Mathematical Methods in the Applied Sciences, Vol. 18, 295–306 (1995)
MOS subject classification: 35 B 30, 35 L 60, 35 Q 20

On the Sharp Singular Limit for Slightly Compressible Fluids

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Communicated by Y. Shibata

We consider the equations of motion to slightly compressible fluids and we prove that solutions converge, in the *strong* norm, to the solution of the equations of motion of incompressible fluids, as the Mach number goes to zero. From a physical point of view this means the following. Assume that we are dealing with a well-specified fluid, so slightly compressible that we assume it to be incompressible. Our result means that the distance between the (continuous) trajectories of the real and of the idealized solution is 'small' with respect to the natural metric, i.e. the metric that endows the data space.

1. Introduction, notations, and results

In [2] we have proved the convergence in the *strong norm* of the solution to the equations of inviscid, compressible fluids to the solution of the equations of inviscid, incompressible fluids as the Mach number goes to zero. The method employed there has the merit of being applicable to boundary value problems. However, in the space periodic case (as well as in the whole space case) the proofs can be substantially simplified. The aim of this paper is to illustrate this new approach in the simplest framework to avoid supplementary technicalities.

Before going on, let us introduce the main notation. Here Ω denotes the n-dimensional torus $[0, 1[^n$. Hence, functions defined in Ω are periodic in each space variable x_i, with period equal to 1. We denote by $|\cdot|_p$ and $\|\cdot\|_m$ the canonical norms in the space $L^p = L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, and in the L^2-Sobolev space $H^m = H^m(\Omega)$, respectively. We set $\|\cdot\| = \|\cdot\|_0 = |\cdot|_2$. By $\|\cdot\|_{m,T}$ we denote the canonical norm in the space $L^\infty_T(H^m) = L^\infty(0, T; H^m)$. We set $C_T(H^m) = C([0, T]; H^m)$. Integrals $\int_\Omega f(x)\,dx$ are denoted simply by $\int f(x)\,dx$.

We denote by k_0 the smallest integer larger than $n/2$ and by k a fixed integer, satisfying $k \geqslant k_0 + 1$. Below, we introduce positive constants c_1, c_2, c_3. Positive constants that depend only on c_1 are denoted by C_1. Similarly, $C_2 = C_2(c_1, c_2)$ and $C_3 = C_3(c_1, c_2, c_3)$. Universal constants are denoted by c. Distinct constants C_j of the same type (same j) are denoted by the same symbol. Hence, $cC_1 = C_1$, $C_1 + C_2 = C_2$, and so on.

CCC 0170–4214/95/040295–12
Received 28 February 1994

Let us return to our problem. In order to exhibit the main lines in our proof it is convenient to restrict the problem to the simplest case. In particular, we will assume here that

$$p(\lambda, \rho) = \lambda^2 \rho,$$

where λ^{-1} is the Mach number. It is worth noting, however, that the result proved here holds also if $p(\lambda, \rho) = \lambda^2 p(\rho)$ where $p \in C^{k+2}(\mathbb{R}^+; \mathbb{R})$ and $p'(s) > 0$ for each $s \in \mathbb{R}^+$. We could also assume that the law of state $p(\rho, \lambda)$ satisfies general assumptions, similar to those in [1]; see also [4]. However, this is not a main point in the mathematical treatment of the problem. In any case note that the significant assumption is $\lim_{\lambda\to\infty} p'(\lambda, \bar{\rho}_0) = \infty$, where $\bar{\rho}_0$ denotes the mean density of the fluid in Ω and p' is the derivative of p with respect to ρ. Under the above law of state the equations of motion are

$$\begin{aligned} &\rho_t^\lambda + v^\lambda \cdot \nabla \rho^\lambda + \rho^\lambda \nabla \cdot v^\lambda = 0, \\ &\rho^\lambda [v_t^\lambda + (v^\lambda \cdot \nabla) v^\lambda] + \lambda^2 \nabla \rho^\lambda = 0, \\ &\rho^\lambda(0, x) = \bar{\rho}_0 + \rho_0^\lambda(x), \; v^\lambda(0, x) = v_0^\lambda(x). \end{aligned} \tag{1.1}$$

It is convenient to replace ρ by $g = \log(\rho/\bar{\rho}_0)$. By assuming that $\bar{\rho}_0 = 1$ and by setting

$$g = \log \rho,$$

equation (1.1) becomes

$$\begin{aligned} &g_t^\lambda + v^\lambda \cdot \nabla g^\lambda + \nabla \cdot v^\lambda = 0, \\ &v_t^\lambda + \lambda^2 \nabla g^\lambda + (v^\lambda \cdot \nabla) v^\lambda = 0, \\ &g^\lambda(0, x) = g_0(x), \quad v^\lambda(0, x) = v_0(x), \end{aligned} \tag{1.2}$$

where $g_0^\lambda(x) = \log(1 + \rho_0^\lambda(x))$. Note that convergence of ρ^λ to 1 and ρ_t^λ to 0 is equivalent to convergence of g^λ to 0 and g_t^λ to 0, respectively. We will prove convergence of g_t^λ to 0 (hence of ρ_t^λ to 1) in $C_T(H^{k-1})$. Convergence of v_t^λ to w_t in this last space holds only under supplementary conditions.

The limit equations to system (1.2) are

$$\begin{aligned} &\nabla \cdot w = 0, \\ &w_t + (w \cdot \nabla) w + \nabla \pi = 0, \\ &w(0, x) = w_0(x). \end{aligned} \tag{1.3}$$

Finally, the assumptions on the initial data are the following:

$$\|v_0^\lambda\|_{k_0+1} \leqslant c_1, \quad \lambda \|g_0^\lambda\|_{k_0+1} \leqslant c_1, \tag{1.4}$$

$$\|v_0^\lambda\|_k \leqslant c_2, \quad \lambda \|g_0^\lambda\|_k \leqslant c_2, \tag{1.5}$$

$$\lambda \|\nabla \cdot v_0^\lambda\|_0 \leqslant c_3, \quad \lambda^2 \|\nabla g_0^\lambda\|_0 \leqslant c_3. \tag{1.6}$$

For convenience $\lambda \geqslant 1$. Under the hypotheses (1.4), (1.5) there exist in $[0, T]$ solutions (v^λ, g^λ) of problem (1.2), where $T = c/c_1$. These solutions belong to $C_T(H_k) \cap C_T^1(H^{k-1})$. Our main result is the following.

Theorem A. *Assume that* (1.4)–(1.6) *hold and that*

$$\lim_{\lambda\to\infty} (\|v_0^\lambda - w_0\|_k + \lambda\|g_0^\lambda\|_k) = 0. \tag{1.7}$$

Then the solutions (v^λ, g^λ) *of problem* (1.2) *converge in the strong norm to the solution* $(w, 0)$ *of problem* (1.3). *More precisely,*

$$\lim_{\lambda\to\infty} (\|v^\lambda - w\|_{k,T}^2 + \|g^\lambda\|_{k,T}^2 + \lambda^2\|\nabla g^\lambda\|_{k-1,T}^2 + \|g_t^\lambda\|_{k-1,T}^2) = 0. \tag{1.8}$$

Other convergence results follow from the above theorem, from compacity arguments connected to the *a priori* estimates, and from the equations. In particular, v_t^λ converges to w_t and $\lambda^2\nabla g$ converges to $\nabla\pi$ both in $L_T^\infty(H_0)$ weak-* (note that $v_t^\lambda + \lambda^2\nabla g \to w_t + \nabla\pi$ in $C_T(H^{k-1})$).

In a forthcoming paper we will deal with inviscid and viscous fluids with variable viscosity coefficients $\nu\in[0,\nu_0]$, $\mu\in[0,\mu_0]$. The limit will be taken simultaneously as $\lambda\to\infty$, $\nu\to\bar\nu$, μ stays bounded, and the initial data converge to initial data in H^k. In particular, we will show that under assumptions (1.4)–(1.7) the solution of the compressible equations converges strongly in $C(0,T;H^k)$ to the solution of the incompressible equations. Moreover, if $\bar\nu > 0$, convergence holds also in the strong $L^2(0,T;H^{k+1})$ norm. A related weak convergence theorem, under stronger assumptions, was proved in [4].

2. The weak convergence theorem

In this section we prove Theorem 1.1 below. The proofs given in this section follow the proof of the weak convergence theorem in [5], to which we refer the reader. However, our result is proved under weaker assumptions. For that reason, and also for completeness, we give self-contained proofs of the estimates (we note that the existence of the solution of problem (1.2), for each fixed λ, follows from the *a priori* estimates by using standard methods). The first estimate is

$$\|fg\| \leqslant c\|f\|_{r-s}\|g\|_s \quad \text{if} \quad 0\leqslant s\leqslant r \quad \text{and } r > n/2. \tag{2.1}$$

In fact, if $r-s>n/2$ then $H^{r-s}\subsetneq L^\infty$ and (2.1) follows. If $r-s=n/2$ then $s>0$. Hence $H^s\subsetneq L^p$ for some $p>2$. Moreover $H^{r-s}\subsetneq L^{2p/(p-2)}$. Consequently, $\|fg\|\leqslant\|f\|_{2p/(p-2)}\|g\|_p$. If $s\geqslant n/2$ we argue as above by replacing s by $r-s$. Finally, if $r-s<n/2$ and $s>n/2$ set $1/p = 1/2-(r-s)/n$, $1/q = 1/2 - s/n$. Since $p,q\in[2,\infty[$ and $1/p+1/q<1/2$ there are reals $p_0,q_0\in[2,\infty[$ such that $1/p_0+1/q_0 = 1/2$, $H^{r-s}\subsetneq L^{p_0}$, $H^s\subsetneq L^{q_0}$.

Next, we recall the Gagliardo–Nirenberg inequalities (see [3, 6]):

$$|D^j h|_{2r/j} \leqslant c|h|_\infty^{1-j/r}\|Dh\|_{r-1}^{j/r} \quad \text{if } 0\leqslant j\leqslant r. \tag{2.2}$$

For convenience, in the sequel, we drop the λ's from the notations g^λ and v^λ, except in some main equations. We start by proving that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_m^2(t) \leqslant cE_{k_0+1}(t)E_m^2(t), \tag{2.3}$$

where $m \geqslant k_0 + 1$ and

$$E_m^2(t) = \lambda^2 \| g \|_m^2 + \| v \|_m^2.$$

Let $|\alpha| = m$ and apply $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, to equation $(1.2)_1$. Then, multiply both sides by $\lambda^2 D^\alpha g$ and integrate on Ω. Taking into account that

$$\int (v \cdot \nabla D^\alpha g) D^\alpha g \, dx = -\tfrac{1}{2} \int (\nabla \cdot v) |D^\alpha g|^2 \, dx$$

and that $H^{k_0} \subsetneq L^\infty$, it readily follows that

$$\frac{\lambda^2}{2} \frac{d}{dt} \| D^\alpha g \|^2 - \lambda^2 \int (D^\alpha v) \cdot \nabla (D^\alpha g) \, dx$$
$$\leqslant c\lambda^2 \| v \|_{k_0+1} \| g \|_m^2 + c\lambda^2 \| g \|_m \sum_{\ell=0}^{m-1} \| (D^{m-\ell} v)(D^{\ell+1} g) \|, \tag{2.4}$$

where simplified but clear notation is used. Next, Gagliardo–Nirenberg's inequalities (2.2) show that

$$|D^\ell (Dg)|_{2(m-1)/\ell} \leqslant c |Dg|_\infty^{1-\ell/(m-1)} |D^2 g|_{m-2}^{\ell/(m-1)}$$
$$\leqslant c |g|_{k_0+1}^{1-\ell/(m-1)} |g|_m^{\ell/(m-1)}$$

and that

$$|D^{m-\ell-1}(Dv)|_{2(m-1)/(m-\ell-1)} \leqslant c \| v \|_{k_0+1}^{1-(m-\ell-1)/(m-1)} \| v \|_m^{(m-\ell-1)/(m-1)}.$$

Hence, by using Hölder's inequality we verify that $\| (D^{m-\ell})(D^{\ell+1} g) \|$ is bounded by the product of the right-hand sides in the two last inequalities. It readily follows that

$$\lambda \| (D^{m-\ell} v)(D^{\ell+1} g) \| \leqslant c E_{k_0+1}(t) E_m(t).$$

Consequently, from (2.4) we obtain

$$\frac{\lambda^2}{2} \frac{d}{dt} \| D^\alpha g \|^2 - \lambda^2 \int (D^\alpha v) \cdot \nabla (D^\alpha g) \, dx \leqslant c E_{k_0+1}(t) E_m^2(t). \tag{2.5}$$

Obviously, this last estimate also holds if $0 \leqslant |\alpha| < m$.

Next, we apply D^α to equation $(1.2)_2$, multiply by $D^\alpha v$ and integrate on Ω. Calculations similar to those done above show that

$$\frac{1}{2} \frac{d}{dt} \| D^\alpha v \|^2 + \lambda^2 \int (\nabla D^\alpha g) \cdot (D^\alpha v) \, dx \leqslant c \| v \|_{k_0+1} \| v \|_m^2.$$

By adding, side by side, these inequalities and the inequalities (2.5), for all α such that $0 \leqslant |\alpha| \leqslant m$, (2.3) follows. □

Next, we show that under assumptions (1.4) there is a $T > 0$ (more precisely, $T = c/c_1$) such that a solution $(g, v) \in C(0, T; H^{k_0+1})$ of problem (1.2) exists and satisfies the uniform estimate

$$\lambda^2 \| g \|_{k_0+1,T}^2 + \| v \|_{k_0+1,T}^2 \leqslant 8 c_1^2. \tag{2.6}$$

Proof. From (2.3), by setting $m = k_0 + 1$, it follows that

$$E_{k_0+1}(t) \leqslant [E_{K_0+1}^{-1}(0) - ct]^{-1}.$$

Hence (2.6) holds provided that

$$T \leqslant \frac{1}{2c} E_{k_0+1}^{-1}(0) = \frac{1}{2c}(\lambda^2 \| g_0 \|_{k_0+1}^2 + \| v_0 \|_{k_0+1}^2)^{-1/2}.$$

Note that the right-hand side of the last equation is bounded by $(2\sqrt{(2)}c)^{-1}/c_1$. □

Now, we return to equation (2.3). Let $m = k > k_0 + 1$ and $t \in [0, T]$. Since $E_{k_0+1}(t)$ is bounded in $[0, T]$ the following result readily follows.

Lemma 2.1. *If* (1.4), (1.5) *hold, then*

$$\lambda^2 \| g \|_{k,T}^2 + \| v \|_{k,T}^2 \leqslant C_1(\| \lambda^2 \| g_0 \|_k^2 + \| v_0 \|_k^2) \leqslant C_2. \tag{2.7}$$

Note that from (2.7) and (1.2) it follows that $v_t \in C_T(H^{k-1})$ and that $\| v_t \|_{k-1,T} \cong C_2 \lambda$. However, we aim at a uniform estimate for v_t^λ in order to be able to pass to the limit in (1.2) as λ goes to ∞. In this direction, the estimate (2.8) below shows that (1.6) is just the natural additional assumption.

Lemma 2.2. *Assume that hypotheses* (1.4) *and* (1.5) *hold. Then*

$$\lambda^2 \| g_t \|_{0,T}^2 + \| v_t \|_{0,T}^2 \leqslant C_2(\lambda^4 \| \nabla g_0 \|^2 + \lambda^2 \| \nabla \cdot v_0 \|^2 + \| v_0 \|^2). \tag{2.8}$$

If, moreover, (1.6) *holds then*

$$\lambda^2 \| g_t \|_{0,T}^2 + \| v_t \|_{0,T}^2 \leqslant C_3. \tag{2.9}$$

Proof. From equations (1.2) it follows that

$$g_{tt} + v \cdot \nabla g_t + \nabla \cdot v_t + v_t \cdot \nabla g = 0,$$
$$v_{tt} + \lambda^2 \nabla g_t + (v \cdot \nabla) v_t + v_t \cdot \nabla g = 0.$$

By multiplying the first of the above equations by $\lambda^2 g_t$, the second by v_t, by integrating on Ω, and by adding side by side the two equations obtained it readily follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\lambda^2 \| g_t \|^2 + \| v_t \|^2) \leqslant C_1(\lambda^2 \| g_t \|^2 + \| v_t \|^2),$$

for each $t \in [0, T]$. Hence,

$$\lambda^2 \| g_t \|_{0,T}^2 + \| v_t \|_{0,T}^2 \leqslant C_1(\lambda^2 \| g_t(0) \|^2 + \| v_t(0) \|^2).$$

By using equations (1.2) in order to express $g_t(0)$ and $v_t(0)$ in terms of the initial data g_0 and v_0, the thesis follows. □

The above lemmas together with well-known compact embedding theorems show the following result. The symbol '$\rightharpoonup$' denotes convergence in weak-* topologies.

Theorem 2.1. *Assume that hypotheses* (1.4)–(1.6) *hold and that* $\lim_{\lambda\to\infty} \| v_0^\lambda - w_0 \|_k = 0$. *Then*

$$\begin{aligned} &v^\lambda \rightharpoonup w \quad \text{in } L_T^\infty(H^k) \text{ and in } C_T(H^{k-\varepsilon}), \quad \varepsilon > 0,\\ &g^\lambda \to 0 \quad \text{in } C_T(H^k),\\ &v_t^\lambda \rightharpoonup w_t \quad \text{in } L_T^\infty(H^0),\\ &g_t^\lambda \to 0 \quad \text{in } C_T(H^{k-1-\varepsilon}), \quad \varepsilon > 0,\\ &\nabla\cdot v^\lambda \to 0 \quad \text{in } C_T(H^{k-1-\varepsilon}),\\ &\lambda^2 \nabla g^\lambda \rightharpoonup \nabla\pi \quad \text{in } L_T^\infty(H^0). \end{aligned} \tag{2.16}$$

Proof. The proof is an easy exercise. From (2.7) and (2.9) one gets

$$\lim_{\lambda\to 0} \lambda^{1-\varepsilon}(\| g^\lambda \|_{k,T} + \| g_t^\lambda \|_{0,T}) = 0. \tag{2.11}$$

In particular $(2.10)_2$ holds and $g_t^\lambda \to 0$ in $C_T(H^0)$. Next (for subsequences . . .) v^λ converges to some w in $L_T^\infty(H^k)$ with respect to the weak-* topology, since v^λ is bounded in $L_T^\infty(H^k)$ which is just the strong dual of $L_T^1(H^k)$. Moreover, v^λ is bounded in $W^{1,\infty}(H^0)$. Since H^k is compactly embedded in H^0, Ascoli–Arzelà's theorem shows compactness in $C_T(H^0)$. Compactness in $C_T(H^{k-\varepsilon})$ follows, since $\| \cdot \|_{k-\varepsilon,T} \leqslant \| \cdot \|_{0,T}^{\varepsilon/k} \| \cdot \|_{k,T}^{1-\varepsilon/k}$.

Similarly, v_t^λ (subsequences . . .) converges in $L_T^\infty(H^0)$ with respect to the weak-* topology. Clearly, if $v^\lambda \rightharpoonup w$ then the above limit must be w_t.

Next, from $-\nabla\cdot v^\lambda = g_t^\lambda + v^\lambda\cdot\nabla g^\lambda$ it follows that $\nabla\cdot v^\lambda \to 0$ in $C_T(H^0)$. In particular, it must be $\nabla\cdot w = 0$. From $(2.10)_1$, $(2.10)_5$ follows. The above equation together with $(2.10)_1$ and $(2.10)_2$ yields $(2.10)_4$.

On the other hand, the left-hand side of the equation $v_t^\lambda + (v^\lambda\cdot\nabla)v^\lambda = -\lambda^2\nabla g^\lambda$ is a gradient and converges in the $L_T^\infty(H^0)$ weak-* topology (always for subsequences). Hence the limit must be a gradient. This shows $(2.10)_6$. It is understood that the above argument has been developed for suitable subsequences. However, by passing to the limit in equations (1.2), as $\lambda \to \infty$, we verify that limits are solutions of (1.3). Since the regular solution of (1.3) is unique, it follows the convergence of the whole 'sequence' to the same limit.

Finally, we remark that

$$\lambda^{1+1/(k-1)} \| \nabla g^\lambda \|_{k-2} \leqslant C_3. \tag{2.12}$$

In fact, equations $(1.2)_2$, (2.7) and (2.9) show that $\lambda^2 \| \nabla g \|_{0,T} \leqslant C_3$. Since $\lambda \| \nabla g \|_{k-1,T} \leqslant C_2$, the result follows by interpolation. □

3. Proof of Theorem A

We start by remarking that, under the hypotheses of Theorem 1.1, equations (2.10) and (2.12) show that

$$\lim_{\lambda\to 0} (\| v^\lambda - w \|_{k-1,T}^2 + \| g^\lambda \|_{k-1,T}^2 + \lambda^2 \| \nabla g^\lambda \|_{k-2,T}^2) = 0. \tag{3.1}$$

In the sequel this result will be applied with k replaced by $k + 1$.

Next, we consider the Fourier series

$$u_0(x) = \sum_{\xi} \hat{u}_0(\xi)\, e^{2\pi i\xi\cdot x},$$

where the Fourier coefficients are given by

$$\hat{u}_0(\xi) = \int e^{-2\pi i\xi\cdot x} u_0(x)\, dx,$$

and $\xi = (\xi_1, \ldots, \xi_n)$. The ξ_i's are nonnegative integers and $|\xi|$ denotes the Euclidean norm of ξ. For each $s \in \mathbb{R}_0^+$ one has

$$\| u_0 \|_s^2 = \sum_{\xi} (1 + |\xi|^2)^s |\hat{u}_0(\xi)|^2.$$

Let $\delta \in]0, 1]$, define operators T^δ by

$$(T^\delta u_0)(x) = \sum_{|\xi| < 1/\delta} \hat{u}_0(\xi) e^{2\pi i\xi\cdot x},$$

and set

$$v_0^{\lambda,\delta} = T^\delta v_0^\lambda, \quad g_0^{\lambda,\delta} = T^\delta g_0^\lambda, \quad w_0^\delta = T^\delta w_0. \tag{3.2}$$

Note that T^δ is a linear operator on H^s for each $s \in \mathbb{R}_0^+$; moreover

$$|||T^\delta|||_{(s,s)} \leqslant 1,$$

where $|||\cdot|||_{(s,m)}$ denotes the canonical norm of bounded linear operators from H^s to H^m. Furthermore, T^δ commutes with the divergence operator. It follows that, for each δ, the initial data $v_0^{\lambda,\delta}, g_0^{\lambda,\delta}$ satisfy hypotheses (1.4)–(1.6) exactly with the same constants c_1, c_2, c_3. Hence, the solution $(v^{\lambda,\delta}, g^{\lambda,\delta})$ of the problem

$$\begin{aligned} &g_t^{\lambda,\delta} + (v^{\lambda,\delta}\cdot\nabla)g^{\lambda,\delta} + \nabla\cdot v^{\lambda,\delta} = 0,\\ &v_t^{\lambda,\delta} + \lambda^2\nabla g^{\lambda,\delta} + (v^{\lambda,\delta}\cdot\nabla)v^{\lambda,\delta} = 0,\\ &v^{\lambda,\delta}(0,x) = v_0^{\lambda,\delta}(x), \quad g^{\lambda,\delta}(0,x) = g_0^{\lambda,\delta}(x) \end{aligned} \tag{3.3}$$

satisfies (uniformly with respect to λ and to δ) the estimates

$$\begin{aligned} &\lambda^2 \| g^{\lambda,\delta} \|_{k,T}^2 + \| v^{\lambda,\delta} \|_{k,T}^2 \leqslant C_2,\\ &\lambda^2 \| g_t^{\lambda,\delta} \|_{0,T}^2 + \| v_t^{\lambda,\delta} \|_{0,T}^2 \leqslant C_3. \end{aligned} \tag{3.4}$$

Next, we study the problem (3.3) in the space H^{k+1}, for fixed δ. First of all one easily shows, by using the definition, that

$$\begin{aligned} &|||T^\delta|||_{(s,m)} \leqslant (2/\delta)^{m-s},\\ &|||T^\delta - I|||_{(m,s)} \leqslant \delta^{m-s}, \end{aligned} \tag{3.5}$$

where $0 \leqslant s \leqslant m$. In particular, it follows that

$$\begin{aligned} &\lambda \| g_0^{\lambda,\delta} \|_{k_0+1} \leqslant c_1, \quad \| v_0^{\lambda,\delta} \|_{k_0+1} \leqslant c_1,\\ &\lambda \| g_0^{\lambda,\delta} \|_{k+1} \leqslant 2c_2/\delta, \quad \| v_0^{\lambda,\delta} \|_{k+1} \leqslant 2c_2/\delta,\\ &\lambda \| \nabla\cdot v_0^{\lambda,\delta} \|_0 \leqslant c_3, \quad \lambda^2 \| \nabla g_0^{\lambda,\delta} \|_0 \leqslant c_3, \end{aligned} \tag{3.6}$$

and that

$$\lambda \| g_0^{\lambda,\delta} \|_{k+1} \leqslant (2/\delta) \lambda \| g_0 \|_k. \tag{3.7}$$

Moreover,

$$\| v_0^{\lambda,\delta} - w_0^{\delta} \|_{k+1} \leqslant (2/\delta) \| v_0^{\lambda} - w_0 \|_k. \tag{3.8}$$

Since $(v_0^{\lambda}, \lambda g_0^{\lambda}) \to (w_0, 0)$ in H^k it follows, in particular, that $(v_0^{\lambda,\delta}, \lambda g_0^{\lambda,\delta}) \to (w_0^{\delta}, 0)$ in H^{k+1}, for each *fixed* δ.

The above results show that the hypotheses of Theorem 1.1 are fulfilled by the data of problem (3.3) (not uniformly with respect to δ) if we replace k by $k+1$ and c_2 by $2c_2/\delta$. Hence, by (3.1), one has

$$\lim_{\lambda \to \infty} (\| v^{\lambda,\delta} - w^{\delta} \|_{k,T}^2 + \| g^{\lambda,\delta} \|_{k,T}^2 + \lambda^2 \| \nabla g^{\lambda,\delta} \|_{k-1,T}^2) = 0, \tag{3.9}$$

for each fixed δ, where w^{δ} denotes the solution of the incompressible equations

$$\begin{aligned} &\nabla \cdot w^{\delta} = 0, \\ &\partial_t w^{\delta} + (w^{\delta} \cdot \nabla) w^{\delta} + \nabla \pi^{\delta} = 0, \\ &w^{\delta}(0,x) = w_0^{\delta}(x). \end{aligned} \tag{3.10}$$

The following estimates will be useful in the sequel:

$$\begin{aligned} &\| v_0^{\lambda,\delta} - v_0^{\lambda} \|_k^2 \leqslant 2 \| v_0^{\lambda} - w_0 \|_k^2 + 2 \sum_{|\xi| \geqslant 1/\delta} (1 + |\xi|^2)^k |\hat{w}_0(\xi)|^2, \\ &\| g_0^{\lambda,\delta} - g_0^{\lambda} \|_k^2 \leqslant \| g_0^{\lambda} \|_k^2. \end{aligned} \tag{3.11}$$

We prove the first estimate. One has

$$\| v_0^{\lambda,\delta} - v_0^{\lambda} \|_k^2 = \sum_{|\xi| \geqslant 1/\delta} (1 + |\xi|^2)^k |\hat{v}_0^{\lambda}(\xi)|^2.$$

Hence the left-hand side is bounded by

$$2 \sum_{|\xi| \geqslant 1/\delta} (1 + |\xi|^2)^k |\hat{v}_0^{\lambda}(\xi) - \hat{w}_0(\xi)|^2 + 2 \sum_{|\xi| \geqslant 1/\delta} (1 + |\xi|^2)^k |\hat{w}_0(\xi)|^2,$$

whence the result follows.

From convenience, sometimes we will write $g, v, g^{\delta}, v^{\delta}$ instead of $g^{\lambda}, v^{\lambda}, g^{\lambda,\delta}, v^{\lambda,\delta}$. Set

$$\bar{g} = g^{\lambda,\delta} - g^{\lambda}, \quad \bar{v} = v^{\lambda,\delta} - v^{\lambda}.$$

In the sequel the hypotheses in Theorem A are assumed without further mention. We start by proving the following result (δ_k^m denotes the Kronecker's symbol).

Theorem 3.1. *Let* $0 \leqslant m \leqslant k$. *One has*

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\lambda^2 \| \bar{g} \|_m^2 + \| \bar{v} \|_m^2 \leqslant C_2 (\lambda^2 \| \bar{g} \|_m^2 + \| \bar{v} \|_m^2) \\ + \delta_k^m C_2 (\lambda \| \nabla g^{\lambda,\delta} \|_k + \| v^{\lambda,\delta} \|_{k+1}) (\lambda |\bar{g}|_\infty + |\bar{v}|_\infty)(\lambda \| \bar{g} \|_k + \| \bar{v} \|_k). \end{aligned} \tag{3.12}$$

Proof. In the following, recall that $\| v \|_{k,T}, . \lambda \| g \|_{k,T}, \| v^{\delta} \|_{k,T}$ and $\lambda \| g^{\delta} \|_{k,T}$ are bounded by a constant of type C_2.

By taking the difference between equations (3.3) and (1.2) we show that

$$\bar{g}_t + v\cdot\nabla\bar{g} + \nabla\cdot\bar{v} = -\bar{v}\cdot\nabla g^\delta,$$
$$\bar{v}_t + \lambda^2\nabla\bar{g} + (v\cdot\nabla)\bar{v} = -(\bar{v}\cdot\nabla)v^\delta. \tag{3.13}$$

Next, we apply the operator D^α, for $|\alpha| = m$, to equation $(3.13)_1$. This yields, with a simplified notation,

$$D^\alpha\bar{g}_t + v\cdot\nabla D^\alpha\bar{g} + \sum_{\ell=0}^{m-1}(D^{m-\ell}v)(D^{\ell+1}\bar{g}) + \nabla\cdot D^\alpha\bar{v} = -D^m(\bar{v}\cdot\nabla g^\delta).$$

Then, by multiplying both sides of this equation by $\lambda^2 D^\alpha\bar{g}$, by integrating on Ω and by employing suitable techniques, we prove that

$$\frac{\lambda^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^\alpha\bar{g}\|^2 - \lambda^2\int D^\alpha\bar{v}\cdot\nabla D^\alpha\bar{g}$$
$$\leqslant C_2\lambda^2\|\bar{g}\|_m^2 + C_2\lambda\|\bar{v}\|_m\|\bar{g}\|_m + \delta_k^m C_2\lambda^2\|\nabla g^\delta\|_k|\bar{v}|_\infty\|\bar{g}\|_k. \tag{3.14}$$

In fact,

$$\left|\int(v\cdot\nabla D^\alpha\bar{g})D^\alpha\bar{g}\,\mathrm{d}x\right| = \tfrac{1}{2}\left|\int(\nabla\cdot v)|D^\alpha\bar{g}|^2\,\mathrm{d}x\right| \leqslant C_2\|\bar{g}\|_m^2.$$

Moreover, from (2.1) it readily follows that

$$\|(D^{m-\ell}v)(D^{\ell+1}\bar{g})\| \leqslant \|D^{m-\ell}v\|_{k-m+\ell}\|D^{\ell+1}\bar{g}\|_{m-\ell-1} \leqslant C_2\|\bar{g}\|_m. \tag{3.15}$$

Finally,

$$\|D^m(\bar{v}\cdot\nabla g^\delta)\| \leqslant c\|\bar{v}\|_m\|Dg^\delta\|_{k-1} + c\delta_k^m|\bar{v}|_\infty\|\nabla g^\delta\|_k. \tag{3.16}$$

If $m \leqslant k-1$, (3.16) follows from (2.1). If $m = k$, (2.1) shows that the L^2-norm of each term of the expansion of $D^k(\bar{v}\cdot\nabla g^\delta)$ is bounded by the first term on the right-hand side of (3.16) except for the term $\bar{v}\cdot D^k\nabla g^\delta$, the L^2-norm of which is bounded by the second term on the right-hand side of (3.15).

Next apply D^α to equation $(2.13)_2$, multiply by $D^\alpha\bar{v}$ and integrate on Ω. As above, we show here that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^\alpha\bar{v}\|^2 + \lambda^2\int\nabla D^\alpha\bar{g}\cdot D^\alpha\bar{v}\,\mathrm{d}x$$
$$\leqslant C_2\lambda\|\bar{g}\|_m\|\bar{v}\|_m + C_2\|\bar{v}\|_m^2 + \delta_k^m C_2\lambda^2\|\nabla g^\delta\|_k|\bar{g}|_\infty\|\bar{v}\|_k$$
$$+ c\delta_k^m\|v^\delta\|_{k+1}|\bar{v}|_\infty\|\bar{v}\|_k. \tag{3.17}$$

Clearly, (3.17) also holds when $|\alpha| < m$. By adding side by side (for all α such that $0 \leqslant |\alpha| \leqslant m$) equations (3.14) and (3.17), (3.12) follows. □

Next, fix $\beta_0 \in]0,1[$ such that $\beta_0 < k_0 - n/2$. Since $k_0 - \beta_0 > n/2$ one has $|\cdot|_\infty \leqslant c\|\cdot\|_{k_0-\beta_0}$. By interpolation in L^2-Sobolev spaces one gets

$$|\cdot|_\infty \leqslant c\|\cdot\|_{k_0-1}^{\beta_0}\|\cdot\|_{k_0}^{1-\beta_0}. \tag{3.18}$$

We prove the following result.

Proposition 3.1. *For each $\delta\in]0,1]$ and each λ the following estimate holds.*

$$(\lambda|\bar{g}|_{\infty,T}+|\bar{v}|_{\infty,T})(\lambda\|g^{\lambda,\delta}\|_{k+1,T}+\|v^{\lambda,\delta}\|_{k+1,T}) \leqslant C_2\delta^{k-k_0-1+\beta_0}. \tag{3.19}$$

Proof. Set

$$G_m^2(t)=\lambda^2\|\bar{g}\|_m^2+\|\bar{v}\|_m^2. \tag{3.20}$$

Assume that $0\leqslant m\leqslant k-1$. From (3.12) it readily follows that $G_m^2(t)\leqslant C_2G_m^2(0)$, $\forall t\in[0,T]$, i.e.

$$\lambda^2\|\bar{g}\|_{m,T}^2+\|\bar{v}\|_{m,T}^2\leqslant C_2(\lambda^2\|g_0^\delta-g_0\|_m^2+\|v_0^\delta-v_0\|_m^2).$$

By using this inequality with $m=k_0$ and with $m=k_0-1$, and also by taking into account (3.18), we prove that

$$\begin{aligned}\lambda^2|\bar{g}|_{\infty,T}^2+|\bar{v}|_{\infty,T}^2 &\leqslant C_2(\lambda^2\|g_0^\delta-g_0\|_{k_0-1}^2+\|v_0^\delta-v_0\|_{k_0-1}^2)^{\beta_0}\\ &\quad\times(\lambda^2\|g_0^\delta-g_0\|_{k_0}^2+\|v_0^\delta-v_0\|_{k_0}^2)^{1-\beta_0}.\end{aligned} \tag{3.21}$$

On the other hand, by using $(3.5)_2$ for $m=k$ and $s=k_0-1$ one gets

$$\lambda^2\|g_0^\delta-g_0\|_{k_0-1}^2+\|v_0^\delta-v_0\|_{k_0-1}^2\leqslant C_2\delta^{2(k-k_0+1)}, \tag{3.22}$$

since $\lambda^2\|g_0\|_k^2+\|v_0\|_k^2\leqslant C_2$. Once again from $(3.5)_2$

$$\lambda^2\|g_0^\delta-g_0\|_{k_0}^2+\|v_0^\delta-v_0\|_{k_0}^2\leqslant C_2\delta^{2(k-k_0)}. \tag{3.23}$$

From (3.21)–(3.23) it follows that

$$\lambda^2|\bar{g}|_{\infty,T}^2+|\bar{v}|_{\infty,T}^2\leqslant C_2\delta^{2(k-k_0+\beta_0)}. \tag{3.24}$$

Furthermore, as already remarked, the estimates (3.6) show that Lemma 2.1 applies in H^{k+1} to the solution (v^δ,g^δ) of problem (3.3), for each fixed δ. Hence

$$\lambda^2\|g^\delta\|_{k+1,T}^2+\|v^\delta\|_{k+1,T}^2\leqslant C_1(\lambda^2\|g_0^\delta\|_{k+1}^2+\|v_0^\delta\|_{k+1}^2).$$

By taking into account $(3.5)_1$ with $m=k+1$ and $s=k$ it follows that

$$\lambda^2\|g^\delta\|_{k+1,T}^2+\|v^\delta\|_{k+1,T}^2\leqslant C_2/\delta^2.$$

This inequality, together with (3.24), yields (3.19). □

Proof of Theorem A. Equations (3.12) and (3.19) show that

$$\frac{\mathrm{d}}{\mathrm{d}t}G_k^2(t)\leqslant C_2G_k^2(t)+C_2\delta^{k-k_0-1+\beta_0}G_k(t).$$

Hence $G_k(t)\leqslant C_2(G_k(0)+\delta^{\beta_0})$, i.e.

$$\lambda^2\|\bar{g}\|_{k,T}^2+\|\bar{v}\|_{k,T}^2\leqslant C_2(\lambda^2\|g_0^\delta-g_0\|_k^2+\|v_0^\delta-v_0\|_k^2+\delta^{2\beta_0}).$$

By taking into account the estimates (3.11) one gets

$$\lambda^2\|\bar{g}\|_{k,T}^2+\|\bar{v}\|_{k,T}^2\leqslant C_2(\lambda^2\|g_0\|_k^2+\|v_0-w_0\|_k^2+\hat{h}(\delta)),$$

where

$$\hat{h}(\delta) = \sum_{|\xi| \geqslant 1/\delta} (1 + |\xi|^2)^k |\hat{w}_0(\xi)|^2 + \delta^{2\beta_0}$$

satisfies $\lim \hat{h}(\delta) = 0$ as $\delta \to 0$.

Next, let $\varepsilon > 0$ be (arbitrarily) fixed. In correspondence with this ε let us fix a $\delta(\varepsilon)$ such that $\hat{h}(\delta) < \varepsilon$ if $\delta \leqslant \delta(\varepsilon)$. Everywhere in the sequel δ denotes this particular value $\delta(\varepsilon)$. One has, for each λ,

$$\lambda^2 \| g^{\lambda,\delta} - g^\lambda \|_{k,T}^2 + \| v^{\lambda,\delta} - v^\lambda \|_{k,T}^2 \leqslant C_2(\lambda^2 \| g_0^\lambda \|_k^2 + \| w_0 - v_0^\lambda \|_k^2) + \varepsilon.$$

Since $\lim_{\lambda\to\infty} (\lambda \| g_0^\lambda \|_k + \| w_0 - v_0^\lambda \|_k) = 0$ there is a $\lambda_0(\varepsilon)$ such that

$$\lambda^2 \| g^{\lambda,\delta} - g^\lambda \|_{k,T}^2 + \| v^{\lambda,\delta} - v^\lambda \|_{k,T}^2 \leqslant 2\varepsilon \quad \text{if } \lambda > \lambda_0(\varepsilon). \tag{3.25}$$

On the other hand, (3.9) shows that there is a λ_1, which depends only on ε (since $\delta = \delta(\varepsilon)$), such that

$$\| v^{\lambda,\delta} - w^\delta \|_{k,T}^2 \| g^{\lambda,\delta} \|_{k,T}^2 + \lambda^2 \| \nabla g^{\lambda,\delta} \|_{k-1,T}^2 < \varepsilon \tag{3.26}$$

if $\lambda > \lambda_1(\varepsilon)$. Next, since

$$\begin{aligned} &\| v^\lambda - v^\mu \|_{k,T} + \| g^\lambda - g^\mu \|_{k,T} + \| \nabla(\lambda g^\lambda - \mu g^\mu) \|_{k-1,T} \\ &\quad \leqslant \| v^\lambda - v^{\lambda,\delta} \|_{k,T} + \| g^\lambda - g^{\lambda,\delta} \|_{k,T} + \| \nabla(\lambda g^\lambda - \lambda g^{\lambda,\delta}) \|_{k-1,T} \\ &\qquad + \| v^{\lambda,\delta} - w^\delta \|_{k,T} + \| g^{\lambda,\delta} \|_{k,T} + \| \nabla \lambda g^{\lambda,\delta}) \|_{k-1,T} \\ &\qquad + \| v^{\mu,\delta} - w^\delta \|_{k,T} + \| g^{\mu,\delta} \|_{k,T} + \| \nabla \mu g^{\mu,\delta}) \|_{k-1,T} \\ &\qquad + \| v^{\mu,\delta} - v^\mu \|_{k,T} + \| g^{\mu,\delta} - g^\mu \|_{k,T} + \| \nabla(\mu g^{\mu,\delta} - \mu g^\mu) \|_{k-1,T}, \end{aligned}$$

one has

$$\| v^\lambda - v^\mu \|_{k,T}^2 + \| g^\lambda - g^\mu \|_{k,T}^2 + \| \nabla(\lambda g^\lambda - \mu g^\mu) \|_{k-1,T}^2 \leqslant c\varepsilon \tag{3.28}$$

if $\lambda, \mu > \max\{\lambda_0(\varepsilon), \lambda_1(\varepsilon)\}$. This shows that, as $\lambda \to \infty$, v^λ, g^λ and $\lambda \nabla g^\lambda$ are Cauchy 'sequences' with respect to the above strong norms. Hence, convergence takes place with respect to these norms. Finally, the convergence of g_t^λ to 0 in $C_T(H^{k-1})$ follows directly from equation $(1.2)_1$. □

Remark. Instead of using the concept of Cauchy sequence and the inequality (3.28) we may simply use the estimate

$$\begin{aligned} &\| v^\lambda - w \|_{k,T} + \| g^\lambda \|_{k,T} + \lambda^2 \| \nabla g^\lambda \|_{k-1,T} \\ &\quad \leqslant \| v^\lambda - v^{\lambda,\delta} \|_{k,T} + \| g^\lambda - g^{\lambda,\delta} \|_{k,T} + \| \nabla(\lambda g^\lambda - \lambda g^{\lambda,\delta}) \|_{k-1,T} \\ &\qquad + \| v^{\lambda,\delta} - w^\delta \|_{k,T} + \| g^{\lambda,\delta} \|_{k,T} + \| \nabla \lambda g^{\lambda,\delta} \|_{k-1,T} + \| w^\delta - w \|_{k,T}. \end{aligned}$$

This way is more natural and elegant. However it requires to show that

$$\lim_{\delta\to 0} \| w^\delta - w \|_{k,T}^2 = 0$$

in order to know that $\| w^\delta - w \|_{k,T}^2 < \varepsilon$ if $\delta < \delta_0(\varepsilon)$, for a suitable δ_0. This result can be proved by a simplification of the above method (since the parameter λ is not present; moreover the problems are incompressible). Recall that w and w^δ are the solutions of (1.3) and (3.10), respectively, and note that $\lim_{\delta\to 0} \| w_0^\delta - w_0 \|_k = 0$.

References

1. Beirão da Veiga, H., 'An L^p-theory for the n-dimensional, stationary, compressible Navier–Stokes equations, and the incompressible limit for compressible fluids. The equilibrium solution', *Comm. Math. Phys.*, **109**, 229–248 (1987).
2. Beirão da Veiga, H., 'On the singular limit for slightly compressible fluids', *Calculus of Variations and PDE*, **2**, 205–218 (1994).
3. Gagliardo, E., 'Ulteriori proprietá di alcune classi di funzioni in piú variabili', *Ricerche di Mat.*, **8**, 24–51 (1959).
4. Klainerman, S. and Majda, A., 'Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids', *Comm. Pure Appl. Math.*, **34**, 481–524 (1981).
5. Klainerman, S. and Majda, A., 'Compressible and incompressible fluids', *Comm. Pure Appl. Math.*, **35**, 629–653 (1982).
6. Nirenberg, L., 'On elliptic partial differential equations', *Ann. Sc. Norm. Sup. Pisa*, **13**, 115–162 (1959).