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Perturbation Theorems for Linear Hyperbolic Mixed Problems and Applications to the Compressible Euler Equations

H. BEIRÃO DA VEIGA
Centro Linceo Interdisciplinare
Accademia Nazionale dei Lincei

Abstract

The main result of this paper (which is completely new, apart from our previous and less general result proved in reference [9]) states that the nonlinear system of equations (1.11) (or, equivalently, (1.10)) that describes the motion of an inviscid, compressible (barotropic) fluid in a bounded domain Ω, gives rise to a strongly well-posed problem (in the Hadamard classical sense) in spaces $H^k(\Omega)$, $k \geqq 3$; see Theorem 1.4 below. Roughly speaking, if $(a_n, \phi_n) \to (a, \phi)$ in $H^k \times H^k$ and if $f_n \to f$ in $\mathscr{L}^2(0, T; H^k)$, then $(v_n, g_n) \to (v, g)$ in $\mathscr{C}(0, T; H^k \times H^k)$. The method followed here (see also [8]) also applies to the non-barotropic case $p = p(\rho, s)$ (see [10]) and to other nonlinear problems. These results are based upon an improvement of the structural-stability theorem for linear hyperbolic equations. See Theorem 1.2 below. Added in proof: The reader is referred to [29], Part I, for a concise explanation of some fundamental points in the method followed here. © 1993 John Wiley & Sons, Inc.

1. Introduction

The core of this paper concerns the improvement of the structural-stability theorem for linear hyperbolic equations and the proof of the continuous dependence on the data, in strong norm, of the solutions of a large class of initial-boundary value problems. The results are new even for initial-value problems. Here, we are interested in applying the above general method to the barotropic Euler equations of fluid dynamics. This leads to the (linear) structural-stability Theorem 1.2 and to the (nonlinear) strong continuous data dependence Theorem 1.4.

Main Notation

Ω is an open, bounded, connected subset of $\mathbb{R}^n$, $n \geqq 2$, locally situated on one side of its boundary Γ, a differentiable manifold of class C^{k+2}. In the sequel k denotes a fixed integer such that $k > n/2 + 1$. We denote by ν the unit outward normal to the boundary Γ and by ∂_ν differentiation in the ν direction. We set $Q_T = [0, T] \times \Omega$, $\Sigma_T = [0, T] \times \Gamma$.

We denote by H^ℓ, ℓ a non-negative integer, the space $H^\ell(\Omega)$ endowed with the canonical norm $\|\cdot\|_\ell$ defined by $\|u\|_\ell^2 = \sum \|\partial^\alpha u\|^2$, where the summation is extended over the multi-indices α such that $0 \leqq |\alpha| \leqq \ell$ and $\|\cdot\| = \|\cdot\|_0$ denotes the L^2-norm in Ω. Moreover

$$|||u|||_\ell^2 = \sum_{j=0}^{\ell} \left\|\partial_t^j u\right\|_{\ell-j}^2 , \qquad |||u|||'^2_\ell = \sum_{j=0}^{\ell-1} \left\|\partial_t^j u\right\|_{\ell-j}^2 .$$

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We also use, on Γ, fractional Sobolev spaces $H^{\ell-1/2}(\Gamma)$ denoted here by $\mathscr{H}^{\ell-1/2}$. The norm in this space is denoted by the symbol $\langle\langle\,\cdot\,\rangle\rangle_{\ell-1/2}$. We set

$$\langle\langle\langle u\rangle\rangle\rangle^2_{\ell-1/2}=\sum_{j=0}^{\ell-1}\langle\langle\partial_t^j u\rangle\rangle^2_{\ell-j-1/2}\,.$$

In the sequel we use the notation

$$C^j_T(X)=C^j([0,T];X)\,,\quad L^p_T(X)=L^p(0,T;X)\,,$$

and so on. We define

$$\mathscr{C}_T(H^\ell)=\bigcap_{j=0}^{\ell}C^j_T(H^{\ell-j})\,,\qquad \mathscr{C}'_T(H^\ell)=\bigcap_{j=0}^{\ell-1}C^j_T(H^{\ell-j})\,,$$

$$\mathscr{L}^p_T(H^\ell)=\bigcap_{j=0}^{\ell}W^{j,p}_T(H^{\ell-j})\,,\qquad \mathscr{L}'^p{}_T(H^\ell)=\bigcap_{j=0}^{\ell-1}W^{j,p}_T(H^{\ell-j})\,,$$

$$\mathscr{C}_T(\mathscr{H}^{\ell-1/2})=\bigcap_{j=0}^{\ell-1}C^j_T(\mathscr{H}^{\ell-j-1/2})\,,\qquad \mathscr{L}^p_T(\mathscr{H}^{\ell-1/2})=\bigcap_{j=0}^{\ell-1}W^{j,p}_T(\mathscr{H}^{\ell-j-1/2})\,.$$

The norms in these functional spaces are the following:

$$|||u|||^2_{\ell,T}=\sup_{0\le t\le T}|||u(t)|||^2_\ell\,,\qquad |||u|||'^2{}_{\ell,T}=\sup_{0\le t\le T}|||u(t)|||'^2{}_\ell\,,$$

$$[u]^2_{\ell,T}=\int_0^T|||u(t)|||^2_\ell\,dt\,,\qquad [u]'^2{}_{\ell,T}=\int_0^T|||u(t)|||'^2{}_\ell\,dt\,,$$

$$\langle\langle\langle u\rangle\rangle\rangle^2_{\ell-1/2,T}=\sup_{0\le t\le T}\langle\langle\langle u(t)\rangle\rangle\rangle^2_{\ell-1/2}\,,$$

$$\langle u\rangle^2_{\ell-1/2,T}=\int_0^T\langle\langle\langle u(t)\rangle\rangle\rangle^2_{\ell-1/2}\,dt\,,$$

where "sup" denotes the essential supremum and $p=2$.

The above notation will be used both for scalar and for vector fields. This convention applies to all notation used in the sequel. In particular, we write $v,g\in X$, even if v is a vector, and g a scalar.

Given an arbitrary function $f(t,x)$ we denote by $f(t)$, for each fixed t, the function $f(t,\,\cdot\,)$. We denote by $\mathscr{L}(X,Y)$ the B-space of bounded linear operators from the B-space X into the B-space Y.

Obvious notation will be used without any definition.

In the following, we often deal with positive "constants" that, in fact, depend (increasingly) on various characteristic quantities such as the norms of the coefficients of the differential operators used in the sequel. Hence, for convenience, we denote by $\lambda = \lambda(\,\cdot\,, \ldots, \,\cdot\,)$ generic real, non-negative functions which are increasing functions of each single (real, non-negative) variable. They will be called λ-functions. Since we are not particularly interested in their explicit form, we shall often denote distinct λ-functions by the same symbol λ.

Some classes of λ-functions, particularly important in the sequel, will be denoted by specific symbols such as, for instance, the P's, Q's, and R's defined in equations (1.4) and (1.8).

In the following, many equations will be considered in connection with the problem (1.1) and also the problem (1.1′); see below. In order to save space, we use the following convention.

Convention. Let $(m \cdot n)$ denote, as usual, the n-th equation in the m-th section. We denote by$(m \cdot n')$ the equation obtained by replacing everywhere in equation $(m \cdot n)$ the elements v, h, F, ϕ, ψ, and g by v', h', F', ϕ', ψ', and g', respectively.

The paper is divided into two parts. In Part I we treat the second-order linear, nonhomogeneous hyperbolic mixed problem

$$(1.1)\qquad \begin{cases} (\partial_t + v \cdot \nabla)^2 g - \nabla \cdot (h \nabla g) = F \quad \text{in } Q_T\,, \\ \qquad\qquad\qquad\qquad \partial_\nu g = G \quad \text{on } \Sigma_T\,, \quad (g, \partial_t g)(0) = (\phi, \psi)\,. \end{cases}$$

Here it is assumed that

$$(1.2)\qquad v, h \in \mathscr{L}'^{\infty}{}_T(H^k)\,,$$

and

$$(1.3)\qquad v \cdot \nu = 0 \quad \text{on } \Sigma_T\,, \qquad h \geqq m > 0 \quad \text{on } Q_T\,,$$

where m is a positive constant. In Part I, T is an arbitrarily large real positive number and the symbols P and Q denote generic λ-functions of the following types, respectively:

$$(1.4)\qquad \begin{aligned} P &= P\left(m^{-1}\,,\ |||v|||'_{k-1,T}\,, |||h|||'_{k-1,T}\right)\,, \\ Q &= Q\left(m^{-1}\,,\ |||v|||'_{k,T}\,, |||h|||'_{k,T}\right)\,. \end{aligned}$$

Before stating the results proved in Part I, we recall some definitions concerning compatibility conditions. Let ℓ be an integer, $1 \leqq \ell \leqq k$, assume that $g \in \mathscr{C}_T(H^\ell)$ is a solution of (1.1), and denote by $\{\partial\, j_t g(0)\}$ the expression, in terms of ϕ, ψ, F, v, and h, formally obtained by solving the equations (1.1) for $\partial_t^j g(0)$. Then, the following equations must be satisfied.

$$\partial_\nu \left\{ \partial_t^j g(0) \right\} = \partial_t^j G(0) \quad \text{on } \Gamma , \tag{1.5}$$

for $j = 0, 1, \dots, \ell - 2$. These are the compatibility conditions up to order $\ell - 2$, for problem (1.1). Under suitable assumptions, they are also sufficient in order to obtain a $\mathscr{C}_T(H^\ell)$ solution. Note that the compatibility conditions involve data and coefficients, but not eventual solutions.

One has the following auxiliary result, more or less well known.

THEOREM 1.1. *Let $1 \leqq \ell \leqq k$ and assume that*

$$(\phi, \psi) \in H^\ell \times H^{\ell-1} , \qquad F \in \mathscr{L}_T^2(H^{\ell-1}) , \qquad G \in \mathscr{L}_T^2(\mathscr{H}^{\ell-1/2}) , \tag{1.6}$$

and that the hypothesis (1.2), (1.3), *and* (1.5) *are satisfied. Then, if $\ell < k$, there is a solution $g \in \mathscr{C}_T(H^\ell)$ of problem* (1.1). *Moreover (for suitable P and Q, having the form* (1.4)) *one has*

$$\begin{aligned} |||g(t)|||_\ell^2 + [g]_{\ell,t}^2 &\leqq P\, e^{Qt} \left(\|\phi\|_\ell^2 + \|\psi\|_{\ell-1}^2 + |||F(0)|||_{\ell-2}^2 \right) \\ &\quad + Q\, e^{Qt} \left([F]_{\ell-1,t}^2 + \langle G \rangle_{\ell-1/2,t}^2 \right) , \end{aligned} \tag{1.7}$$

for each $t \in [0, T]$. If $\ell = k$, the solution g belongs to $\mathscr{C}_T'(H^k)$; moreover (1.7) *holds provided that we replace its left-hand side by $|||g(t)|||'^2_k + [g]'^2_{k,t}$. Or, alternatively, $u \in \mathscr{C}_T(H^k)$ and* (1.7) *holds if $v \in \mathscr{C}_T(H^{k-1})$ and if we allow P to depend on the full norm $|||v|||_{k-1,T}$.*

For regular coefficients, these types of estimates were proved by S. Miyatake (see [18], [19], [20]) for a large class of second-order equations. In order to apply these estimates to significant nonlinear problems, however, it is crucial to prove them in the presence of and in terms of nonregular coefficients v and h. In fact, the particular form of the "constants" P and Q (see (1.4)) and the use of primed norms (see remark in Section 5) are essential for proving the existence Theorem 1.3. For completeness we give (see Section 2) the proof of the estimate (1.7). For $n = k = 3$ see reference [5]. For general results see reference [27]. Readers mainly interested in the central contributions and the general aspects of our paper should skip Section 2.

The main result in Part I is a sharp structural-stability theorem that establishes the strong-continuous dependence of the solution g of problem (1.1) in terms of the coefficients v and h; see Theorem 1.2 below. Here, together with the problem (1.1), we shall consider the similar problem

$$(1.1') \qquad \begin{cases} (\partial_t + v' \cdot \nabla)^2 g' - \nabla \cdot (h' \nabla g') = F' & \text{in } Q_T \ , \\ \partial_\nu g' = G' & \text{on } \Sigma_T \ , \\ (g', \partial_t g')(0) = (\phi', \psi') \ , \end{cases}$$

where the couple (v', h') satisfy the hypothesis (1.2′), (1.3′) (recall the convention about notation stated at the end of the previous section).

In the sequel, the symbol R_ℓ denotes a generic λ-function of the form

$$(1.8) \qquad \begin{aligned} R_\ell = R_\ell \Big(& m^{-1}, |||v, h, v', h'|||'_{k,T} \ , \quad \|\phi, \phi'\|_\ell \ , \ \|\psi, \psi'\|_{\ell-1} \ , \\ & |||F(0), F'(0)|||_{\ell-2} \ , \quad [F, F']_{\ell-1,T} \ , \ \langle G, G' \rangle_{\ell-1/2,T} \Big) \ , \end{aligned}$$

where, for brevity, $\|\phi, \phi'\|_\ell$ means $\|\phi\|_\ell$ and $\|\phi'\|_\ell$, and similarly for the other functions and norms. Note that constants of type P and Q are particular constants of type R. Sometimes we shall write R instead of R_ℓ, especially when $\ell = k$.

One has the following result:

THEOREM 1.2. *Let v, h and v', h' satisfy, respectively, the assumptions (1.2), (1.3), and (1.2′), (1.3′), assume that $v, h, v', h' \in \mathscr{L}^2_T(H^k)$, and let ϕ, ψ, F, G and ϕ', ψ', F', G' satisfy, respectively, (1.6) and (1.6′), for $\ell = k$. Assume that the compatibility conditions (1.5) and (1.5′) are satisfied for the system (1.1) and (1.1′) up to the order $k-2$. Let $g, g' \in \mathscr{C}_T(H^k)$ be the solutions of these last systems. Then, given $\varepsilon > 0$, there is a real positive $\Lambda(\varepsilon)$, that depends only on ε, on T, and on the particular functions v, h, ϕ, ψ, F, and G, such that (for suitable $R = R_k$) the following estimate holds for each $t \in [0, T]$.*

$$(1.9) \qquad \begin{aligned} & |||g(t) - g'(t)|||_k^2 + [g - g']_{k,t}^2 \\ & \leqq R e^{Rt} \Big\{ \varepsilon + \|\phi - \phi'\|_k^2 + \|\psi - \psi'\|_{k-1}^2 + |||F(0) - F'(0)|||_{k-2}^2 \\ & + |||h(0) - h'(0)|||_{k-1}^2 + |||v(0) - v'(0)|||_{k-1}^2 + \langle\langle G(0) - G'(0) \rangle\rangle_{k-3/2}^2 \\ & + [F - F']_{k-1,t}^2 + [v - v']_{k,t}^2 + [h - h']_{k,t}^2 + \langle G - G' \rangle_{k-1/2,t}^2 + [g - g']_{k,t}^2 \\ & + \Lambda(\varepsilon) \Big([h - h']_{k-1,t}^2 + [v - v']_{k-1,t}^2 \Big) \Big\} \ . \end{aligned}$$

In particular, if one has a family of problems (1.1′) such that $(\phi', \psi', F', G') \to (\phi, \psi, F, G)$ in $H^k \times H^{k-1} \times \mathscr{L}^2_T(H^{k-1}) \times \mathscr{L}^2_T(\mathscr{H}^{k-1/2})$ and $(v', h') \to (v, h)$ in $\mathscr{L}^2_T(H^k) \times \mathscr{L}^2_T(H^k)$, then $g' \to g$ in $\mathscr{C}_T(H^k)$.

A similar result holds for each ℓ such that $1 \leqq \ell \leqq k-1$. See Proposition 3.2. Theorem 1.2 will be proved in Section 3, by following a method introduced in reference [8], where we proved similar results for initial-boundary value problems for first-order hyperbolic systems.

Theorem 1.2 would allow us to prove strong well-posedness results for nonlinear counterparts of problem (1.1). We are, however, interested in studying a more complex problem, namely the motion of compressible inviscid fluids (see below, and Part II). Theorem 1.2 is just the first step in solving this last problem.

Part I ends (see Section 4) with a perturbation result for a transport equation, proved by following a simplification of our method.

In Part II we study the nonlinear system

$$(1.10) \quad \begin{cases} \rho\,(\partial_t v + (v\cdot\nabla)v) + \nabla p(\rho) = f & \text{in } Q_T\,, \\ \partial_t \rho + \nabla\cdot(\rho v) = 0 & \text{in } Q_T\,, \\ v\cdot\nu = 0 & \text{on } \Sigma_T\,, \quad (v,\rho)(0) = (a,\rho_0)\,, \end{cases}$$

where $n = 3$, since this is the significant physical case. Hence $k \geqq 3$. The above system describes the barotropic motion of a compressible inviscid fluid (see [24], [26]). We assume that $p \in C^{k+1}(\mathbb{R}^+;\mathbb{R})$ and that $p'(s) > 0$ for each $s \in \mathbb{R}^+$. Moreover, $\inf \rho_0(x) > 0$ in Ω. By setting

$$g = \log\rho \quad \text{and} \quad h(s) = p'(e^s) \quad \text{for each } s \in \mathbb{R}\,,$$

the above system is transformed into the system

$$(1.11) \quad \begin{cases} \partial_t v + (v\cdot\nabla)v + h(g)\nabla g = f & \text{in } Q_T\,, \\ (\partial_t + v\cdot\nabla)g + \nabla\cdot v = 0 & \text{in } Q_T\,, \\ v\cdot\nu = 0 & \text{on } \Sigma_T\,, \quad (v,g)(0) = (a,\phi)\,. \end{cases}$$

Problems (1.10) and (1.11) are equivalent. We shall establish and prove results for the solution (v,g) of (1.11), and leave to the reader the transposition of these results in terms of (v,ρ); see also [5], [7].

The four by four boundary matrix associated with the first-order hyperbolic system (1.11) has rank two on the boundary. This gives rise to serious obstacles that can be overcome by a suitable device (introduced in references [3] and [5]) for proving an existence theorem for the system (1.10). The first step of this method consists in showing that the system (1.11) is equivalent[1]

[1] Assume that Ω is simply connected. If not, see Remark 1, at the end of this section.

to the system

$$(1.12)\quad \begin{cases} (\partial_t + v\cdot\nabla)(\nabla\times v) - ((\nabla\times v)\cdot\nabla)v + (\nabla\cdot v)(\nabla\times v) = \nabla\times f\,, \\ (\partial_t + v\cdot\nabla)^2 g - \nabla\cdot(h(g)\nabla g) = \sum_{i,j}(\partial_i v_j)(\partial_j v_i) - \nabla\cdot f\,, \\ -\nabla\cdot v = (\partial_t + v\cdot\nabla)g\,; \\ v\cdot\nu = 0\,, \qquad \partial_\nu g = G\,, \quad \text{on } \Sigma_T\,; \\ (g,\partial_t g)(0) = (\phi,\psi)\,, \qquad v(0) = a\,, \end{cases}$$

where, by definition, $G = h(g)^{-1}(\sum_{i,j}(\partial_i\nu_j)v_iv_j + f\cdot n)$ and $\psi = -(a\cdot\nabla\phi + \nabla\cdot a)$ (we extend ν to a neighborhood of Γ, as a C^{k+1} vector field). The equivalence of the systems (1.11) and (1.12) is easy to prove, provided that the functions involved in the calculations are sufficiently regular (as occurs in the sequel). For, set $V \equiv \partial_t v + (v\cdot\nabla)v - h\nabla g - f$. Equation $(1.11)_1$, i.e., $V = 0$, is equivalent (see, for instance, [2] or [13]) to the equations

$$(1.13)\qquad \nabla\times V = 0 \quad\text{and}\quad \nabla\cdot V = 0 \quad \text{in } \Omega\,, \qquad V\cdot\nu = 0 \quad \text{on } \Gamma\,,$$

for each $t \in [0,T]$. By replacing V, in (1.13), by its explicit expression, and by doing straightforward manipulations, we prove the above equivalence; see, for instance, [9], Appendix.

Let us state the existence theorem for the system (1.11).

THEOREM 1.3. *Let $k \geqq 3$, and let Ω and $h(\,\cdot\,)$ be as above. Hence $h \in C^k(\mathbb{R};\mathbb{R}^+)$. Assume that (a,ϕ,f) belong to $H^k\times H^k\times \mathscr{L}^2_T(H^k)$ and satisfy the compatibility conditions up to order $k-1$ for the system* (1.11). *Then, there is a positive constant T such that there exists a (unique) solution $(v,g)\in\mathscr{C}_T(H^k\times H^k)$ of the problem* (1.11) *in Q_T. Moreover*

$$(1.14)\qquad |||v|||'^2_{k,T} + |||g|||'^2_{k,T} \leqq \lambda_1\,, \qquad |||v|||^2_{k,T} + |||g|||^2_{k,T} \leqq \lambda_4\,.$$

The result is valid for any T satisfying

$$(1.15)\qquad \lambda_2 T \leqq 1\,, \qquad \lambda_3[f]'_{k,T} \leqq 1\,.$$

where λ_2 and λ_3 are suitable λ-functions. Here, λ_1, λ_2, and λ_3 have the form $\lambda(\|a\|_k, \|\phi\|_k, |||f(0)|||'_{k-1})$, moreover $\lambda_4 = \lambda(\|a\|_k, \|\phi\|_k, |||f(0)|||_{k-1}, [f]_{k,T})$.

The existence theorem for the mixed problem (for the Cauchy problem see [17] and the references therein) was proved by Ebin (see [11]), in the space $\mathscr{L}^\infty_T(H^3)$, by assuming that the initial velocity is subsonic and the initial density is close to constant. The existence of the solution for arbitrarily large initial data (in the above space) was first proved by us (see [3] and [5]) and (in an independent paper) by Agemi (see [1]). The existence of the solution (in spaces $\mathscr{L}^\infty(0,T;H^3)$) for the non-barotropic case, was proved by Schochet (see [22] and also [23]) by using a different approach which has, however, some ideas in common with our method. Recently, the existence theorem has been extended by Secchi (see [25]) to the case of moving boundaries.

Theorem 1.3 is not substantially new. Our proof is the extension (to arbitrarily large values of k) of that of Theorem 1.1 in reference [5]. Distinct ways for proving this result can be found in references [1] or [22]. With respect to [5], the proof given below follows a simpler and more direct approach even if the main point is still the reduction to the system (1.12). Some points must be treated now more deeply due to the arbitrariness of k. For completeness and for the reader's convenience we insert (see Section 5) the proof of Theorem 1.3 by our method. Readers who are mainly interested in the fundamental aspects of the theory developed here should skip Section 5.

In Part II we prove our second main result, the well-posedness in Hadamard's classical sense of system (1.11) and, consequently, of system (1.10). See Theorem 1.4 below (for $k=3$ and $\Omega=\mathbb{R}^3_+$ this result was proved in reference [9]). In this theorem we assume the existence of a solution (v,g) on some arbitrarily large interval $[0,T_0]$, not necessarily that found in the existence Theorem 1.3. We prove the following result:
the existence Theorem 1.3. We prove the following result:

THEOREM 1.4. *Let (a,ϕ,f) be as in Theorem 1.3, and let $f\in\mathscr{L}^2_{T_0}(H^k)$ for some $T_0>0$. Assume that there is in Q_{T_0} a solution $(v,g)\in\mathscr{C}_{T_0}(H^k\times H^k)$ of problem (1.1). Let (a',ϕ',f') denote a family of data belonging to $H^k\times H^k\times\mathscr{L}^2_{T_0}(H^k)$ and satisfying the compatibility conditions up to order $k-1$ for the system*

$$(1.11')\quad \begin{cases} \partial_t v' + (v'\cdot\nabla)v' + h(g')\nabla g' = f' , \\ (\partial_t + v'\cdot\nabla)g' + \nabla\cdot v' = 0 \qquad \text{in } Q_T \\ v'\cdot\nu = 0 \qquad \text{on } \Sigma_T , \quad (v',g')(0) = (a',\phi') . \end{cases}$$

Then, there exists a neighborhood of (a,ϕ,f) such that for each (a',ϕ',f') in this neighborhood the solution (v',g) of (1.11′) exists and belongs to $\mathscr{C}_{T_0}(H^k\times H^k)$. Moreover, if

$$(1.16)\qquad \lim\,(a',\phi',f') = (a,\phi,f) \quad \text{in } H^k\times H^k\times\mathscr{L}^2_{T_0}(H^k)$$

then

$$(1.17)\qquad \lim\,(v',g') = (v,g) \quad \text{in } \mathscr{C}_{T_0}(H^k\times H^k) .$$

In particular, if $[0,\tau'[$ is the maximal interval of existence of the solution (v',g'), and if $[0,\tau[$ is that of (v,g) one has $\liminf\tau'\geqq\tau$.

As far as we know, there are no similar results in the literature for the system (1.11), or for similar systems. Note, however, that the $\mathscr{L}_T^\infty(H^k)$ uniform estimate for the solutions u' of problem (1.11′) together with the uniqueness of the solution u and a compactness argument yield the convergence of u' to u in $\mathscr{L}_T^\infty(H^k)$ weak- $*$, and hence in $\mathscr{C}_T(H^{k-\varepsilon})$, for each $\varepsilon > 0$. Alternatively, we may use the above uniform estimate, the convergence with respect to a weaker norm (say, $\mathscr{C}_T(H^1)$), and interpolation to obtain the latter result. These arguments are well known in the literature.

It is worth noting that the C^0 dependence on the initial data, proved here, is the most one can expect in Eulerian coordinates. A separate problem is that of the dependence in Lagrangian coordinates. For this case one has at least a C^1 dependence on the initial data, as asserted in the remark on page 483, in [12]; the proof of this result exists in manuscript form.

Our method also allows the proof of the strong continuous dependence of solutions of nonlinear equations with respect to variations of the functional dependence of the coefficients on the solutions; see reference [8]. For the specific problem studied in this paper this means that the thesis of Theorem 1.4 still holds if we replace in the family of systems (1.11) the fixed equation of state $h(\cdot)$ by variables $h'(\cdot)$, provided that $h' \to h$ in $C^k(\mathbb{R})$, uniformly on compact intervals. Our method can also be applied to prove convergence in the *strong norm* of the solutions of the compressible Euler equations to the solution of the incompressible Euler equations as the Mach number goes to zero; see [28]. This remark, together with a sketch of the proof, have been suggested to us by the referee.

The referee also suggests a distinct approach, by remaining in the framework of first-order systems. We intend to profit from his suggestions in the near future.

Remark 1. If Ω is not simply connected, there are N linearly independent solutions $u^\ell(x)$, $\ell = 1, \dots, N$, of system (1.13), where N is the smallest number of cuts needed to make Ω simply connected (see, for instance, [13]). By adding to the system (1.13) the N orthogonality conditions $(V, u^{(\ell)}) = 0$, we get a system that turns out to be equivalent to the single equation $V = 0$. If Ω is not simply connected these orthogonality conditions should be inserted in the system (1.12); see [5] and also [4]. These conditions do not cause mathematical difficulties, nevertheless they complicate the exposition and the notation. For that reason we assume, in Part II, that Ω is simply connected.

PART I

2. Proof of Theorem 1.1

Here, and in the next sections, we shall make continuous use of the following inequalities. Let $r > n/2$, $0 \leqq \ell \leqq r$; $\alpha, \beta \in [\ell, r]$, and $\alpha + \beta = \ell + r$.

Then

$$(2.1)\qquad \begin{cases} \|fg\|_\ell \leqq c\|f\|_\alpha \|g\|_\beta\,, \quad |||fg|||_\ell \leqq c|||f|||_\alpha\,|||g|||_\beta\,, \\ |||fg|||_\ell \leqq c\left(\sum_{p=0}^{\ell} \|\partial_t^p f\|_{\alpha-p}\right)\left(\sum_{p=0}^{\ell} \|\partial_t^p g\|_{\beta-p}\right)\,. \end{cases}$$

For a proof, we refer the reader to [8], Appendix A. In particular, $|||fg|||_\ell \leqq c|||f|||'_\alpha|||g|||_\beta$ if $\alpha > \ell$, and $|||fg|||_\ell \leqq c|||f|||'_\alpha\,|||g|||'_\beta$ if $\alpha, \beta > \ell$.

Suppose $f \in H^\alpha$, $G \in \mathscr{H}^{\beta-1/2}$. Let $\mathscr{R}$ be a linear operator mapping functions defined on Γ into functions defined on Ω, such that $\gamma_0\mathscr{R} =$ identity. Here, γ_0 denotes the trace operator. In the sequel we drop this symbol. If Γ is of class C^{r+1}, it is well known that $\mathscr{R}$ can be constructed in such a way that $\mathscr{R} \in \mathscr{L}(\mathscr{H}^{\beta-1/2}; H^\beta)$ for each $\beta = 1, \dots, r$. Hence, by using $(2.1)_1$, one has

$$(2.1\text{-bis})\qquad \begin{cases} \langle\langle fG\rangle\rangle_{\ell-1/2} \leqq c\|f\|_\alpha\,\langle\langle G\rangle\rangle_{\beta-1/2}\,, \\ \langle\langle\langle fG\rangle\rangle\rangle_{\ell-1/2} \leqq c|||f|||'_\alpha\,\langle\langle\langle G\rangle\rangle\rangle_{\beta-1/2}\,, \end{cases}$$

since $\langle\langle fG\rangle\rangle_{\ell-1/2} \leqq c\|f\mathscr{R}G\|_\ell \leqq c\|f\|_\alpha\|\mathscr{R}G\|_\beta$ and since $\partial_t\mathscr{R} = \mathscr{R}\,\partial_t$.

In the sequel we use the following notation. Here, γ is a positive real number.

$$\|g\|_{\ell,\gamma} = e^{-\gamma t}\|g\|_\ell\,, \qquad |||g|||_{\ell,\gamma} = e^{-\gamma t}|||g|||_\ell\,,$$
$$|||g|||'_{\ell,\gamma} = e^{-\gamma t}|||g|||'_\ell\,, \quad \langle\langle G\rangle\rangle_{\ell-1/2,\gamma} = e^{-\gamma t}\langle\langle G\rangle\rangle_{\ell-1/2}\,,$$

and so on. Moreover,

$$|||g|||_{\ell,\gamma,T} = \sup_{0\leqq t\leqq T}|||g(t)|||_{\ell,\gamma}\,, \qquad [g]^2_{\ell,\gamma,T} = \int_0^T |||g(t)|||^2_{\ell,\gamma}\,dt\,,$$
$$\langle G\rangle^2_{\ell-1/2,\gamma,T} = \int_0^T \langle\langle\langle G(t)\rangle\rangle\rangle^2_{\ell-1/2,\gamma}\,dt\,,$$

and so on. We leave to the reader the following exercise. If H is a real Hilbert space endowed with a Hilbertian norm $|\cdot|$, and if $f \in W^{1,1}_T(H)$, then

$$(2.2)\qquad e^{-2\gamma t}|f(t)|^2 + \gamma\int_0^t e^{-2\gamma s}|f(s)|^2\,ds \leqq |f(0)|^2 + \gamma^{-1}\int_0^t e^{-2\gamma s}|f'(s)|^2\,ds$$

for each $t \in [0, T]$. here, f' denotes df/dt.

Concerning the proof of Theorem 1.1, we shall show just the main point; i.e., the a priori estimate (1.7). Our main concern will be to show that there are λ-functions P and Q (see (1.4)) such that, for each ℓ ($1 \leqq \ell \leqq k-1$) and each $t \in [0, T]$, one has (L is defined below)

$$
\begin{aligned}
|||g(t)|||^2_{\ell,\gamma} + \gamma[g]^2_{\ell,\gamma,t} \leqq P &\left(\|g(0)\|^2_\ell + \|\partial_t g(0)\|^2_{\ell-1} + |||Lg(0)|||^2_{\ell-2} \right) \\
&+ Q\gamma^{-1} \left([Lg]^2_{\ell-1,\gamma,t} \right. \\
&\left. + [g]'^2_{\ell,\gamma,t} + \langle \partial_\nu g \rangle^2_{\ell-1/2,\gamma,t} \right) ,
\end{aligned} \tag{2.3}
$$

provided that $\gamma \geqq \overline{Q}$, for a suitable λ-function $\overline{Q}$. If $\ell = k$, (2.3) holds either by replacing, in the left-hand side of (2.3), $|||\cdot|||_{k,\gamma}$ and $[\cdot]_{k,\gamma,t}$ by $|||\cdot|||'_{k,\gamma}$ and $[\cdot]'_{k,\gamma,t}$ respectively, or by assuming that $P = P(m^{-1}, |||h|||'_{k-1,T}, |||v|||_{k-1,T})$.

By convention, norms with a negative Sobolev index should be dropped from the equations. For instance, if $\ell = 1$, in equation (2.3) we drop the term concerning $Lg(0)$.

Proof of inequality (2.3): For $\ell = 1$, it is well known that (see [18], [19], [20], and [5], Theorem 6.1)

$$
\begin{aligned}
|||g(t)|||^2_{1,\gamma} + \gamma[g]^2_{1,\gamma,t} \leqq P &\left(\|g(0)\|^2_1 + \|\partial_t g(0)\|^2_0 \right) \\
&+ Q\gamma^{-1} \left([Lg]^2_{0,\gamma,t} + \langle \partial_\nu g \rangle^2_{1/2,\gamma,t} \right) ,
\end{aligned} \tag{2.4}
$$

provided that $\gamma \geqq \overline{Q}$. In reference [5] this estimate is proved for the half-space $\mathbb{R}^3_+$. The same proof, however, applies to the half-space $\mathbb{R}^n_+$. Moreover, standard methods of localization and deformation allow us to extend the estimate to open bounded regular subsets Ω. We also refer the reader to the estimate (a), Theorem 2.2 in reference [27].

Concerning the main assumptions in this theorem, we note that our coefficients v and h belong to $C^{1,\mu}(\overline{Q}_T)$, for some real positive $\mu(k, n)$, since $\mathscr{L}'^{\infty}{}_T(H^k)$ is embedded in that space. Our operator L, however, does not satisfy the assumption (A.4) in [27]. Nevertheless the proof can be adapted to our case due to the particular form of the operator L in which, roughly speaking, ∂_t is replaced by $B = \partial_t + v \cdot \nabla$. In order to obtain a suitable Green's formula (which takes the place, here, of that in equation (5.2) of Lemma 5.1, [27]) we expand $(Lu, Bu)_\gamma$ as in [5] equation (5.4), instead of expanding $(Lu, \partial_t u)_\gamma$.

If $\ell = 1$, equation (2.3) follows from (2.4). Now, we assume that (2.3) holds for some $\ell \in [1, k-1]$ and we prove it for $\ell + 1$.

For convenience, we define

$$Bg \equiv (\partial_t + v \cdot \nabla)\, g\ , \qquad Lg \equiv B^2 g - \nabla \cdot (h\, \nabla g)\ .$$

By applying the operator B to both sides of equation $(1.1)_1$, one gets

$$L\delta = B\, L\, g + Dg \tag{2.5}$$

where, by definiton, $\delta \triangleq Bg$ and D is the commutator $[\nabla \cdot (h\nabla \cdot), B]$, i.e.,

$$\begin{aligned} Dg \equiv \sum_{i,j} \{ & h\,(\partial_i v_j)\,(\partial_i \partial_j g) \\ & - v_j\,(\partial_i \partial_j h)\,\partial_i g + \partial_i \left[h\,(\partial_i v_j)\,(\partial_j g) \right] \} - \nabla \cdot [(\partial_t h)\, \nabla g]\ . \end{aligned} \tag{2.6}$$

By using inequalities (2.1) it readily follows that (for each fixed t)

$$|||Dg|||_{\ell-1} \leqq Q |||g|||'_{\ell+1}\ , \quad |||Dg|||_{\ell-2} \leqq P |||g|||'_{\ell+1}\ ,$$

Moreover,

$$|||BLg|||_{\ell-1} \leqq Q |||Lg|||_{\ell}\ , \qquad |||BLg|||_{\ell-2} \leqq P |||Lg|||_{\ell-1}\ .$$

Hence, from (2.5), we get

$$[\, L\delta\,]_{\ell-1,\gamma,t} \leqq Q \left([Lg]_{\ell,\gamma,t} + [g]'_{\ell+1,\gamma,t} \right)\ , \tag{2.7}$$

and also

$$|||L\delta|||_{\ell-2} \leqq P \left(|||Lg|||_{\ell-1} + |||g|||'_{\ell+1} \right)\ . \tag{2.8}$$

Since $\partial_t^2 g = Lg + \nabla \cdot (h\, \nabla g) - \partial_t (v \cdot \nabla g) - v \cdot \nabla\, Bg$, it readily follows that (for each t)

$$|||g|||'_{\ell+1} \leqq P\, (\|g\|_{\ell+1} + \|\partial_t g\|_{\ell} + |||Lg|||_{\ell-1})\ . \tag{2.9}$$

In particular, (2.8) plus (2.9) yield

$$|||L\delta(0)|||_{\ell-2} \leqq P\, (\|g(0)\|_{\ell+1} + \|\partial_t g(0)\|_{\ell} + |||Lg(0)|||_{\ell-1})\ . \tag{2.10}$$

Finally, by using the definition of δ, one shows that

$$\|\delta(0)\|_{\ell} + \|\partial_t \delta(0)\|_{\ell-1} \leqq P\, (\|g(0)\|_{\ell+1} + \|\partial_t g(0)\|_{\ell} + |||Lg(0)|||_{\ell-1})\ . \tag{2.11}$$

Now, we want to prove that

$$\langle \partial_\nu \delta \rangle^2_{\ell-1/2,\gamma,t} \leqq Q \left(\langle \partial_\nu g \rangle^2_{\ell+1/2,\gamma,t} + [g]'^2{}_{\ell+1,\gamma,t} \right)\ . \tag{2.12}$$

By applying the operator ∂_ν to $\delta = \partial_t g + v \cdot \nabla g$, one shows that

$$\partial_\nu \delta = (\partial_t + v \cdot \nabla)(\partial_\nu g) + [(\nu \cdot \nabla)v - (v \cdot \nabla)\nu] \cdot \nabla g \; . \tag{2.13}$$

Note that the operators $\partial_t^j[(v \cdot \nabla) \cdot]$ are tangential. Clearly,

$$\langle\langle\langle \partial_t (\partial_\nu g) \rangle\rangle\rangle_{\ell-1/2} \leqq \langle\langle\langle \partial_\nu g \rangle\rangle\rangle_{\ell+1/2} \; . \tag{2.14}$$

Moreover, (2.1-bis) shows that,

$$\langle\langle\langle (v \cdot \nabla)(\partial_\nu g) \rangle\rangle\rangle_{\ell-1/2} \leqq c |||v|||_{k-1} \langle\langle\langle \partial_\nu g \rangle\rangle\rangle_{\ell+1/2} \; . \tag{2.15}$$

Finally, since Γ is of class C^{k+2}, there is a C^{k+2} function Φ defined on a neighborhood ω of Γ such that $\nu = \nabla\phi$ on Γ. We extend ν to ω as a C^{k+1} vector field, by setting $\nu = \nabla\phi$. Set, for convenience, $(\nu \cdot \nabla)v - (v \cdot \nabla)\nu = w$, in ω. One has

$$\begin{aligned}\langle\langle\langle (w \cdot \nabla g) \rangle\rangle\rangle_{\ell-1/2} &\leqq c\, |||w \cdot \nabla g|||_\ell \leqq c\, |||w|||_{k-1}\, |||g|||'_{\ell+1} \\ &\leqq c\, |||v|||'_k\, |||g|||'_{\ell+1} \; .\end{aligned}$$

From this last equation, together with (2.14) and (2.15) and by taking into account (2.13), one proves (2.12).

By applying the estimate (2.3) to the solution δ of (2.5) and by using (2.7), (2.10), (2.11), (2.12), and (2.15), one gets

$$\begin{aligned}|||\delta(t)|||^2_{\ell,\gamma} + \gamma[\delta]^2_{\ell,\gamma,t} &\leqq P\left(\|g(0)\|^2_{\ell+1} + \|\partial_t g(0)\|^2_\ell + |||Lg(0)|||^2_{\ell-1}\right) \\ &\quad + Q\gamma^{-1}\left([Lg]^2_{\ell,\gamma,t}\right. \\ &\quad \left. + [g]'^2{}_{\ell+1,\gamma,t} + \langle \partial_\nu g \rangle^2_{\ell+1/2,\gamma,t}\right) \; .\end{aligned} \tag{2.16}$$

Now, using the elliptic equation $-\nabla \cdot (h \nabla g) = B\delta + Lg$ one shows that, for each fixed t,

$$\|g\|_{\ell+1} \leqq P\left(\|B\delta + Lg\|_{\ell-1} + \langle\langle \partial_\nu g \rangle\rangle_{\ell-1/2}\right) \; , \tag{2.17}$$

since $h \geqq m$ and $k - 1 > n/2$; see Appendix B. On the other hand, $\|B\delta\|_{\ell-1} \leqq P|||\delta|||_\ell$. Hence

$$\|g(t)\|^2_{\ell+1} \leqq P\left(|||\delta(t)|||^2_\ell + \|Lg(t)\|^2_{\ell-1} + \langle\langle \partial_\nu g(t) \rangle\rangle^2_{\ell-1/2}\right) \; . \tag{2.18}$$

By using the equation $\partial_t g = -v \cdot \nabla g + \delta$ in order to express derivatives $\partial_t^j g$ in terms of lower order derivatives (see Lemma 2.1 below), it readily follows that

$$|||g(t)|||^2_{\ell+1} \leqq P\left(|||\delta(t)|||^2_\ell + \|Lg(t)\|^2_{\ell-1} + \langle\langle \partial_\nu g(t) \rangle\rangle^2_{\ell-1}\right) \; , \tag{2.19}$$

if $\ell < k-1$. If $\ell = k-1$, this estimate holds if we replace $|||g(t)|||_{\ell+1}$ by $|||g(t)|||'_{\ell+1}$ or, alternatively, if we allow P to depend on the full norm $|||v|||_{k-1,T}$. Similar modifications must be done in equations (2.20) and (2.21) below, if $\ell = k-1$. From (2.19) one gets, in particular,

$$[g]^2_{\ell+1,\gamma,t} \leqq P\left([\delta]^2_{\ell,\gamma,t} + [Lg]^2_{\ell-1,\gamma,t} + \langle\partial_\nu g\rangle^2_{\ell-1,\gamma,t}\right) . \tag{2.20}$$

Equations (2.16), (2.19), and (2.20) show that

$$\begin{aligned}
&|||g(t)|||^2_{\ell+1,\gamma} + \gamma[g]^2_{\ell+1,\gamma,t} \\
&\leqq P\left(\|g(0)\|^2_{\ell+1} + \|\partial_t g(0)\|^2_\ell + |||Lg(0)|||^2_{\ell-1}\right) \\
&\quad + Q\gamma^{-1}\left([Lg]^2_{\ell,\gamma,t} + [g]'^2{}_{\ell+1,\gamma,t} + \langle\partial_\nu g\rangle^2_{\ell+1/2,\gamma,t}\right) \\
&\quad + P\Big(\|Lg(t)\|^2_{\ell-1,\gamma} + \gamma[Lg]^2_{\ell-1,\gamma,t} \\
&\quad + \langle\langle\partial_\nu g(t)\rangle\rangle^2_{\ell-1,\gamma} + \gamma\langle\partial_\nu g\rangle^2_{\ell-1,\gamma,t}\Big) .
\end{aligned} \tag{2.21}$$

Now, the inequality (2.2) allows us to drop the last four terms in the right-hand side of (2.21). This shows that (2.3) holds for the value $\ell + 1$.

Finally, we obtain (1.7) from (2.3) by fixing $\gamma = 1 + Q + \overline{Q}$ (or by setting $\gamma = \max\{1, \overline{Q}\}$ and by using Gronwall's lemma).

The following lemma, when $i = \ell + 1$ and also when $i = \ell = k-1$, can be used in order to get (2.19) from (2.18).

LEMMA 2.1. *Let* $1 \leqq \ell \leqq k-1$; $1 \leqq i \leqq \ell+1$. *Let* $\partial_t^2 g = Lg + \nabla\cdot(h\nabla g) - \partial_t(v\cdot\nabla g) - v\cdot\nabla Bg$. *Then*

$$|||g|||^{[i]}_{\ell+1} \leqq \lambda\left(|||v|||^{[i-1]}_{k-1}, |||h|||^{[i-2]}_{k-1}\right)\left(\|g\|_{\ell+1} + \|\partial_t g\|_\ell + |||Lg|||^{[i-2]}_{\ell-1}\right) .$$

Here $|||g|||^{[i]}_\ell \equiv \sum_{p=0}^i \|\partial_t^p g\|_{\ell-i}$. The proof is easily done, by induction on i, using the inequalities (A.6) in [8].

3. Proof of Theorem 1.2

The following result is of capital importance in the sequel.

PROPOSITION 3.1. *Let* $1 \leqq \ell \leqq k-1$ *and assume that* (1.2), (1.3), (1.5), *and* (1.6) *are satisfied. Let* $\varepsilon \in [0,1]$ *be given. There are elements* $(\phi_\varepsilon, \psi_\varepsilon) \in$

$H^{\ell+1} \times H^{\ell}$, $F_\varepsilon \in \mathscr{L}^2_T(H^\ell)$, $G_\varepsilon \in \mathscr{L}^2_T(\mathscr{H}^{\ell+1/2})$ *that satisfy the compatibility conditions* (1.5) *up to the order* $\ell - 1$, *and such that*

$$(3.1) \qquad \begin{cases} \|\phi - \phi_\varepsilon\|^2_\ell \leqq \varepsilon\,, \quad \|\psi - \psi_\varepsilon\|^2_{\ell-1} \leqq \varepsilon\,, \\ \|(F - F_\varepsilon)(0)\|_{\ell-2} \leqq \varepsilon\,, \quad [F - F_\varepsilon]^2_{\ell-1,T} \leqq \varepsilon\,, \\ \langle G - G_\varepsilon \rangle^2_{\ell-1/2,T} \leqq \varepsilon\,. \end{cases}$$

Since the proof is quite involved we shall give it in Appendix A.

PROPOSITION 3.2. *Suppose* $1 \leqq \ell \leqq k-1$. *Assume that* v, h, ϕ, ψ, F, G *satisfy the assumptions* (1.2), (1.3), (1.5), *and* (1.6) *and that* $v', h', \phi', \psi', F', G'$ *satisfy the corresponding assumptions* (1.2′), (1.3′), (1.5′), *and* (1.6′). *Let* g *and* g' *be the solutions of problems* (1.1) *and* (1.1′) *respectively* (*whose existence is guaranteed by Theorem* 1.1). *Given* $\varepsilon > 0$, *there is a real positive* $\Lambda(\varepsilon)$, *that depends only on* ε, *on* ℓ, *on* T, *and on the particular functions* v, h, ϕ, ψ, F, G *such that, for suitable constants* $R = R_\ell$, *one has*

$$(3.2) \qquad \begin{aligned} &|||(g - g')(t)|||^2_\ell + [g - g']^2_{\ell,t} \\ &\leqq R\,e^{Rt}\Big\{\varepsilon + \|\phi - \phi'\|^2_\ell + \|\psi - \psi'\|^2_{\ell-1} \\ &\quad + |||(F - F')(0)|||^2_{\ell-2} + |||(h - h')(0)|||'^2_{k-1} \\ &\quad + |||(v - v')(0)|||'^2_{k-1} + [F - F']^2_{\ell-1,t} \\ &\quad + \langle G - G' \rangle^2_{\ell-1/2,t} + \Lambda(\varepsilon)\left([h - h']^2_{k-1,t} + [v - v']^2_{k-1,t}\right)\Big\}\,. \end{aligned}$$

Proof: Let ϕ_ε, ψ_ε, F_ε, G_ε be as in Proposition 3.1 (note that these functions depend only on ε and on v, h, ϕ, ψ, F, and G) and consider the problem

$$(3.3) \qquad \begin{cases} Lg_\varepsilon = F_\varepsilon \quad \text{in } Q_T\,, \\ \partial_\nu g_\varepsilon = G_\varepsilon \quad \text{on } \Sigma_T\,, \quad (g_\varepsilon, \partial_t g_\varepsilon)(0) = (\phi_\varepsilon, \psi_\varepsilon)\,. \end{cases}$$

From (1.7) it follows that

$$(3.4) \qquad |||g_\varepsilon(t)|||^2_\ell + [g_\varepsilon]^2_{\ell,t} \leqq R\,e^{Qt}$$

and that

$$(3.5) \qquad |||g_\varepsilon(t)|||^2_{\ell+1} + [g_\varepsilon]^2_{\ell+1,t} \leqq Q\,e^{Qt}\Lambda(\varepsilon)$$

for suitable R and Q's, where

$$\Lambda(\varepsilon) = \|\phi_\varepsilon\|^2_{\ell+1} + \|\psi_\varepsilon\|^2_\ell + |||(F_\varepsilon(0))|||^2_{\ell-1} + [F_\varepsilon]^2_{\ell,t} + \langle G_\varepsilon\rangle^2_{\ell+1/2,T} .$$

Note the following. In correspondence to each set of specific elements ε, ϕ, ψ, F, G, v, h we *fix* one particular set of functions ϕ_ε, ψ_ε, F_ε, G_ε, satisfying (3.1). This determines uniquely the solution g_ε of (3.3). Next, recall the definitions of the operators B and L given in Section 2, namely $Bg \equiv (\partial_t + v\cdot\nabla)g$ and $Lg \equiv B^2 g - \nabla\cdot(h\nabla g)$, and define $B'g' \equiv (\partial_t + v'\cdot\nabla)g'$ and $L'g' \equiv B'^2 g' - \nabla\cdot(h'\nabla g')$. Take the difference of the respective sides of equations (1.1′) and (3.3). One gets

$$(3.6)\qquad \begin{cases} L(g'-g_\varepsilon) = F' - F_\varepsilon + \partial_t\left[(v-v')\cdot\nabla g_\varepsilon\right] + (v-v')\cdot\nabla B' g_\varepsilon \\ \qquad\qquad + v\cdot\nabla\left[(v-v')\cdot\nabla g_\varepsilon\right] + \nabla\cdot\left[(h'-h)\nabla g_\varepsilon\right] , \\ \partial_\nu(g'-g_\varepsilon) = G' - G_\varepsilon \quad \text{on } \Sigma_T , \\ (g'-g_\varepsilon)(0) = \phi' - \phi_\varepsilon , \quad \partial_t(g'-g_\varepsilon)(0) = \psi' - \psi'_\varepsilon . \end{cases}$$

Next, applying the estimate (1.7) to the solution $g' - g_\varepsilon$ of problem (3.6), using (2.1) in order to bound the norms of the products in the right-hand side of $(3.6)_1$ and taking into account (3.1), (3.4), and (3.5), one proves that

$$(3.7)\qquad |||(g'-g_\varepsilon)(t)|||^2_\ell + [g'-g_\varepsilon]^2_{\ell,t} \leqq \text{right-hand side of (3.2).}$$

On doing the above manipulations, the $[\ \ ]_{\ell-1,t}$ norms of the last four terms in the right-hand side of (3.6) are estimated as follows (we exemplify just for the first one). Inequality (2.1) shows that

$$|||\partial_t\left[(v-v')\cdot\nabla g_\varepsilon\right]|||_{\ell-1} \leqq |||\partial_t(v-v')|||_{k-2}|||\nabla g_\varepsilon|||_\ell + |||v-v'|||_{k-1}|||\partial_t\nabla g_\varepsilon|||_{\ell-1} ,$$

for each t. Since the right-hand side of this inequality is bounded by

$$c|||v-v'|||_{k-1}\,|||g_\varepsilon|||_{\ell+1} ,$$

one gets

$$\left[\partial_t\left((v-v')\cdot\nabla g_\varepsilon\right)\right]^2_{\ell-1,t} \leqq c|||g_\varepsilon|||^2_{\ell+1,t}[v-v']^2_{k-1,t} .$$

Note that this argument fails for $\ell = k$. See Remark (ii) below.

On the other hand, by taking the difference of the respective sides of equations (1.1) and (3.3), one gets $L(g-g_\varepsilon) = F - F_\varepsilon$ in Q_T, $\partial_\nu(g-g_\varepsilon) = G - G_\varepsilon$ on Σ_T, $(g-g_\varepsilon)(0) = \phi - \phi_\varepsilon$, $\partial_t(g-g_\varepsilon)(0) = \psi - \psi_\varepsilon$. By applying (1.7) to the solution $g - g_\varepsilon$ of these last equations it readily follows that $|||(g-g_\varepsilon)(t)|||^2_\ell + [g-g_\varepsilon]^2_{\ell,t} \leqq \varepsilon\, Q\, e^{Qt}$. This estimate, together with (3.7), proves (3.2).

Remarks. (i) A crucial point in proving Proposition 3.2 is the use of the extra smoothness of the solution g_ε of (3.3), obtained via Proposition 3.1. In fact, if in equation (3.6) we replace g_ε by the solution g of (1.1), the (modified) right-hand side of (3.6) does not belong (in general) to $H^{\ell-1}$. Hence, we cannot get, directly, an H^ℓ energy estimate for the solution $g' - g$.

(ii) Note that the proof of Proposition 3.2 fails if $\ell = k$. In fact, in this case, the solution $g_\varepsilon(t)$ of (3.3) does not belong to H^{k+1} since the coefficients $h(t)$ and $v(t)$ of L do not belong to that space. A natural attempt to overcome this obstacle would be to replace, in (3.3), the coefficients $h(t), v(t) \in H^k$ by coefficients $h_\varepsilon(t), v_\varepsilon(t) \in H^{k+1}$ which are "near" $h(t)$, $v(t)$ in the H^k norm. After straightforward calculations, however, we would find in the right-hand side of (3.2) the term

$$|||g_\varepsilon|||_{k+1,t}^2 \left([h_\varepsilon - h]_{k,t}^2 + [v_\varepsilon - v]_{k,t}^2\right) .$$

For Cauchy problems (and, possibly, for mixed problems if k is sufficiently small) it seems to us that one can construct particular ε-approximations that are adequate to control the above term. In the general case, however, this way is nonviable or, at least, is much harder than application of Proposition 3.2 (with $\ell = k - 1$) to the first-order derivatives of the solution itself, as done in the sequel (or, more generally, to suitable differential expressions depending on the particular problem).

Proof of Theorem 1.2: By applying the operator B to both sides of equation (1.1) and by setting, as above, $\delta \equiv Bg$, one gets (cf. (2.5), (2.13))

$$\text{(3.8)} \qquad \begin{cases} L\delta = \overline{F} & \text{in } Q_T , \\ \partial_\nu \delta = \overline{G} & \text{on } \Sigma_T ; \quad (\delta, \partial_t \delta)(0) = (\bar{\phi}, \bar{\psi}) , \end{cases}$$

where $\overline{F} = BF + Dg$, D is as in (2.6) and $\overline{G}$ is the right-hand side of (2.13), $\bar{\phi} = \psi + v(0) \cdot \nabla \phi$, $\bar{\psi} = F(0) + \nabla \cdot (h \nabla \phi) - v \cdot \nabla(\psi + v \cdot \nabla \phi)$. The compatibility conditions up to order $k-2$ for the system (1.1) yield those up to order $k-3$ for the system (3.8). Alternatively, the compatibility conditions for this last system follow from the fact that δ is a solution, belonging to $\mathscr{C}_T(H^{k-1})$.

By arguing in a similar way for the system (1.1′) we show that

$$\text{(3.8)} \qquad \begin{cases} L\delta = \overline{F} & \text{in } Q_T , \\ \partial_\nu \delta = \overline{G} & \text{on } \Sigma_T ; \quad (\delta, \partial_t \delta)(0) = (\bar{\phi}', \bar{\psi}') , \end{cases}$$

where $\overline{F}'$, $\overline{G}'$, $\bar{\phi}'$, and $\bar{\psi}'$ are defined by replacing the functions v, h, g, F, ϕ, ψ by v', h', g', F', ϕ', ψ' in the definitions of F, G, ϕ, ψ. Now, we apply

Proposition 3.2 to the difference $\delta - \delta'$ of the solutions of the systems (3.8) and (3.8′). Equation (3.2) yields

$$
\begin{aligned}
& |||(\delta-\delta')(t)|||_{k-1}^2 + [\delta-\delta']_{k-1,t}^2 \\
& \leqq R e^{Rt} \Big\{ \varepsilon + \|\bar{\phi} - \bar{\phi}'\|_{k-1}^2 + \|\bar{\psi} - \bar{\psi}'\|_{k-2}^2 \\
(3.9) \qquad & + |||(F-\overline{F})(0)|||_{k-3}^2 + |||(h-h')(0)|||'^2{}_{k-1} + |||(v-v')(0)|||_{k-1}^2 \\
& + \left[\overline{F} - \overline{F}'\right]_{k-2,t}^2 \quad \langle \overline{G} - \overline{G}' \rangle_{k-3/2,t}^2 \\
& + \Lambda(\varepsilon) \left([h-h']_{k-1,t}^2 + [v-v']_{k-1,t}^2 \right) \Big\} \, .
\end{aligned}
$$

where $R = R_k$. Note that, in equation (3.9), the quantity $\Lambda(\varepsilon)$ depends, in principle, on ε, v, h and on the functions $\overline{F}$, $\overline{G}$, $\bar{\phi}$, $\bar{\psi}$. Taking into account the definitions of these last functions, however, one concludes that $\Lambda(\varepsilon)$ depends only on ε, v, h, F, G, ϕ, ψ. Similarly, in equation (3.9) the constants R are λ-functions that, in principle, are of type R_{k-1} (see Definition 1.8) with respect to the variables $\bar{\phi}$, $\bar{\phi}'$, $\bar{\psi}$, $\bar{\psi}'$, $\overline{F}$, $\overline{F}'$, $\overline{G}$, $\overline{G}'$. The definitions of these last functions, however, show that the above R's are, in fact, λ-functions of type R_k with respect to the variables ϕ, ϕ', ψ, ψ', F, F', G, G'.

By taking into account the definitions of $\bar{\phi}$, $\bar{\phi}'$, $\bar{\psi}$, $\bar{\psi}'$, $\overline{F}$, $\overline{F}'$, $\overline{G}$, $\overline{G}'$, and by using the inequalities (2.1), it readily follows that

$$
(3.10) \qquad |||(\delta-\delta')(t)|||_{k-1}^2 + [\delta-\delta']_{k-1,t}^2 \leqq \text{right-hand side of } (1.9) \, .
$$

On the other hand

$$
-\nabla \cdot [h'\nabla(g'-g)] = \nabla \cdot [(h-h')\nabla g] + B(\delta-\delta') + F' - F
$$

or, equivalently,

$$
\begin{aligned}
-\Delta(g-g') = h'^{-1} \{ \nabla h' \cdot \nabla(g'-g) + \nabla \cdot [(h'-h) \cdot \nabla g] \\
+ (\partial_t + v \cdot \nabla)(\delta - \delta') + F - F' \} \, .
\end{aligned}
$$

Since $\|h'^{-1}\|_{k-1} \leqq \lambda(m^{-1}, \|h'\|_{k-1})$, it readily follows that, for each fixed t,

$$
\begin{aligned}
(3.11) \qquad & \|g-g'\|_k^2 \leqq \lambda \Big(\|g-g'\|_{k-1}^2 + \|\delta-\delta'\|_{k-1}^2 + \|\partial_t(\delta-\delta')\|_{k-2}^2 \\
& + \|F-F'\|_{k-2}^2 \Big) + c \langle\langle G - G' \rangle\rangle_{k-3/2}^2 \, ,
\end{aligned}
$$

where $\lambda = \lambda(m^{-1}, \|h'\|_k, \|v\|_k, \|g\|_k)$ is of type R_k. Since $\partial_t(g-g') = \delta - \delta' + v \cdot \nabla(g'-g) + (v'-v) \cdot \nabla g'$, it follows that

$$\begin{aligned} &|||(g-g')(t)|||_k^2 \\ &\leqq R\Big(\|(g-g')(t)\|_{k-1}^2 + |||(\delta-\delta')(t)|||_{k-1}^2 \\ &\quad + \|(F-F')(t)\|_{k-2}^2 + |||(v-v')(t)|||_{k-1}^2 \\ &\quad + \langle\langle (G-G')(t)\rangle\rangle_{k-3/2}^2\Big) . \end{aligned} \tag{3.12}$$

Equations (3.10) and (3.12) yield (1.9). Note that we have used Gronwall's lemma to eliminate a term $R\,e^{Rt}[g-g']_{k,t}^2$ in equation (3.12). This term could be dropped also by working with the γ-norms. For the reader's convenience we choose, here and in the sequel, to use Gronwall's lemma instead of γ-norms.

Finally, the last assertion in Theorem 1.1 follows from (1.9).

4. Remarks on Transport Equations

Consider the transport equation

$$\begin{cases} B\zeta - (\zeta \cdot \nabla)v + (\nabla \cdot v)\zeta = H & \text{in } Q_T , \\ \zeta(0) = \alpha , & \end{cases} \tag{4.1}$$

where $B = \partial_t + v \cdot \nabla$, v satisfies (1.2), (1.3), $\alpha \in H^{k-1}$, and $H \in \mathscr{L}_T^2(H^{k-1})$. Here, ζ, α, and H can be N-vectors, $N \geqq 1$. We assume, however, that $N = n$. Regarding the existence of the solution, we just establish the fundamental a priori estimate. The construction of the solution can be done by well-known methods. See, for instance, [6].

By applying the operator $\partial_t^j \partial^\alpha$ to both sides of equation (4.1), by taking the inner product in L^2 with $\partial_t^j \partial^\alpha \zeta$, by doing standard calculations, and by adding the respective sides of the inequalities obtained for all (j, α) such that $0 \leqq j + |\alpha| \leqq k-1$, we get

$$\frac{d}{dt}|||\zeta(t)|||_{k-1}^2 \leqq c\left(1 + |||v(t)|||_k^2\right)|||\zeta(t)|||_{k-1}^2 + |||H(t)|||_{k-1}^2 .$$

Hence

$$|||\zeta(t)|||_{k-1}^2 \leqq e^{c(1+|||v|||_{k,t}^2)t}\left(|||\zeta(0)|||_{k-1}^2 + [H]_{k-1,t}^2\right) .$$

Finally, by using equation $(4.1)_1$ in order to estimate $|||\zeta(0)|||_{k-1}$, one gets

$$|||\zeta(t)|||_{k-1}^2 \leqq \lambda\left(|||v(0)|||'_{k-1}\right) e^{c(1+|||v|||'^2_{k,t})t} \left(\|\alpha\|_{k-1}^2 + |||H(0)|||_{k-2}^2 + [H]_{k-1,t}^2\right) . \tag{4.2}$$

As claimed above, a solution $\zeta \in \mathscr{C}_T(H^{k-1})$ satisfying (4.2) can be constructed.

Now, we establish the main result of this section. Consider the two systems

$$B\zeta = H \quad \text{in } Q_T , \qquad \zeta(0) = \alpha , \tag{4.3}$$

and

$$B'\zeta' = H' \quad \text{in } Q_T , \qquad \zeta'(0) = \alpha' . \tag{4.3'}$$

One has the following result.

THEOREM 4.1. *Let v and v' satisfy* (1.2) *and* (1.3); *assume that $\alpha, \alpha' \in H^{k-1}$; and that $H, H' \in \mathscr{L}_T^2(H^{k-1})$. Let ζ and ζ' be the solutions in $\mathscr{C}_T(H^{k-1})$ of* (4.3) *and* (4.3′), *respectively. Then to each $\varepsilon > 0$ there corresponds a positive real $C(\varepsilon)$, that depends only on ε and on the particular functions α, H such that*

$$\begin{aligned} |||(\zeta - \zeta')(t)|||_{k-1}^2 \leqq R\, e^{Rt} \Big\{ &\varepsilon + \|\alpha - \alpha'\|_{k-1}^2 + |||(H - H')(0)|||_{k-2}^2 \\ &+ [H - H']_{k-1,t}^2 + C(\varepsilon) \left(|||(v - v')(0)|||'^2_{k-2} \right. \\ &\left. + [v - v']'^2_{k-1,t} \right) \Big\} , \end{aligned} \tag{4.4}$$

for suitable λ-functions $R = R(|||v; v'|||'_{k,t})$.

Proof: Consider the equation

$$B\zeta_\varepsilon = H_\varepsilon \quad \text{in } Q_T , \qquad \zeta_\varepsilon(0) = \alpha_\varepsilon , \tag{4.5}$$

where $\alpha_\varepsilon \in H^k$, $H_\varepsilon \in \mathscr{L}_T^2(H^k)$, and

$$\|\alpha - \alpha_\varepsilon\|_{k-1}^2 \leqq \varepsilon , \quad |||(H - H_\varepsilon)(0)|||_{k-2}^2 \leqq \varepsilon , \quad [H - H_\varepsilon]_{k-1,t}^2 \leqq \varepsilon .$$

By arguing as done above in proving (4.2), one easily shows that

$$|||\zeta_\varepsilon|||'_{k,t} \leqq R\, e^{Rt}\, C(\varepsilon) ,$$

where R and $C(\varepsilon)$ are as above. Here we use all the indexes (j,α) such that $0 \leqq j+|\alpha| \leqq k$, $j \neq k$. By taking the difference of the respective sides of (4.3′) and (4.5), and by applying the solution $\zeta'-\zeta_\varepsilon$ of the equation $B'(\zeta'-\zeta_\varepsilon) = H'-H_\varepsilon+[(v-v')\cdot\nabla]\zeta_\varepsilon$ the estimate (4.2), one easily proves that $|||(\zeta'-\zeta_\varepsilon)(t)|||_{k-1}^2$ is bounded by the right-hand side of (4.4). On the other hand, by taking the difference between (4.3) and (4.4), and by applying the estimate (4.2) to $\zeta-\zeta_\varepsilon$, one proves that $||||(\zeta-\zeta_\varepsilon)(t)|||_{k-1}^2$ is bounded by the left-hand side of (4.4). Hence, (4.4) holds.

PART II

5. Proof of Theorem 1.3

The proof will be done by solving the system (1.12), which is equivalent to the system (1.11). As noted in the Introduction, we assume for convenience that Ω is simply connected, the general case being easily treated by introducing standard devices (see [4] and [5]). Denote by $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ the connected components of Γ, such that the Γ_j's, for $j=1,\dots,m$, are inside of Γ_0 and outside of one another. Recall that a regular vector field $\xi(x)$ is a curl in Ω if and only if $\nabla\cdot\xi=0$ in Ω and the surface integrals of ξ over each Γ_i, $i=1,\dots,m$, vanish.

Outline of Proof: Consider the following systems, where ϑ, ξ, and q are defined in Q_T and where $\vartheta(0)=\nabla\cdot a$.

$$\nabla\cdot v=\vartheta\,,\quad \nabla\times v=\xi\quad \text{in } Q_T\,;\quad v\cdot\nu=0\quad \text{on } \Sigma_T\,. \tag{5.1}$$

$$\begin{cases} (\partial_t+v\cdot\nabla)^2 g-\nabla\cdot(h(q)\nabla g) \\ \qquad\qquad = \sum_{i,j}(\partial_i v_j)(\partial_j v_i)-\nabla\cdot f & \text{in } Q_T\,, \\ \partial_\nu g = h(q)^{-1}\left(\sum_{i,j}(\partial_i\nu_j)v_iv_j+f\cdot n\right) & \text{on } \Sigma_T\,, \\ g(0)=\phi\,, \\ \partial_t g(0) = -(a\cdot\nabla\phi+\nabla\cdot a)\,. \end{cases} \tag{5.2}$$

$$\begin{cases} \partial_t\zeta+(v\cdot\nabla)\zeta-(\zeta\cdot\nabla)v+(\nabla\cdot v)\zeta=\nabla\times f & \text{in } Q_T\,, \\ \zeta(0)=\nabla\times a\,, \end{cases} \tag{5.3}$$

$$\delta = -(\partial_t+v\cdot\nabla)\,g\,.\ ^2 \tag{5.4}$$

Note that we reobtain the system (1.12) by replacing (ϑ, ξ, q) by (δ, ζ, g) in the above equations. Hence, our aim will be to prove the existence of a solution of the above equations such that $(\delta, \zeta, g) = (\vartheta, \xi, q)$. We shall argue as follows. Given (ϑ, ξ, q) in a suitable set $\mathbb{K}$, we get $v = v[\vartheta, \xi]$ by solving the problem (5.1). Then, we get $g = g[v, q]$ and $\zeta = \zeta[v]$ by solving respectively (5.2) and (5.3). Finally $\delta = \delta[v, g]$ is defined by equation (5.4). This procedure defines a map S on $\mathbb{K}$ by setting $S(\theta, \xi, q) = (\delta, \zeta, g)$. A fixed point for the map S is the desired solution.

The Set $\mathbb{K}$

This set will depend on two positive parameters T and A to be fixed later on. Consider the following constraints.

$$|||\vartheta|||_{k-1,T} \leqq A\,, \quad |||\xi|||_{k-1,T} \leqq A\,, \quad |||q|||'_{k,T} \leqq A\,. \tag{5.5}$$

$$\begin{cases} \partial_t^j \vartheta(0) = \nabla \cdot \left\{ \partial_t^j v(0) \right\}\,, & \partial_t^j \xi(0) = \nabla \times \left\{ \partial_t^j v(0) \right\}\,, \\ \partial_t^j q(0) = \{ \partial_t^j g(0) \}\,, & \text{for } j = 0, \ldots, k-2\,.^{3} \end{cases} \tag{5.6}$$

$$\begin{cases} \displaystyle\int_\Omega \vartheta(t,x)\,dx = 0\,; \\ \nabla \cdot \xi(t,x) = 0\,; \qquad \displaystyle\int_{\Gamma_i} \xi(t,x)\,d\Gamma = 0\,, \quad i = 1, \ldots, m\,, \end{cases} \tag{5.7}$$

for each $t \in [0, T]$. It is worth noting that the functions on the right-hand side of (5.6) are the formal values obtained (in terms of a and ϕ) from equations (1.11). We set

$$\mathbb{K}(A,T) \equiv \Big\{ (\vartheta, \xi, q) \in \mathscr{L}_T^\infty(H^{k-1}) \times \mathscr{L}_T^\infty(H^{k-1}) \times \mathscr{L}'^{\infty}{}_T(H^k) : \text{(5.5), (5.6), (5.7) hold.} \Big\}\,. \tag{5.8}$$

The convex set $\mathbb{K}$ is closed in the Banach space $\chi \equiv C_T(H^0 \times H^0 \times H^1)$. This follows easily by taking into account that strongly closed balls in Banach spaces $L_T^\infty(X)$ are compact with respect to the weak-$*$ topology, if X is a Hilbert space.

[2] Note the change of sign with respect to the δ in Part I.

[3] We could also consider the value $j = k - 1$, but it is not necessary here.

Remarks on the Printed Norms

The "curl-divergence method" requires the use of the printed norms in order to prove the existence of the above fixed point in $\mathbb{K}$. In fact, the solutions v of (5.1) belong to $\mathscr{L}'^{\infty}{}_T(H^k)$ but not to $\mathscr{L}^{\infty}_T(H^k)$. But the v corresponding to the fixed point does belong to $\mathscr{L}^{\infty}_T(H^k)$.

Remarks on P and Q

In the system (5.2), the function $h(t,x)$ is given by a composite function, namely $h(q(t,x))$, where $h(\,\cdot\,)$ was defined in Section 1. On applying below the estimate (1.7) to the solution g of (5.2) it will be useful to express m and the norms of h in terms of norms of q as follows. Since $h(\,\cdot\,)$ is positive and increasing and since the norm of q in $L^\infty(Q_T)$ is bounded by a constant c times $\|q\|_{k-1,T}$, the condition $(1.3)_2$ holds by setting $m = \min h(s)$, for $|s| \leqq c\|q\|_{k-1,T}$. In particular m^{-1} is a λ-function that depends only on $\|q\|_{k-1,T}$. On the other hand, since $h(\,\cdot\,)$ is of class C^k, it readily follows that the norms in the space $L^\infty(Q_T)$ of the functions $h^{(\ell)}(q(t,x))$, $\ell = 0, 1, \ldots, k$, are bounded by $\lambda(\|q\|_{k-1,T})$, for suitable λ-functions. Here $h^{(\ell)}(\,\cdot\,)$ denotes the derivative of order ℓ of $h(\,\cdot\,)$. In the end, one finds that there are suitable λ-functions such that $|||h(q)|||'_{k-1,T} \leqq \lambda(|||q|||'_{k-1,T})$, $|||(h(q)|||'_{k,T} \leqq \lambda(|||q|||'_{k,T})$, and $|||h(q)^{-1}|||'_{k,T} \leqq \lambda(|||q|||'_{k,T})$. Hence, the λ-functions P and Q, defined by equation (1.4), are now of the following types:

$$P = P(|||v,q|||'_{k-1,T}) \,, \qquad Q = Q(|||v,q|||'_{k,T}) \,. \tag{5.9}$$

In the sequel, the symbol d denotes a generic λ-function of the form

$$d = d\left(\|a\|_k \,,\ \|\phi\|_k \,,\ |||f(0)|||'_{k-1}\right) \,.$$

K Is Not Empty

Here, we show that if $A \geqq d$, for a suitable d, then $\mathbb{K}(A,T)$ is not empty. Since $\{\partial_t^j v(0)\} \in H^{k-j}$ and $\{\partial_t^j(0)\} \cdot \nu = 0$ on Γ $(j = 0, \ldots, k-2)$, there is a $w \in \mathscr{C}_T(H^k)$ such that $\partial_t^j w(0) = \{\partial_t^j v(0)\}$ for $j = 0, \ldots, k-2$; $\partial_t^{k-1} w(0) = \partial_t^k w(0) = 0$; and $w \cdot \nu = 0$ on Σ_T. This correspondence can be chosen to be linear and continuous. Since

$$\sum_{j=0}^{k-2} \left\| \left\{ \partial_t^j v(0) \right\} \right\|_{k-j} \leqq d \,,$$

for some d, it follows that $|||w|||_{k,T} \leqq d$ for some d. We set $\vartheta = \nabla \cdot w$, $\xi = \nabla \times w$. Similarly, we construct $q \in \mathscr{C}_T(H^k)$ such that $\partial_t^j q(0) = \{\partial_t^j g(0)\}$, for $j = 0, \ldots, k-2$, $\partial_t^{k-1} q(0) = \partial_t^k q(0) = 0$, and $|||q|||_{k,T} \leqq d$. The element (ϑ, ξ, q) belongs to $\mathbb{K}(A,T)$.

The Map S

Let $(\vartheta, \xi, q) \in \mathbb{K}$ and define $v = v[\vartheta, \xi]$ as the solution of the system (5.1). Clearly,

$$|||v|||'_{k,T} \leqq c\,A\,, \qquad |||v|||'_{k-1,T} \leqq d + c\,A\,T^{1/2}\,. \tag{5.10}$$

Note that $|||v(0)|||'_{k-1} \leqq d$ since $\partial_t^j v(0) = \{\partial_t^j v(0)\}$, for $j = 0, \dots, k-2$, as follows from (5.1), $(5.6)_1$, and $(5.6)_2$.[4] Next, we consider the system (5.2), where $v = v[\vartheta, \xi]$. Since $\partial_t^j v(0) = \{\partial_t^j v(0)\}$ and $\partial_t^j q(0) = \{\partial_t^j g(0)\}$, for $j = 0, \dots, k-2$, it follows that the compatibility conditions for the system (5.2) are verified up to order $k-2$; we postpone the proof of this main point to the end of this section. Hence, a solution $g = g[v, q]$ exists, belongs to $\mathscr{C}_T(H^k)$, and satisfies the estimate (1.7) for $\ell = k$. This last estimate shows that

$$|||g|||'^2{}_{k,T} \leqq d\,P\,e^{Qt} + Q\,e^{Qt}\left([f]'^2{}_{k,T} + A^4 T + A^2 T\right)\,. \tag{5.11}$$

Consider now the solution $\zeta = \zeta[v]$ of problem (5.3). By applying the estimate (4.2) we prove that

$$|||\zeta|||^2_{k-1,T} \leqq d\,e^{Qt}\left(1 + [f]'^2{}_{k,T}\right)\,, \tag{5.12}$$

for suitable λ-functions d and Q. Next, we prove that

$$\nabla \cdot \zeta(t,x) = 0 \quad \text{and} \quad \int_{\Gamma_i} \zeta(t,x)\,d\Gamma = 0\,, \qquad i = 1, \dots, m\,, \tag{5.13}$$

for each $t \in [0, T]$. In fact, from the identity

$$(v \cdot \nabla)\,\zeta - (\zeta \cdot \nabla)\,v = (\nabla \cdot \zeta)\,v - (\nabla \cdot v)\zeta - \nabla \times (v \times \zeta)$$

it follows that

$$\partial_t \zeta + (\nabla \cdot \zeta)v = \nabla \times (v \times \zeta + f)\,. \tag{5.14}$$

By applying the operator divergence to both sides of this last equation, by using the identity $\nabla \cdot [(\nabla \cdot \zeta)v] = v \cdot \nabla(\nabla \cdot \zeta) + (\nabla \cdot v)(\nabla \cdot \zeta)$, and by taking into account that $\zeta(0) = \nabla \times a$, one gets

$$\begin{cases} \partial_t(\nabla \cdot \zeta) + v \cdot \nabla(\nabla \cdot \zeta) + \vartheta(\nabla \cdot \zeta) = 0 & \text{in } Q_T\,, \\ (\nabla \cdot \zeta)(0) = 0\,. \end{cases}$$

[4]Note that the functions $\partial_t^j v(0)$ concern the solution $v = v[\vartheta, \xi]$ of problem (5.1). In contrast, the functions $\{\partial_t^j v(0)\}$ are obtained from (1.11), by formal manipulations.

This transport equation has a unique solution, $\nabla \cdot \zeta \equiv 0$. On the other hand, (5.14) together with Stokes' theorem shows that

$$\frac{d}{dt}\int_{\Gamma_i} \zeta \cdot \nu \, d\Gamma = 0 .$$

Hence

$$\int_{\Gamma_i} \zeta \cdot \nu \, d\Gamma = \int_{\Gamma_i} (\nabla \times a) \cdot \nu \, d\Gamma = 0 , \tag{5.15}$$

for each $i = 1, \dots, m$, and each $t \in [0, T]$.

Finally, let $\delta = \delta[v, g]$ be the solution of (5.4). Clearly,

$$|||\delta|||'_{k-1,T} \leqq c\left(1 + |||v|||'_{k-1,T}\right) |||g|||'_{k,T} \, . \tag{5.16}$$

Actually, this estimate holds if all occurrences of $||| \cdot |||'$ are replaced by $||| \cdot |||$. Since we want an estimate independent of $\|\partial_t^{k-1} v\|_{0,T}$, however, we shall argue as follows. Equations $(5.2)_1$ and (5.4) give an explicit expression for $\partial_t \delta$ which, by differentiation, gives an expression for $\partial_t^{k-1}\delta$. To this last expression we apply inequalities (2.1) to prove that

$$\begin{aligned} \|\partial_t^{k-1}\delta\|_0 \leqq c|||v|||'_{k-1}\, |||\delta|||'_{k-1} + c|||h(q)|||'_{k-1}\, |||g|||'_k \\ + |||f|||'_{k-1} + c\|\partial_t^{k-2}(\partial v)^2\|_0 \end{aligned} \tag{5.17}$$

for each t. For convenience, we denote by $(\partial v)^2$ the first term on the right-hand side of $(5.2)_1$. Fix a real $\mu \in]0, 1[$ such that $k > \mu + 1 + n/2$. Interpolation theorems, [16], Theorem 9.6, show that $\| \cdot \|_{k-1-\mu-j} \leqq \| \cdot \|^{\mu}_{k-2-j} \cdot \| \cdot \|^{1-\mu}_{k-1-j}$, for each $j = 0, \dots, k-2$. On the other hand, since $k - 1 - \mu > n/2$, one has

$$\|\partial_t^{k-2}(\partial v)^2\|_0 \leqq \sum_{j=0}^{k-2} \binom{k-2}{j} \|\partial_t^{k-j-2}(\partial v)\|_j \cdot \|\partial_t^j(\partial v)\|_{k-1-\mu-j} \, . \tag{5.18}$$

Consequently, the left-hand side of the above inequality is bounded by

$$c(|||v|||'_{k-1})^{1+\mu}(|||v|||'_k)^{1-\mu}$$

Finally, (5.16), (5.17), and (5.18) show that

$$|||\delta|||^2_{k-1,T} \leqq P\left(|||g|||'^2{}_{k,T} + |||v|||'_k{}^{2(1-\mu)}\right) . \tag{5.19}$$

Existence of the Fixed Point

Since $|||q(0)|||'_{k-1} \leqq d$, one has $|||q|||'_{k-1,T} \leqq d + c\,A\,T^{1/2}$. This estimate, together with $(5.5)_3$ and (5.10), shows that in all the above estimates the λ-functions P and Q have the form $P = P(d + c\,AT^{1/2})$ and $Q = Q(A)$, respectively. Now it is an elementary exercise to prove that we can choose λ-functions $\lambda_1(d)$, $\lambda_2(d, A)$, and $\lambda_3(d, A)$ such that if $A \geqq \lambda_1$, and if $\lambda_2 T \leqq 1$, $\lambda_3 [f]'_{k,T} \leqq 1$, then the right-hand sides of equations (5.11), (5.12), and (5.19) are less than or equal to A^2. Hence $(\delta, \zeta, g) = S(\vartheta, \xi, q)$ satisfies (5.5).

It is easily proved (by choosing, if necessary, a smaller value for T, still defined by a condition of the form $\lambda(d, A)T \leqq 1$) that the map S is a contraction of $\mathbb{K}$ with respect to the χ-norm. For a similar proof, see that of Lemma 2.3 in reference [7]. Hence, in order to prove the existence of a fixed point in $\mathbb{K}$ for the map S, it remains to prove that δ satisfies $(5.7)_1$, which is the missing condition in order to show that $S(\mathbb{K}) \subset \mathbb{K}$. In general, however, δ does not satisfy $(5.7)_1$. We shall avoid this obstacle as follows.

LEMMA 5.1. *Let* $(\delta, \zeta, g) = S(\vartheta, \xi, q)$. *Then,*

$$\frac{d}{dt} \int_\Omega \delta\, dx = \int_\Omega \vartheta(\delta - \vartheta)\, dx \quad \textit{in } [0, T]\,, \qquad \textit{and} \qquad \int_\Omega \delta(0, x)\, dx = 0\,.$$

Proof: The divergence theorem together with equation $(5.2)_2$ shows that

$$\int_\Omega \nabla \cdot (h(q)\nabla g)\, dx = \int_\Gamma \left[\sum_{i,j} (\partial_i \nu_j)\, v_i v_j + f \cdot \nu \right] d\Gamma\,. \tag{5.20}$$

Moreover,[5]

$$\begin{cases} \nabla \cdot [(v \cdot \nabla)v - f] = \sum_{i,j} (\partial_i v_j)(\partial_j v_i) + v \cdot \nabla\theta - \nabla \cdot f & \text{in } \Omega\,, \\ [(v \cdot \nabla)v - f] \cdot \nu = -\sum_{i,j} (\partial_i \nu_j)\, v_i v_j - f \cdot \nu & \text{on } \Gamma\,, \end{cases}$$

[5]Since $v \cdot \nu = 0$ on Γ, the vector fields v and $\nabla(v \cdot \nu)$ defined on Γ, are orthogonal. Consequently,

$$0 = \sum_{i,j} v_i \partial_i \left(v_j \nu_j\right) = [(v \cdot \nabla)v] \cdot \nu + \sum_{i,j} \left(\partial_i \nu_j\right) v_i v_j\,.$$

hence, by the divergence theorem,

$$(5.21)\qquad \begin{aligned} &-\int_\Omega \sum_{i,j} (\partial_i v_j)\,(\partial_j v_i)\,dx + \int_\Omega \nabla\cdot f\,dx - \int_\Omega v\cdot\nabla\vartheta\,dx \\ &= \int_\Gamma \left[\sum_{i,j} (\partial_i \nu_j)\,v_i\cdot v_j + f\cdot\nu\right] d\Gamma\,. \end{aligned}$$

By integrating both sides of equation $(5.2)_1$ over Ω, by taking into account the equation (5.4), and by using the equations (5.20) and (5.21), one proves the first assertion in the lemma. The second assertion is immediate, since $\delta(0) = \nabla\cdot a$ and $a\cdot\nu = 0$ on Γ.

Now, we define the linear operator

$$\pi u = u - |\Omega|^{-1}\int_\Omega u(y)\,dy\,.$$

For each $\ell \geqq 0$ the operator π has norm equal to 1 in H^ℓ, and hence in $L^\infty_T(H^\ell)$. Since $\pi\,\partial_t = \partial_t\pi$, it follows, in particular, that π has norm equal to 1 in $\mathscr{L}^\infty_T(H^{k-1})$. On the other hand, $\pi\,\partial_t^j\vartheta(0) = \partial_t^j\vartheta(0)$, for $j = 0,\dots,k-1$. It readily follows that the map $\overline{S} = (\pi\times id\times id)\circ S$ (i.e., $\overline{S}(\vartheta,\xi,q) = (\pi\delta,\zeta,g)$ if $(\delta,\zeta,g) = S(\vartheta,\xi,q)$) has a fixed point in $\mathbb{K}$. Let $(\vartheta,\xi,q) = (\pi\delta,\zeta,g)$ be this fixed point. In order to show that it is also a fixed point for the map S, it suffices to show that $\pi\delta = \delta$. This is easily proved, as follows. Set

$$y(t) = |\Omega|^{-1}\int_\Omega \delta(t,x)\,dx\,.$$

Since $\vartheta = \pi\delta$, one has $\delta(t,x) - \vartheta(t,x) = y(t)$. By using the above lemma, it follows that

$$y'(t) = -\left(|\Omega|^{-1}\int_\Omega \vartheta(t,x)\,dx\right) y(t) = 0 \quad \text{for each } t\in[0,T]\,,$$

and that $y(0) = 0$. Hence $y(t)$ vanishes identically on $[0,T]$.

We note that the λ-functions $\lambda_2(d,A)$ and $\lambda_3(d,A)$ depend only on d if we set $A = \lambda_1(d)$. This shows $(1.14)_1$ and (1.15) in Theorem 1.3 (the λ_1 in Theorem 1.3 is a constant times the above λ_1). By using the equations $(1.11)_1$ and $(1.11)_2$, we estimate the norms $\|\partial_t^k v\|_0$ and $\|\partial_t^k g\|_0$. This yields $(1.14)_2$.

The Compatibility Conditions

Here we prove that if the compatibility conditions (from now on, c.c.) up to order $k-1$ hold for the system (1.11), then the c.c. up to order $k-2$ for the system (5.2) are satisfied.

The c.c. for (1.11) are

$$\{\partial_t^{j+1} v(0)\}\Big|_\Gamma \cdot \nu = 0\,, \qquad j = -1, 0, \ldots, k-2\,, \tag{5.22$_{j+1}$}$$

where, by definition, $\{v(0)\} = a$, $\{g(0)\} = \phi$, and the $\{\partial_t^j v(0)\}$ are defined as follows. Differentiate the equation $(1.11)_1$ (with respect to t) up to order $k-2$, the equation $(1.11)_2$ up to order $k-3$, set $t = 0$ in these equations, and formally solve the system obtained in the above way for the unknowns $\partial_t^j v(0)$ ($j = 1, \ldots, k-1$) and $\partial_t^j g(0)$ ($j = 1, \ldots, k-2$). Note that these solutions, that are obtained by recurrence on j, depend only on a, ϕ, and $\partial_t^j f(0)$, $j = 0, \ldots, k-2$. These solutions are, by definition, the functions $\{\partial_t^j v(0)\}$ and $\{\partial_t^j g(0)\}$. For brevity, from now on, we set $\{F(0)\} = \{F\}$.

By using equation $(1.11)_1$ one shows that

$$\{\partial_t^{j+1} v\} = \left\{\partial_t^j \left(f - (v\cdot\nabla)v - h(g)\nabla g\right)\right\}\,, \tag{5.23$_j$}$$

for $0 \leqq j \leqq k-2$. Hence, the c.c. for (1.11) are $\{v\}\cdot\nu = 0$ on Γ, plus

$$\left\{\partial_t^j \left(f - (v\cdot\nabla)v - h(g)\nabla g\right)\right\}\Big|_\Gamma \cdot \nu = 0\,, \qquad j = 0, \ldots, k-2\,. \tag{5.24$_{j+1}$}$$

Let us now consider the c.c. for the system (5.2). It is worth noting that here $\partial_t^j v(0)$ and $\partial_t^j q(0)$ are not defined via a formal calculation from the equation (5.2) but, on the contrary, are the values of the functions $\partial_t^j v(t)$ and $\partial_t^j q(t)$, at $t = 0$. Since the function $v(t)$, the solution of (5.1), satisfies the condition $v\cdot\nu = 0$ on Σ_T, one has $\Sigma(\partial_i \nu_j)v_i v_j = -[(v\cdot\nabla)v]\cdot\nu$ on Σ_T. Hence, the boundary condition $(5.2)_2$ can be written in the equivalent form

$$(h(q)\nabla\hat{g} + (v\cdot\nabla)v - f)\cdot\nu = 0 \quad \text{on} \quad \Sigma_T\,, \tag{5.25}$$

where we use the symbol $\hat{g}$ to indicate the g appearing in (5.2), in order to distinguish it from the g that appears in (1.11). The c.c. up to order $k-2$ for the system (5.2) are

$$\left\{\partial_t^j \left(h(q)\nabla\hat{g} + (v\cdot\nabla)v - f\right)\right\}\Big|_\Gamma \cdot \nu = 0\,, \quad \text{for } j = 0, \ldots, k-2\,. \tag{5.26$_j$}$$

Note that the highest order derivatives, with respect to t, that appear in $(5.26)_j$ are those of order $k-2$. Since

$$\partial_t^j v(0) = \{\partial_t^j v\}\,, \quad \partial_t^j q(0) = \{\partial_t^j g\}\,, \tag{5.27$_j$}$$

for $j = 0, \dots, k-2$, it is superfluous to distinguish between the couple (g, v) in equations (5.24_{j+1}) and the couple (q, v) in equations (5.26_j). If we are able to prove that

$$(5.28_j) \qquad \left\{\partial_t^j \hat{g}\right\} = \left\{\partial_t^j g\right\}, \quad \text{for } j = 0, \dots, k-2,$$

then, the c.c. (5.26_j) are satisfied since they coincide with the c.c. (5.24_{j+1}), which hold by assumption.

Proof of (5.28_j): For $j = 0$ one has $\{g\} = \phi = \{\hat{g}\}$. For $j = 1$, equation $(1.11)_1$ shows that $\{\partial_t g\} = \psi$, hence $(5.28)_1$ holds. Assume now, for some $j_0 \in [1, k-3]$, that the hypothesis (5.28_j) holds for each $j \in [0, j_0]$. We want to show that $(5.28)_{j_0+1}$ holds.

The function $\{\partial_t^{j_0+1} g\}$ is defined by formally solving the equation

$$(5.29) \qquad \partial_t^{j_0}[(\partial_t + v \cdot \nabla)g + \nabla \cdot v] = 0,$$

for the unknown $\partial_t^{j_0+1} g(0)$. Equations (1.11) show that

$$\begin{aligned}\partial_t(\nabla \cdot v) &= -\nabla \cdot [(v \cdot \nabla)v + h(g)\nabla g - f] \\ &= (v \cdot \nabla)(\partial_t + v \cdot \nabla)g - \sum(\partial_i v_j)(\partial_j v_i) - \nabla \cdot [h(g)\nabla g] + \nabla \cdot f.\end{aligned}$$

Hence, equation (5.29) can be written in the form

$$(5.30) \quad \partial_t^{j_0-1}\left[(\partial_t + v \cdot \nabla)^2 g - \nabla \cdot (h(g)\nabla g) - \sum (\partial_i v_j)(\partial_j v_i) + \nabla \cdot f\right] = 0.$$

On the other hand, the function $\{\partial_t^{j_0+1} \hat{g}\}$ is defined by solving the equation

$$(5.31) \quad \partial_t^{j_0-1}\left[(\partial_t + v \cdot \nabla)^2 \hat{g} - \nabla \cdot (h(q)\nabla \hat{g}) - \sum (\partial_i v_j)(\partial_j v_i) + \nabla \cdot f\right] = 0$$

for the unknown $\partial_t^{j_0+1} \hat{g}(0)$. Since $(5.28)_j$ holds (by assumption) for each $j \leqq j_0$ and since $(5.27)_j$ holds, the equation (5.30) for the unknown $\partial_t^{j_0+1} g(0)$ coincides with the equation (5.31) for the unknown $\partial_t^{j_0+1} \hat{g}(0)$. Hence $(5.28)_{j_0+1}$ holds.

Remark. The c.c. up to order $k-1$ for the system (1.11) coincide with the c.c. up to order $k-2$ for the system (1.12). These last c.c. are $\partial_\nu\{\partial_t^j g\} = \partial_t G(0)$, $j = 0, \dots, k-2$, plus $\{v\} \cdot \nu = 0$.

6. Proof of Theorem 1.4

Fix a constant c_0 such that the norms $\|v\|_{k,T_0}$, $\|g\|_{k,T_0}$, $|||f|||_{k-1,T_0}$, and $[f]_{k,T_0}$ are bounded by $c_0 - 1$ and let λ_2, λ_3, and λ_4 (see the statement of

Theorem 1.3) denote the particular values of these λ-functions when all their arguments are equal to c_0. Fix $T > 0$ satisfying $\lambda_2 T \leqq 1$ and $\lambda_3[f]_{k,[t_0,t_0+T]} \leqq 1/2$ for every $t_0 \in [0, T_0[$. If $t_0 + T > T_0$, replace $t_0 + T$ by T_0. In the sequel we take into consideration only those f' for which $|||f' - f|||_{k-1,T_0} \leqq 1$ and $[f' - f]_{k,T_0} \leqq 1/2$. Hence, $|||f'|||_{k-1,T_0} \leqq c_0$ and $\lambda_3[f']_{k,[t_0,t_0+T]} \leqq 1$, for every t_0 as above. Consequently, Theorem 1.3 applies to the solution (v', g') on intervals $[t_0, t_0+T]$, provided that the "initial data" $(v'(t_0), g'(t_0))$ satisfies the conditions $\|v'(t_0)\|_k \leqq c_0$, $\|g'(t_0)\|_k \leqq c_0$. It readily follows that, in order to prove Theorem 1.4, it suffices to prove the thesis for the particular intervals $[mT, (m+1)T]$, $m = 0, 1, \dots, [T_0/T]$, hence just for the interval $[0, T]$.

As in Section 5, instead of studying the systems (1.11) and (1.11′) directly, we shall study the equivalent systems (1.12) and (1.12′). We start by applying Theorem 1.2 to the pair of equations $(1.12)_2$ and $(1.12')_2$. This yields the estimate (1.9) in which, according to (1.12), one has $h = h(g(t,x))$, $F = \sum(\partial_i v_j)(\partial_j v_i) - \nabla \cdot f$, $\psi = -(a \cdot \nabla\phi + \nabla \cdot a)$, $G = h(g)^{-1}(\sum(\partial_i \nu_j)v_i v_j + f \cdot n)$, and similarly for h', F', ψ', G'. By taking into account these expressions and the above setup, one shows that $R = R_k$ (see (1.8)) and T are here fixed constants. In the end, one gets

$$
\begin{aligned}
|||g(t)-g'(t)|||_k^2 \leqq c \Big\{ & \varepsilon + \|a-a'\|_k^2 + \|\phi-\phi'\|_k^2 + |||f(0)-f'(0)|||_{k-1}^2 \\
& + [f-f']_{k,t}^2 + [v-v']_{k,t}^2 + [g-g']_{k,t}^2 \\
& + \Lambda(\varepsilon)\Big([v-v']_{k-1,t}^2 + [g-g']_{k-1,t}^2\Big) \Big\} .
\end{aligned} \tag{6.1}
$$

In order to prove (6.1) we have to take into account the definitions of F, ψ, and G in terms of a, ϕ, f, v, and g, and the expressions of the derivatives $\partial_t^j v(0)$ and $\partial_t^j g(0)$ in terms of a, ϕ, and $\partial_t^j f(0)$. These expressions can be obtained directly from the equations $(1.11)_1$, $(1.11)_2$, or from (1.12). We also take into account that $|||h(g) - h(g')|||_\ell \leqq c|||g - g'|||_\ell$, for each t, and we use inequalities (2.1). Similar devices should be used in order to prove the estimates below. We deem it unnecessary to call these devices to the reader's attention again.

By applying Theorem 4.1, one proves that

$$
\begin{aligned}
|||\zeta(t) - \zeta'(t)|||_{k-1}^2 \leqq c \Big\{ & \varepsilon + \|a - a'\|_k^2 + |||f(0) - f'(0)|||_{k-1}^2 \\
& + [f - f']_{k,t}^2 + [v - v']_{k,t}^2 \\
& + C(\varepsilon)\Big(|||v(0) - v'(0)|||_{k-2}^2 + [v - v']_{k-1,t}^2\Big) \Big\} .
\end{aligned} \tag{6.2}
$$

Elliptic regularization shows that

$$
|||v(t)-v'(t)|||'^2{}_k \leqq c\Big(|||\zeta(t)-\zeta'(t)|||_{k-1}^2 + |||\nabla \cdot v(t) - \nabla \cdot v'(t)|||_{k-1}^2\Big) , \tag{6.3}
$$

since $(v - v') \cdot \nu = 0$ on Γ, for each t. Moreover, by using equations $(1.11)_2$ and $(1.11')_2$, one shows that

$$|||\nabla \cdot v(t) - \nabla \cdot v'(t)|||_{k-1}^2 \leqq c \left(|||g(t) - g'(t)|||_k^2 + |||v(0) - v'(0)|||_{k-1}^2 + [v - v']_{k,t}^2 \right) . \tag{6.4}$$

From equations (6.3), (6.1), (6.2), and (6.4) it readily follows that

$$\begin{aligned} &|||v(t) - v'(t)|||_k^2 \\ &\leqq c \Big\{ \varepsilon + \|a - a'\|_k^2 + \|\phi - \phi'\|_k^2 \\ &\quad + |||f(0) - f'(0)|||_{k-1}^2 + [f - f']_{k,t}^2 + [v - v']_{k,t}^2 \\ &\quad + [g - g']_{k,t}^2 + C_1(\varepsilon) \Big(\|a - a'\|_{k-1}^2 + \|\phi - \phi'\|_{k-1}^2 \\ &\quad + |||f(0) - f'(0)|||_{k-2}^2 + [v - v']_{k-1,t}^2 + [g - g']_{k-1,t}^2 \Big) \Big\} , \end{aligned} \tag{6.5}$$

where $C_1(\varepsilon) = \max\{C(\varepsilon), \Lambda(\varepsilon)\}$. Note that

$$|||v - v'|||_k \leqq c \left(|||v - v'|||_k' + |||g - g'|||_k + |||f - f'|||_{k-1} \right),$$

by equations $(1.11)_1$ and $(1.11')_1$.

From (6.1) and (6.5) one shows that the left-hand side of equation (6.6) below is bounded by the right-hand side of equation (6.5). Hence, by Gronwall's lemma,

$$\begin{aligned} &|||v(t) - v'(t)|||_k^2 + |||g(t) - g'(t)|||_k^2 \\ &\leqq c \Big(\varepsilon + \|a - a'\|_k^2 + \|\phi - \phi'\|_k^2 \\ &\quad + |||f(0) - f'(0)|||_{k-1}^2 + [f - f']_{k,t}^2 \Big) \\ &\quad + c\, C_1(\varepsilon) \Big(\|a - a'\|_{k-1}^2 + \|\phi - \phi'\|_{k-1}^2 \\ &\quad + |||f(0) - f'(0)|||_{k-2}^2 \\ &\quad + [v - v']_{k-1,t}^2 + [g - g']_{k-1,t}^2 \Big) , \end{aligned} \tag{6.6}$$

for each $t \in [0, T]$.

Assume now that (1.16) holds. It is a simple exercise (see, for instance, the proof of Lemma 4.1 in reference [9]) to show that

$$\|v(t) - v'(t)\|_0^2 + \|g(t) - g'(t)\|_0^2 \leqq c \left(\|a - a'\|_0^2 + \|\phi - \phi'\|_0^2 + [f - f']_{0,t}^2 \right) .$$

Since

$$[\cdot]_{k-1,T} \leqq c[\cdot]_{k,T}^{(k-1)/k}[\cdot]_{0,T}^{1/k} ,$$

and since the norms $[v']_{k,T}$ and $[g']_{k,T}$ are uniformly bounded, it follows that

$$\begin{aligned} & |||v(t) - v'(t)|||_k^2 + |||g(t) - g'(t)|||_k^2 \\ & \leqq c\Big\{\varepsilon + \|a - a'\|_k^2 + \|\phi - \phi\|_k^2 \\ & \quad + |||f(0) - f'(0)|||_{k-1}^2 + [f - f']_{k,t}^2\Big\} \\ & \quad + c\,C_1(\varepsilon)\left(\|a - a'\|_0^2 + \|\phi - \phi'\|_0^2 + [f - f']_{0,t}^2\right)^{1/k} \\ & \quad + c\,C_1(\varepsilon)\left(\|a - a'\|_{k-1}^2 + \|\phi - \phi'\|_{k-1}^2 + |||f(0) - f'(0)|||_{k-2}^2\right) . \end{aligned} \tag{6.7}$$

Given a real positive σ, fix $\varepsilon = \sigma/(2c)$. One has

$$|||v - v'|||_{k,T}^2 + |||g - g'|||_{k,T}^2 \leqq \sigma$$

provided that

$$\|a - a'\|_k^2 + \|\phi - \phi'\|_k^2 + |||f(0) - f'(0)|||_{k-1}^2 + [f - f']_{k,T}^2 \leqq \delta ,$$

where $\delta > 0$ satisfies the equation $c\delta + c\,C_1(\sigma/(2c))(\delta^{1/k} + \delta) \leqq \sigma/2$.

Appendix A

Proof of Proposition 3.1: For convenience, we shall assume that $\Omega = \mathbb{R}_+^n$. The general case can be reduced to the previous one by standard methods. On the other hand, the lack of regularity of the coefficients v and h made necessary some additional control on the calculations done below. This control, left to the reader, is done by using the inequalities (2.1) and (2.1-bis).

In the sequel, x_n is the normal direction to the boundary of $\mathbb{R}_+^n$. For convenience, we set $x' = (x_1, \dots, x_{n-1})$, $y = x_n$. We start by writing the compatibility conditions (1.5) in a more explicit form. Equation (1.1) is a particular case of an equation of the form

$$\begin{aligned} \partial_t^2 g = \vartheta \partial_y^2 g + \sum c\partial_{x'}\partial_y g + \sum c\,\partial_{x'}\partial_t g + \sum c\,\partial_x g \\ + \sum c\,\partial_t g + c\,g + F , \end{aligned} \tag{7.1}$$

where $c = c(t,x)$ denotes distinct functions. The boundary and initial conditions are those in (1.1). Above, and in the sequel, we use some abbreviated but clear notation. In equation (7.1) $\vartheta(t,x)$ satisfies $\vartheta \geqq m > 0$ on the boundary Σ_T. In the particular case of equation (1.1), one has $\vartheta = h - v_n^2$.

By differentiating both sides of (7.1) $j-2$ times with respect to t ($j \geqq 2$), and by setting $t=0$, we get the formal expression

$$\begin{aligned}\left\{\partial_t^j g(0)\right\} &= \vartheta\,\partial_y^2 \left\{\partial_t^{j-2} g(0)\right\} + \sum c\,\partial_{x'} \left\{\partial_t^{j-1} g(0)\right\} \\ &+ \sum c\,\partial_{x'}\partial_y \left\{\partial_t^{j-2} g(0)\right\} \\ &+ \sum_{0\leq r+s\leq j-1} c\,\partial_x^r \{\partial_t^s g(0)\} + \partial_t^{j-2} F(0)\,. \end{aligned} \tag{7.2}$$

From these expressions and from the initial conditions $\{g(0)\} = \phi$, $\{\partial_t g(0)\} = \psi$, we define (by recurrence) the functions $\{\partial_t^j g(0)\}$, which are, in fact, linear differential expressions in the vector (ϕ, ψ, F).

LEMMA 1. *One has* $\{g(0)\} = \phi$, $\{\partial_t g(0)\} = \psi$. *Moreover, for* $j \geqq 2$,

$$\begin{aligned}\left\{\partial_t^j g(0)\right\} &= \vartheta^{j/2}\,\partial_y^j \phi + \sum_{|r|+s=0}^{j}{}^{*}\, \alpha_{r,s}^{(j)}\,\partial_{x'}^r\,\partial_y^s \phi \\ &+ \sum_{|r|+s=0}^{j-1} \beta_{r,s}^{(j)}\,\partial_{x'}^r\,\partial_y^s \psi + \sum_{|i|=0}^{j-2} \gamma_i^{(j)}\,\partial^i F\,, \end{aligned} \tag{7.3}$$

if j is even, and

$$\begin{aligned}\left\{\partial_t^j g(0)\right\} &= \vartheta^{(j-1)/2}\,\partial_y^{j-1} \psi + \sum_{|r|+s=0}^{j} \alpha_{r,s}^{(j)}\,\partial_{x'}^r\,\partial_y^s \phi \\ &+ \sum_{|r|+s=0}^{j-1}{}^{*}\, \beta_{r,s}^{(j)}\,\partial_{x'}^r\,\partial_y^s \psi + \sum_{|i|=0}^{j-2} \gamma_i^{(j)}\,\partial^i F\,, \end{aligned} \tag{7.4}$$

if j is odd.

Above, $i = (i_0, \ldots, i_n)$ and $r = (r_1, \ldots, r_{n-1})$ are multi-indices, $\partial^i = \partial_t^{i_0}\partial_{x_1}^{i_1}\ldots\partial_y^{i_n}$ denotes a derivative of order $|i| = i_0 + \cdots + i_n$, $\partial_{x'}^r = \partial_{x_1}^{r_1}\ldots\partial_{x_{n-1}}^{r_{n-1}}$ denotes a derivative of order $|r| = r_1 + \cdots + r_{n-1}$, and the symbol $\sum^*$ means that the derivative of highest order with respect to the normal variable y is not present in the summation.

The proof of the lemma is easily done by induction on j, starting from the value $j=2$.

By using the equations (7.3) and (7.4), one easily shows that the compatibility conditions (1.5) have the following form:

$$\partial_y \phi = G(0)\,, \qquad \partial_y \psi = \partial_t G(0)\,, \quad \text{on } \Gamma\,; \tag{7.5}$$

$$\vartheta^{j/2}\partial_y^{j+1}\phi+\sum_{|r|+s=0}^{j+1}{}^{*}\,\hat{\alpha}_{r,s}^{(j)}\,\partial_{x'}^{r}\,(\partial_y^s\phi)$$

$$(7.6)\qquad +\sum_{|r|+s=0}^{j}\hat{\beta}_{r,s}^{(j)}\,\partial_{x'}^{r}(\partial_y^s\psi)$$

$$+\sum_{|i|=0}^{j-1}\hat{\gamma}_i^{(j)}\,\partial^i F(0)=\partial_t^j G(0)\,,\quad \text{on }\Gamma\,,$$

if j is even, and

$$\vartheta^{(j-1)/2}\partial_y^j\psi\quad+\sum_{|r|+s=0}^{j+1}\hat{\alpha}_{r,s}^{(j)}\,\partial_{x'}^{r}\,(\partial_y^s\phi)$$

$$(7.7)\qquad +\sum_{|r|+s=0}^{j}{}^{*}\,\hat{\beta}_{r,s}^{(j)}\,\partial_{x'}^{r}\,(\partial_y^s\psi)$$

$$+\sum_{|i|=0}^{j-1}\hat{\gamma}_i^{(j)}\,\partial^i F(0)=\partial_t^j G(0)\,,\quad \text{on }\Gamma\,,$$

if j is odd. Since $\vartheta\geqq m>0$ on the boundary, we can substitute, one at a time, the functions $\partial_y\phi$, $\partial_y\psi$, $\partial_y^2\phi$, $\partial_y^2\psi$, and so on, in the subsequent equations. Hence the normal derivatives $\partial_y^j\phi$ and $\partial_y^j\psi$ can be written in terms of the tangential derivatives of ϕ and ψ, and of F and G. In fact one easily proves by induction that

$$(7.8)\qquad \partial_y^{j+1}\phi=\sum_{|r|=0}^{j+1}a_r^{(j)}\,\partial_{x'}^{r}\,\phi+\sum_{|r|=0}^{j}b_r^{(j)}\,\partial_{x'}^{r}\psi+\sum_{|i|=0}^{j-1}c_i^{(j)}\,\partial^i F$$

$$+\sum_{|q|=0}^{j}\partial^q G(0)\equiv P^{(j)}(\phi,\psi,F,G)\,,$$

if $j\geqq 2$ is even, and

$$(7.9)\qquad \partial_y^{j}\psi=\sum_{|r|=0}^{j+1}a_r^{(j)}\,\partial_{x'}^{r}\,\phi+\sum_{|r|=0}^{j}b_r^{(j)}\,\partial_{x'}^{r}\psi+\sum_{|i|=0}^{j-1}c_i^{(j)}\,\partial^i F$$

$$+\sum_{|q|=0}^{j}\partial^q G(0)\equiv P^{(j)}(\phi,\psi,F,G)\,,$$

if $j\geqq 2$ is odd, where $q=(q_0,q_1,\dots,q_{n-1})$, and $\partial^q=\partial_t^{q_0}\,\partial_{x_1}^{q_1}\dots\partial_{x_{n-1}}^{q_{n-1}}$.

Now let $1 \leqq \ell \leqq k-1$, and assume that $(\phi, \psi, F, G) \in H^\ell \times H^{\ell-1} \times \mathscr{L}_T^2(H^{\ell-1}) \times \mathscr{L}_T^2(\mathscr{H}^{\ell-1/2})$ satisfy the compatibility conditions up to order $\ell-2$ for the problem (7.1). This means that (7.5), (7.8), and (7.9) hold for $j \leqq \ell - 2$.

We want to prove that there is a sequence

$$(7.10) \qquad (\phi_n, \psi_n, F_n, G_n) \in H^{\ell+1} \times H^\ell \times \mathscr{L}_T^2(H^\ell) \times \mathscr{L}_T^2(\mathscr{H}^{\ell+1/2})$$

such that

$$(7.11) \qquad \begin{aligned} &(\phi_n, \psi_n, F_n, G_n) \to (\phi, \psi, F, G) \\ &\qquad \text{in } H^\ell \times H^{\ell-1} \times \mathscr{L}_T^2(H^{\ell-1}) \times \mathscr{L}_T^2(\mathscr{H}^{\ell-1/2}) , \end{aligned}$$

and that the compatibility conditions (7.5), (7.8), and (7.9) are satisfied, up to order $\ell-1$, by each set of data $(\phi_n, \psi_n, F_n, G_n)$. The method followed here in order to prove the existence of the above sequence was suggested by reference [21]. We fix sequences F_n, G_n $\bar{\phi}_n$, $\bar{\psi}_n$, $n \in \mathbb{N}$, satisfying (7.10) and (7.11) and we look for ϕ_n and ψ_n of the form $\phi_n = \bar{\phi}_n + \lambda_n$, $\psi_n = \bar{\psi}_n + \mu_n$. Hence our problem becomes: to find λ_n and μ_n such that

$$(7.12) \qquad (\lambda_n, \mu_n) \in H^{\ell+1} \times H^\ell , \qquad (\lambda_n, \mu_n) \to 0 \quad \text{in } H^\ell \times H^{\ell-1} ,$$

and such that

$$(7.13) \qquad \partial_y \lambda_n = G_n(0) - \partial_y \bar{\phi}_n \equiv a_n^{(0)} ; \quad \partial_y \mu_n = \partial_t G_n(0) - \partial_y \bar{\psi}_n \equiv a_n^{(1)} ,$$

$$(7.14) \qquad \partial_y^{j+1} \lambda_n = P^{(j)} (\phi_n, \psi_n, F_n, G_n) - \partial_y^{j+1} \bar{\phi}_n \equiv a_n^{(j)} ,$$

if j is even, and

$$(7.15) \qquad \partial_y^j \mu_n = P^{(j)}(\phi_n, \psi_n, F_n, G_n) - \partial_y^j \bar{\psi}_n \equiv a_n^{(j)} ,$$

if j is odd. Above, $2 \leqq j \leqq \ell - 1$.

Since the $P^{(j)}$'s are linear differential operators of order $j+1$ with respect to ϕ, of order j with respect to ψ and to G, and of order $j-1$ with respect to F, it follows from (7.10) and (7.11) that

$$(7.16) \qquad \begin{cases} a_n^{(j)} \in \mathscr{H}^{\ell-j-1/2} , & \text{for } 0 \leqq j \leqq \ell - 1 , \\ a_n^{(j)} \to 0 \quad \text{in } \mathscr{H}^{\ell-j-3/2} , & \text{for } 0 \leqq j \leqq \ell - 2 . \end{cases}$$

Now, we prove the existence of the above λ_n. The existence of μ_n is proved in a similar way. We set $\hat{a}_n^{(j)} = a_n^{(j)}$ if j is even, $\hat{a}_n^{(j)} = 0$ if j is odd ($\hat{a}_n^{-1} = 0$). Let R be a linear continuous map, R: $\mathscr{H}^{\ell+(1/2)} \times \mathscr{H}^{\ell-(1/2)} \times \cdots \times \mathscr{H}^{1/2} \to$

$H^{\ell+1}$, such that $\partial_y^s R(u_0, u_1, \ldots, u_\ell) = u_s$ for $s = 0, 1, \ldots, \ell$, and that the map $(u_0, u_1, \ldots, u_{\ell-1}) \to R(u_0, u_1, \ldots, u_{\ell-1}, 0)$ is continuous on $\mathscr{H}^{\ell-(1/2)} \times \cdots \times \mathscr{H}^{1/2}$ with values in H^ℓ. Such a map exists (see [14], Theorem 2.5.7). Fix, in correspondence to each $n \in \mathbb{N}$, a function $\bar{a}_n^{(\ell-1)} \in C_0^\infty(\Gamma)$ such that $\langle\langle \bar{a}_n^{(\ell-1)} - \hat{a}_n^{(\ell-1)} \rangle\rangle_{1/2} \leqq n^{-1}$. Set $\lambda_n' = R(\hat{a}_n^{(-1)}, \hat{a}_n^{(0)}, \ldots, \hat{a}_n^{(\ell-2)}, 0)$ and $\lambda_n'' = R(0, \ldots, 0, \hat{a}_n^{(\ell-1)} - \bar{a}_n^{(\ell-1)})$. Clearly, $\lambda_n', \lambda_n'' \in H^{\ell+1}$, $\lambda_n' \to 0$ in H^ℓ, and $\lambda_n'' \to 0$ in $H^{\ell+1}$. Moreover, $\partial_y^{j+1}(\lambda_n' + \lambda_n'') = \hat{a}_n^{(j)}$, $j = -1, 0, \ldots, \ell-2$, and $\partial_y^\ell(\lambda_n' + \lambda_n'') = \hat{a}_n^{(\ell-1)} - \bar{a}_n^{(\ell-1)}$. Set $\lambda_n = \lambda_n' + \lambda_n'' + \lambda_n'''$. For our purpose, it is sufficient to prove that there exists $\lambda_n''' \in H^{\ell+1}$, for each $n \in \mathbb{N}$, such that $\lambda_n''' \to 0$ in H^ℓ and such that

$$\partial_y^{j+1} \lambda_n''' = 0 \quad \text{on } \Gamma \quad \text{for } j = 0, \ldots, \ell-2\, ; \; \partial_y^\ell \lambda_n''' = \bar{a}_n^{(\ell-1)} . \tag{7.17}$$

Let $\theta \in C_0^\infty([0,\infty)]$ be a function such that $\theta(t) = 1$ for each t in a neighborhood of the origin, and such that $\theta(t) = 0$ if $t \geqq 1$. Set $\theta_n(y) = (\ell!)^{-1} y^\ell \theta(ny)$, and define $\lambda_n'''(x) = \theta_n(y)\, \bar{a}_n^{(\ell-1)}(x')$. Clearly $\lambda_n''' \in C_0^\infty(\mathbb{R}_+^n)$. Moreover, $\theta_n^{(j+1)}(0) = 0$, if $j = 0, \ldots, \ell-1$, and $\theta_n^{(\ell)}(0) = 1$. Hence, the equations (7.17) are satisfied. Let us show that $\lambda_n''' \to 0$ in H^ℓ, as $n \to \infty$.

We assume, without loss of generality (elements in the sequence can be repeated if necessary), that $\langle\langle \bar{a}_n^{(\ell-1)} \rangle\rangle_\ell^2 \leqq$ const. $n^{1/2}$. Then, if $|r| + s \leqq \ell$ (where r is the multi-index $r = (r_1, \ldots, r_{n-1})$) one has

$$\|\partial_y^s \partial_{x'}^r \lambda'''\|_0^2 \leqq \left(\int_0^{1/n} |\partial_y^s \theta_n(y)|^2 \, dy \right) \langle\langle \bar{a}_n^{(\ell-1)} \rangle\rangle_\ell^2 .$$

Since the right-hand side is bounded by const. $n^{-1/2}$, it tends to zero as n tends to infinity.

Appendix B

For the reader's convenience, we give here a proof of the estimate (2.17). The point here is that we do not allow dependence of P on $\|h\|_{k,T}$ (but only on $\|h\|_{k-1,T}$). By writing equation (2.17) in the form

$$-\Delta g = h^{-1}(\nabla h \cdot \nabla g + B\delta + Lg) \tag{8.1}$$

we find that

$$\|g\|_{\ell+1} \leqq \lambda(m^{-1}, \|h\|_{k-1}) \, (\|\nabla h \cdot \nabla g\|_{\ell-1} + \|B\delta + Lg\|_{\ell-1}) + c\langle\langle \partial_\nu g \rangle\rangle_{\ell-1/2} , \tag{8.2}$$

where (here and in the sequel) c denotes a generic positive constant. Fix real numbers $r > n/2$ and $a \in]0,1]$ such that $k-1 = r+a$. We want to show that

$$(8.3) \qquad \lambda\|\nabla h \cdot \nabla g\|_{\ell-1} \leqq (c\lambda\|h\|_{k-1})^{\ell/a}\varepsilon^{-(\ell-a)/a}\|\nabla g\|_0 + \varepsilon\|g\|_{\ell+1} ,$$

for arbitrary, positive ε. Then, by fixing $\varepsilon = 1/2$, estimating $\|\nabla g\|_0$ (using (8.4)), and by using (8.2), one proves (2.17). Let us prove (8.3). From (2.1) it readily follows that $\|\nabla h \cdot \nabla g\|_{\ell-1} \leqq c\|\nabla h\|_{r-1+a}\|h\|_{k-1}$. Hence, by using interpolation in H^s spaces, $s \in \mathbb{R}^+$, one shows that

$$\|\nabla h \cdot \nabla g\|_{\ell-1} \leqq c\|h\|_{k-1}\|\nabla g\|_0^{a/\ell}\|\nabla g\|_\ell^{(\ell-a)/\ell} .$$

Consequently,

$$\lambda\|\nabla h \cdot \nabla g\|_{\ell-1} \leqq \left[(c\lambda\|h\|_{k-1})^{\ell/a}\ \varepsilon^{-(\ell-a)/a}\|\nabla g\|_0\right]^{a/\ell} (\varepsilon\,\|g\|_{\ell+1})^{(\ell-a)/a} ,$$

and (8.3) follows.

The reader should note that one could give a more elegant proof of (2.17) by dealing directly with the equations

$$(8.4) \qquad -\nabla \cdot (h\nabla g) = B\delta + Lg \quad \text{in } \Omega\,; \quad h\,\partial_\nu g = hG \quad \text{in } \Gamma\,.$$

In practice, this proof consists in checking that the manipulations done using standard methods of proof of (2.17) for solutions g of (8.4) are still valid when the coefficient h belongs to H^{k-1}.

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