{1}{2}\,(\overline{p}-\overline{n})^2 - \frac{1}{2}\,f^2$ (however, we will estimate the corresponding term in equation (2.3) in a different way). From (2.3) and (2.4) one gets

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\int(\mu_2\overline{p}^2+\mu_1\,\overline{n}^2)\,dx + \rho\mu_0^2\int|\nabla(\overline{p},\overline{n})|^2\,dx + \\
&\frac{\mu_0^2}{4a}\int(\overline{p}-\overline{n})^2\,(\overline{p}+\overline{n})\,dx + \frac{\mu_0^2}{a}\,k\int(\overline{p}-\overline{n})^2\,dx \\
&\leq \frac{\mu_0^2}{4a}\int|f|^2\,(\overline{p}+\overline{n})\,dx + \frac{\mu_0^2}{a}\,k\int|f|\,|\overline{p}-\overline{n}|\,dx + \int R(p,n)\,(\mu_2\overline{p}+\mu_1\overline{n})\,dx \ .
\end{aligned} \tag{2.5}$$

Next, we estimate the last term in the above inequality. In the specific case (1.9), one has

$$R(p,n)\,(\,\mu_2\overline{p}+\mu_1\overline{n}) \leq b(\overline{p}+\overline{n}) \ . \tag{2.6}$$

In the general case (1.7), (1.8) one has $R(p,n)^+ \leq \delta_0(\overline{p}+\overline{n}+2k)$ for $k \geq \max\{N_0, N_1\}$. Hence

$$R(p,n)\,(\mu_2\overline{p}+\mu_1\overline{n}) \leq \mu_3\delta_0(\overline{p}+\overline{n})^2 + 2\delta_0\mu_3 k(\overline{p}+\overline{n}) \ .$$

By taking into account (1.15) one gets

$$R(p,n)\,(\mu_2\overline{p}+\mu_1\overline{n}) \leq \frac{\rho\mu_0^2}{4c_0}\,(\overline{p}+\overline{n})^2 + 2\delta_0\mu_3\,k\,(\overline{p}+\overline{n}) \ . \tag{2.7}$$

Hence, by (1.13), (2.5) shows that

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\int(\mu_2\overline{p}^2+\mu_1\overline{n}^2)\,dx + \frac{\rho\mu_0^2}{2}\int(|\nabla\overline{p}|^2+|\nabla\overline{n}|^2)\,dx \\
&\leq \frac{\mu_0^2}{4a}\int f^2(\overline{p}+\overline{n})\,dx + \frac{\mu_0^2}{a}\,k\int|f|\,(\overline{p}+\overline{n})\,dx + 2\delta_0\mu_3 k\int(\overline{p}+\overline{n})\,dx \ .
\end{aligned} \tag{2.8}$$

In the specific case (1.9) we could replace $2\delta_0\mu_3 k$ by b. However (for convenience) we rather prefer replacing $2\delta_0\mu_3$ by b and assuming that $k \geq 1$. Under this assumption (2.8) holds for each $k \geq M$.

Next, we define

$$A_k(t) = \{x \in \Omega : p(t,x) > k\} \,\cup\, \{x \in \Omega : n(t,x) > k\} \ . \tag{2.9}$$

We set

$$\alpha = \frac{(N+2)r}{4N} \quad , \quad \beta = \frac{(N+2)r}{(N+2)r - 4N} \tag{2.10}$$

if $N \geq 3$, and

$$\alpha = \frac{4+3r}{8} \quad , \quad \beta = \frac{4+3r}{3r-4} \tag{2.11}$$

if $N = 2$. Note that $\alpha^{-1} + \beta^{-1} = 1$ and that $2/(\hat{2}\beta) = 1 + \chi$.

By Hölder's inequality one gets

$$\int f^2(\overline{p} + \overline{n})\, dx \leq \|\overline{p} + \overline{n}\|_{2^*} \|f^2\|_{\hat{2}\alpha} |A_k(t)|^{\frac{1}{\hat{2}\beta}} \ .$$

Since $\|\overline{p} + \overline{n}\|_{2^*} \leq \sqrt{2} c_1 \|\nabla(\overline{p}, \overline{n})\|$ it follows that

$$\int f^2\, (\overline{p} + \overline{n})\, dx \leq c_1^2 \varepsilon\, \|\nabla(\overline{p}, \overline{n})\|^2 + \frac{1}{2\varepsilon}\, \|f^2\|^2_{\hat{2}\alpha}\, |A_k(t)|^{1+\chi} \ . \tag{2.12}$$

By setting $\varepsilon = a\rho / 2c_1^2$ one obtains

$$\frac{\mu_0^2}{4a} \int f^2(\overline{p} + \overline{n})\, dx \leq \frac{\rho \mu_0^2}{8}\, \|\nabla(\overline{p}, \overline{n})\|^2 + \frac{c_1 \mu_0^2}{4a^2 \rho}\, \|f\|_r^4\, |A_k(t)|^{1+\chi} \ .$$

Note that $r = 2\hat{2}\alpha$. In this section, for convenience, we denote by C positive constants that depend, at most, on the constants $N, c_0, c_1, a, D_1, D_2, \mu_1, \mu_2, r$ and $|\Omega|$. The same symbol C can be used to denote distinct constants, even in the same formula. It is worth noting that all the constants C that appear in the sequel can be easily estimated (as in (2.12)).

By replacing in equation (2.12) f^2 by kf we show that

$$k \int |f|\, (\overline{p}, \overline{n})\, dx \leq c_1^2 \varepsilon \|\nabla(\overline{p}, \overline{n})\|^2 + \frac{k^2}{2\varepsilon}\, \|f\|^2_{r/2} |A_k(t)|^{1+\chi} \ , \tag{2.13}$$

and by replacing in (2.13) f by $2\delta_0$ we show that

$$2\delta_0 k \int (\overline{p} + \overline{n})\, dx \leq c_1^2 \varepsilon\, \|\nabla(\overline{p}, \overline{n})\|^2 + \frac{2\delta_0^2}{\varepsilon}\, k^2 |\Omega|^{\frac{2}{\hat{2}\alpha}}\, |A_k(t)|^{1+\chi} \ . \tag{2.14}$$

By making obvious choices for ε in the above estimates, we show from (2.8) that

$$\frac{d}{dt} \int (\mu_2 \overline{p}^2 + \mu_1 \overline{n}^2)\, dx + \nu \int (\mu_2 \overline{p}^2 + \mu_1 \overline{n}^2)\, dx \leq C \left[\|f\|_r^4 + k^2 (\|f^2_{r/2} + 1) \right]\ |A_k(t)|^{1+\chi} \ , \tag{2.15}$$

where $\nu = \rho \mu_0^2 / 4 c_0 \mu_3$. We have used also Poincaré's inequality (1.13). Recall that $\hat{2}\alpha = r/2$. Since we are not looking for the sharpest estimates, we replace in (2.15) the term $\|f\|_r^4$ by $k^2 \|f\|_r^4$. We get the simplified expression

$$y_k'(t) + \nu y_k(t) \leq C k^2 g(t)\, |A_k(t)|^{1+\chi} \tag{2.16}$$

where

$$\begin{cases} y_k(t) = \int (\mu_2 \overline{p}^2 + \mu_1 \overline{n}^2)\, dx \ , \\ g(t) = 1 + \|f(t)\|_r^4 \ . \end{cases} \tag{2.17}$$

Note that $y_k(0) = 0$ since $k \geq M$. It readily follows from (2.16) that

$$y_k(t) \leq C k^2 \int_0^t e^{-\nu(t-s)} g(s)\, |A_k(s)|^{1+\chi}\, ds \ . \tag{2.18}$$

In particular

$$y_k(t) \le \frac{C}{\nu} k^2 \left(\sup_{0\le t<+\infty} g(t) \right) \left(\sup_{0\le t<+\infty} |A_k(t)|^{1+\chi} \right) \quad . \tag{2.19}$$

Let now $h > k \ge M$. One has

$$y_k(t) \ge \mu_4 (h-k)^2 \, |A_h(t)| \quad . \tag{2.20}$$

Set

$$\Phi(h) = \sup_{0\le t<+\infty} |A_h(t)|^{1/2} \quad , \tag{2.21}$$

for each $h \ge M$. Note that $\Phi(h) \le |\Omega|^{1/2}$. From (2.19) and (2.20) it follows that

$$\Phi(h) \le \frac{\gamma \, k}{h-k} \Phi(k)^{1+\chi} \quad , \quad \text{for } h > k \ge M \quad , \tag{2.22}$$

where

$$\gamma = \left(\frac{C}{\nu \mu_4} \right)^{1/2} \left(1 + \|f\|^2_{r,\infty} \right) \quad . \tag{2.23}$$

The proof of the following result is postponed to the end of this section.

Lemma 2.1. *Let $\Phi(\xi)$ be a function defined for $\xi \ge M$, nonegative and decreasing (not necessarily strictly decreasing) such that, for $h > k \ge M$, the estimate (2.22) holds. Then $\Phi(2d) = 0$ where*

$$d = M + 2^{\frac{2}{\chi} + \frac{1}{\chi^2}} \gamma^{1+\frac{1}{\chi}} \, \Phi(M)^{1+\chi} M \quad . \tag{2.24}$$

M and γ are nonegative constants.

Application of the above lemma shows that $|A_{2d}(t)| = 0$ for $t \in [0, +\infty[$, hence $p^{(2d)}$ and $n^{(2d)}$ vanish on Q. This shows that (1.19) holds when $s = +\infty$. Note that $\Phi(M) \le |\Omega|^{1/2}$. □

Next, we show that (1.19) holds if

$$f \in L^s \left(0, +\infty ; \, L^r \right) \quad , \tag{2.25}$$

for some $s \ge 4$. In fact, from (2.18) and from Hölder's inequality one gets

$$y_k(t) \le C k^2 \left(\frac{1}{\nu} + \frac{1}{(\nu \vartheta')^{1/\vartheta'}} \|f\|^4_{r,4\vartheta;t} \right) \sup_{0\le t<+\infty} |A_k(t)|^{1+\chi} \quad , \tag{2.26}$$

where $\vartheta \ge 1$ and $1/\vartheta + 1/\vartheta' = 1$. Hence (2.22) holds, where now γ is given by

$$\gamma = \left(\frac{C}{\nu \mu_4} \right)^{1/2} \left(1 + \frac{\nu^{1/2\vartheta}}{\vartheta'^{1/2\vartheta'}} \|f\|^2_{r,4\vartheta} \right) \quad . \tag{2.27}$$

Note that the right hand side of (2.27) coincides with that of (2.23) provided that $\vartheta = +\infty, \vartheta' = 1$. □

Finally, we prove the Lemma 2.1. We will prove a slightly more general result, that can be useful in other situations. This kind of results turn back to ideas of De Giorgi, and were developed by other authors in particular O. A. Ladyzhenskaya and G. Stampacchia. See Stampacchia (1963) and Ladyzhenskaya, Solonnikov and and Ural'ceva (1968).

Lemma 2.2. *Let $\phi(\xi)$ be a function defined for $\xi \geq M$, nonegative and decreasing (not necessarily strictly) such that for $h > k \geq M$ the estimate*

$$\phi(h) \leq \frac{\gamma k^{\vartheta}}{(h-k)^{\alpha}} \phi(k)^{1+\chi} \tag{2.28}$$

holds. Here, γ, α and χ are positive constants. Moreover $\vartheta < \alpha(1+\chi)$. Then $\phi(2d) = 0$, where $d > M$ is the root of the equation

$$d = M + \lambda M^{\vartheta/\alpha}\, d^{\frac{\vartheta-\alpha}{\chi\alpha}} \tag{2.29}$$

and

$$\lambda^{\alpha} = 2^{\frac{\alpha+\vartheta}{\chi}+\frac{\alpha}{\chi^2}}\, \gamma^{1+\frac{1}{\chi}} \phi(M)^{1+\chi} \ . \tag{2.30}$$

Proof: Set $k_j = d(2-2^{-j})$, $j = 0, 1, 2, \ldots$. We want to show that

$$\phi(k_j) \leq \left[\frac{d^{\alpha-\vartheta}}{2^{\alpha(j+1+1/\chi)+\vartheta}\, \gamma}\right]^{1/\chi} \ . \tag{2.31}$$

Since $\lim_{j\to+\infty} k_j = 2d$, then (2.31) implies that $\phi(2d) = 0$. Equation (2.28) for $h = k_0$ and $k = M$ shows that

$$\phi(k_0) \leq \frac{\gamma M^{\vartheta}}{(d-M)^{\alpha}} \phi(M)^{1+\chi} \ . \tag{2.32}$$

By replacing $(d-M)^{\alpha}$ by the value obtained from equation (2.29) it readily follows that the right hand side of (2.32) is equal to the right hand side of (2.31) for $j = 0$. Next, by supposing that (2.31) holds for some $j \geq 0$ and by using (2.28), we prove that

$$\phi(k_{j+1}) \leq \frac{2^{\vartheta+(j+1)\alpha}\gamma}{d^{\alpha-\vartheta}} \left[\frac{d^{\alpha-\vartheta}}{2^{\alpha(j+1+1/\chi)+\vartheta}\, \gamma}\right]^{1+1/\chi} \ . \tag{2.33}$$

Straightforward calculations show that the right hand side of (2.33) is equal to the right hand side of (2.31) if here we replace j by $j+1$.

3. EXISTENCE

In the sequel we denote by μ a real fixed number, larger than M_0. If w is a real function we set

$$\hat{w} = \begin{cases} \mu & \text{if } w \geq \mu \ , \\ w & \text{if } 0 \leq w \leq \mu \ , \\ 0 & \text{if } w \leq 0 \ . \end{cases} \tag{3.1}$$

We assume that ϕ, ψ and U are as in Section 1. We denote by T a fixed, positive (arbitrarily large) real number and we assume that $f \in L^1(0,T;L^2)$. Let

$$q, m \in L^2(0,T;L^2) \tag{3.2}$$

and consider the auxiliary problem (see Gajewski and Gröger, 1986)

$$\begin{cases} \dfrac{\partial p}{\partial t} - \nabla \cdot D_1(\nabla p + \mu_1 \hat{q} \nabla u) = R(\hat{q}, \hat{m}) \ , \\ \dfrac{\partial n}{\partial t} - \nabla \cdot (D_2 \nabla n - \mu_2 \hat{m} \nabla u) = R(\hat{q}, \hat{m}) \ , \\ -\nabla \cdot (a \nabla u) = f + \hat{q} - \hat{m} \ , \end{cases} \tag{3.3}$$

with boundary conditions

$$\begin{cases} p=\phi(x) \ \ , n=\psi(x) & \text{on} \ \ (0,T)\times D \ \ , \\ (D_1\nabla p+\mu_1\hat{q}\nabla u)\cdot\nu=(D_2\nabla n-\mu_2\hat{m}\nabla u)\cdot\nu=0 & \text{on} \ \ (0,T)\times N \ \ , \end{cases} \tag{3.4}$$

$$\begin{cases} u=U(x) \ \ \text{on} \ \ (0,T)\times D \ \ , \\ \dfrac{\partial u}{\partial\nu}=0 \quad \text{on} \ \ (0,T)\times B \ \ , \end{cases} \tag{3.5}$$

and initial conditions

$$p(0,x)=p_0(x) \ \ , \ \ n(0,x)=n_0(x) \ \ \text{in} \ \ \Omega \ . \tag{3.6}$$

The first step is to show the existence of a fixed point $(p,n)=(q,m)$. Let q and m be given. By setting $u=U+w$, the problem $(3.3)_3$, (3.5) is formulated in the following weak form. We look for $w\in V$ such that

$$\int a\ \nabla w\cdot\nabla v\ dx=-a\int\nabla U\cdot\nabla v\ dx+\int(f+\hat{q}-\hat{m})v\ dx\ ,\ \ \forall\, v\in V\ . \tag{3.7}$$

The symmetric bilinear form on the left hand side of (3.7) is continuous and coercive over the Hilbert space V, moreover the right hand side of (3.7) defines a bounded linear functional on V. By Riesz-Fréchet representation theorem, the problem (3.7) admits a unique solution $w\in V$. One easily shows that the weak solution $u=U+w$ of $(3.3)_3$, (3.5) satisfies

$$\|\nabla u\|\le 2\|\nabla U\|+c_0^{1/2}a^{-1}\ (\|f\|+\|\hat{q}\|+\|\hat{m}\|)\ \ . \tag{3.8}$$

Next, we study the problems $(3.3)_1$ and $(3.3)_2$ with boundary and initial conditions (3.4) - (3.6), in the following weak form. We set $p=y+\phi$, $n=z+\psi$, and we look for y and z in $L^2(0,T;V)$ with y' and z' in $L^2(0,T;V')$ (see, for instance Dautray- Lions, 1992, chap. XVIII, §1, specially sections 1 and 2) such that $y(0)=p_0-\phi$, $z(0)=n_0-\psi$ and

$$\begin{aligned} &\langle y'(t),v\rangle+D_1(\nabla y,\nabla v)=-D_1(\nabla\phi+\mu_1\hat{q}\nabla u,\nabla v)+(R(\hat{q},\hat{m}),v)\ ,\ \forall\, v\in V\ \ , \\ &\langle z'(t),v\rangle+(D_2(\nabla z,\nabla v)=-(D_2\nabla\psi-\mu_2\hat{m}\nabla u,\nabla v)+(R(\hat{q},\hat{m}),v)\ ,\ \forall\, v\in V\ . \end{aligned} \tag{3.9}$$

Since $D_1\nabla\phi+\mu_1\hat{q}\nabla u$ and $R(\hat{q},\hat{m})$ belong to $L^2(0,T;L^2)$, in $(3.9)_1$ they act on V as elements of $L^2(0,T;V')$. It follows (see, for instance Dautray- Lions, 1992, Chap. XVIII §3, Theorem 2) that the above problems admit unique solutions y and z. Hence, the problems $(3.3)_1$ and $(3.3)_2$ with boundary and initial conditions (3.4) and (3.6) have unique solutions p and n in $L^2(0,T;H^1)$ with p' and n' in $L^2(0,T;V')$. □

Next we show the existence of a fixed point. By setting $v=y(t)$ in equation $(3.9)_1$ it follows that

$$\frac{1}{2}\frac{d}{dt}\|y\|^2+D_1\|\nabla y\|^2\le D_1\left(\|\nabla\phi\|+\mu_1\|\hat{q}\|_\infty\|\nabla u\|\right)\|\nabla y\|+c_0\|R(\hat{q},\hat{m})\|\|\nabla y\|\ \ .$$

By taking (3.8) into account, it readily follows that the right hand side of the above inequality is bounded by $C\,(1+\|\nabla\phi\|+\|\nabla U\|+\|f\|)\,\|\nabla y\|$, hence is bounded also by

$\frac{1}{2} D_1 \|\nabla y\|^2 + (C/\mu_1)(1 + \|\nabla\phi\| + \|\nabla U\| + \|f\|)^2$, where the constant C may depend on a, D_1, μ_1 and μ. Hence

$$\frac{d\|y(t)\|^2}{dt} + D_1\|\nabla y\|^2 \leq \frac{2C}{\mu_1}(1 + \|\nabla\phi\| + \|\nabla U\| + \|f\|)^2 . \tag{3.10}$$

It readily follows that

$$\sup_{0\leq t\leq T} \|y(t)\|^2 + D_1 \int_0^T \|\nabla y(t)\|^2\, dt \leq 2\|y(0)\|^2 + \frac{4C}{\mu_1}\int_0^T (1 + \|\nabla\phi\| + \|\nabla U\| + \|f\|)^2\, dt .$$

Hence, there is a constant B_0, independent of the pair $(q, m) \in L^2(0,T;L^2)$, such that the norms of the solution y of $(3.9)_1$ in the spaces $C(0,T;L^2)$, $L^2(0,T;V)$ and $L^2(0,T;L^2)$ are bounded by B_0. A similar result holds for z. From these bounds and from the equations $(3.9)_1$ and $(3.9)_2$ it follows that the norms of y' and z' in $L^2(0,T;V')$ are uniformly bounded. Since $p = y + \phi$ and $n = z + \psi$, a similar result holds for p and n. Denote by B_1 an uniform bound for these norms and set

$$\mathbb{K}_T := \left\{(q,m) : \|q\|_{L^2(H^1)} \leq B_1,\ \|m\|_{L^2(H^1)} \leq B_1,\ \|q'\|_{L^2(V')} \leq B_1,\ \|m'\|_{L^2(V')} \leq B_1\right\} ,$$

where $L^2(X) = L^2(0,T;X)$. $\mathbb{K}_T$ is a closed, convex, compact set with respect to the $L^2(L^2)$ topology. The map $S : (q,m) \to (p,n)$ satisfies $S(\mathbb{K}_T) \subset \mathbb{K}_T$. Moreover, S is continuous on $\mathbb{K}_T$ with respect to the $L^2(L^2)$ norm, as follows from standard arguments. Hence, Schauder's fixed point theorem guarantees the existence of a fixed point on $\mathbb{K}_T$ for the map S, i.e. $(p,n) = (q,m)$. Clearly this fixed point is a weak solution of the problem

$$\begin{cases} \dfrac{\partial p}{\partial t} - \nabla\cdot(D_1\nabla p + \mu_1\hat{p}\nabla u) = R(\hat{p},\hat{n}) \\ \dfrac{\partial n}{\partial t} - \nabla\cdot(D_2\nabla n - \mu_2\hat{n}\nabla u) = R(\hat{p},\hat{n}) \\ -\nabla(a\nabla u) = f + \hat{p} - \hat{n} \end{cases} \tag{3.11}$$

$$\begin{cases} p = \phi(x) \quad , \quad n = \psi(x) \quad \text{on } (0,T)\times D , \\ (D_1\nabla p + \mu_1\hat{p}\nabla u)\cdot\nu = (D_2\nabla n - \mu_2\hat{n}\nabla u)\cdot\nu = 0 \quad \text{on } (0,T)\times B , \end{cases} \tag{3.12}$$

$$\begin{cases} u = U(x) \quad \text{on } (0,T)\times D , \\ \dfrac{\partial u}{\partial\nu} = 0 \quad \text{on } (0,T)\times B , \end{cases} \tag{3.13}$$

$$p(0,x) = p_0(x) \quad , \quad n(0,x) = n_0(x) \quad \text{in } \Omega \tag{3.14}$$

Next, we show (by following Gajewski and Gröger, 1986) that the solution (p,n) satisfies $p \geq 0$, $n \geq 0$. Multiply the equation $(3.11)_1$ by $p^- = \min\{p, 0\}$ and integrate on Ω. By taking into account that $(\hat{p}\nabla u, \nabla p^-) = 0$ and that $(R(\hat{p},\hat{n}), p^-) \leq 0$ (recall (1.8)) one shows that $d\|p^-(t)\|^2/dt \leq 0$. Since $p^-(0) = 0$, it follows that $p^- \equiv 0$. A similar proof shows that $n^- \equiv 0$.

Finally, we prove that the solutions p and n of problem (3.11) - (3.14) are bounded from above by the right hand side ℓ of equation (1.19). Since $\mu > \ell$, this shows that

$\hat{p} = p$ and $\hat{n} = n$. Hence (p, n) solves (1.1)-(1.4) and satisfies (1.19), for arbitrarily large T. At this point the reader should recall the explanation about the proof of the Theorem 1.1. given just after the statement of that theorem. According to that explanation, we show here how to modify the proof of (1.19) given in Section 2 in order to adapt it to the solution (p, n) of problem (3.11)-(3.14).

Set $\hat{v} = \min\{v, \mu\}$ where μ is above, and set $\bar{\hat{v}} = \max\{\hat{v} - k, 0\}$. In the sequel $k \in [M_0, \mu]$. Next, instead of multiplying $(1.1)_1$ by $\overline{p}$ (as done in Section 2) we multiply $(3.11)_1$ by $\bar{\hat{p}}$. As in Section 2, we prove (2.1) where now (p, n) is replaced by $(\hat{p}, \hat{n})$. Hence (2.2) and (2.2 ′) hold by replacing (p, n) by $(\hat{p}, \hat{n})$. From now on, all the calculations done in Section 2 hold if we replace (p, n) by $(\hat{p}, \hat{n})$ since they depend just on (2.2) and (2.2 ′). This shows that $\hat{p} \leq \ell$ and $\hat{n} \leq \ell$. By chosing $\mu > \ell$, one gets $(\hat{p}, \hat{n}) = (p, n)$. □

The proof of the uniqueness of the solution follows the standard argument. If (p, n, u) and (q, m, v) are two solutions of problem (1.1)-(1.4) one easily shows that

$$\frac{1}{2}\frac{d}{dt}\|p-q\|^2 + D_1 \int |\nabla(p-q)|^2\, dx + \frac{\mu_1}{2a}\int (f+p-n)(p-q)^2\, dx +$$
$$+\mu_1 \int q\nabla(u-v)\cdot\nabla(p-q)dx = \int [R(p,n) - R(q,m)](p-q)\, dx \ .$$

Since p, n, q and m are bounded and R is locally Lipschitz continuous, one has

$$\frac{1}{2}\frac{d}{dt}\|p-q\|^2 + D_1 \int |\nabla(p-q)|^2\, dx \leq C\int (p-q)^2\, dx + C\int |f|(p-q)^2\, dx +$$
$$+C\|\nabla(u-v)\|\,\|\nabla(p-q)\| + C\|n-m\|^2 \ .$$

Moreover, $\|\nabla(u-v)\| \leq C\|(p-q)+(n-m)\|$ and

$$\int |f|\,(p-q)^2\, dx \leq \|f\|_N\,\|p-q\|\,\|p-q\|_{2^*} \leq \frac{D_1}{2}\,\|\nabla(p-q)\|^2 + C\,\|f\|_N^2\,\|p-q\|^2 \ .$$

(Here, and below, if $N = 2$ replace N and 2^* by 4). By using similar results for $n - m$ one shows that

$$\frac{d}{dt}\left(\|p-q\|^2 + \|n-m\|^2\right) + C\left(\|\nabla(p-q)\|^2 + \|\nabla(n-m)\|^2\right)$$
$$\leq C\left(1+\|f\|_N^2\right)\left(\|p-q\|^2 + \|n-m\|^2\right) \ .$$

Since $f \in L^2_{\text{loc}}(0, +\infty;\, L^N)$ one has

$$\|p(t)-q(t)\|^2 + \|n(t)-m(t)\|^2 \leq \left(\|p(0)-q(0)\|^2 + \|n(0)-m(0)\|^2\right)\, e^{C\int_0^t (1+\|f\|_N^2)\, ds}.$$

Hence, $(p, q) = (n, m)$.

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