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On the Existence Theorem for the Barotronic Motion of a Compressible Inviscid Fluid in the Half-Space (*).

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Summary. – *We give here an existence theorem, see Theorem* 1.1, *for the solution of the system of equations* (1.1) *that describes the motion of a compressible inviscid fluid in the half space. Moreover, we establish some sharp estimates, see Theorem* 3.2, *for the solution of the linear second order hyperbolic mixed problem* (3.1) *in terms of suitable norms of the coefficients. These estimates play a main rule here and in reference* [BV3], *where a first proof of Hadamard's classical well-posedness for the above nonlinear system of equations is given; see also* [BV4]. *Here, we adapt and simplify the method followed in our previous paper* [BV1].

Notations.

$N = \{\text{positive integers}\}$; $\quad N_0 = NU\{0\}$; $\quad R^+ = \{\text{positive reals}\}$.

$\Omega = R^3_+ = R^2 \times R^+$; $\quad Q_T = [0, T] \times R^3_+$, $\; T \in]0, 1]$.

$\Gamma = R^2 \times \{0\}$; $\quad \Sigma_T = [0, T] \times \Gamma$; $\quad \nu = (0, 0, 1)$.

$z = (y_1, y_2, x) = (y, x)$, the generic point in R^3.

$\partial_t = \partial/\partial t$; $\quad \partial_i = \partial/\partial y_i$ for $i = 1, 2$; $\quad \partial_3 = \partial/\partial x$.

∂_y denotes either ∂_1 and ∂_2.

$\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (N_0)^3$, a multi-index; $\quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$.

$$(v \cdot \nabla) w = \sum_{i=1}^{3} v_i (\partial_i w).$$

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We use the abbreviate notation

$$\int F = \int_{\boldsymbol{R}_+^3} F(z)\,dz .$$

We set $X = X(\boldsymbol{R}_+^3)$ for every functional space $X(\boldsymbol{R}_+^3)$ consisting of functions defined on $\boldsymbol{R}_+^3$. For instance, L^p means $L^p(\boldsymbol{R}_+^3)$. Moreover,

$$|\cdot|_p, \qquad \text{norm in } L^p, \qquad p \in [1, +\infty],$$

$$\|\cdot\|, \qquad \text{norm in } H^0 = L^2,$$

$$\|D^l u\|^2 = \sum_{|\alpha| = l} \|D^\alpha u\|^2,$$

$$\|u\|_k^2 = \sum_{l=0}^{k} \|D^l u\|^2,$$

$$\|D^s u\|_k^2 = \sum_{l=s}^{s+k} \|D^l u\|^2,$$

$$[u]_k^2 = \sum_{l=1}^{k} \|D^l u\|^2 .$$

The symbol D concerns always derivatives with respect to the space variables. Sometimes we will use the symbol D in a formal (but convenient) way, specially when dealing with estimates. For instance, we take leave to write expression like

$$|\partial_x (\nabla \times u)(x) + \Delta u(x)| \leq c |D^2 u(x)| \qquad \text{or} \qquad \int |(\nabla \times u) \cdot \nabla v| \leq c \int |Du|\,|Dv| .$$

$C_0^\infty(\overline{\boldsymbol{R}_+^3})$ is the subset of $C^\infty(\overline{\boldsymbol{R}_+^3})$ consisting of functions having compact support contained in $\overline{\boldsymbol{R}_+^3}$, the closure of $\boldsymbol{R}_+^3$. H^k is the completion of $C_0^\infty(\overline{\boldsymbol{R}_+^3})$ with respect to the norm $\|\cdot\|_k$. We made (here and elsewhere) usual identifications and follow usual conventions. Hence

$$H^k = \{u \in L^2 : D^\alpha u \in L^2, \ \text{ for all } \alpha \text{ such that } |\alpha| \leq k\} .$$

Since $\boldsymbol{R}_+^3$ is unbounded, it is convenient to make use of spaces $\tilde{H}^k$, defined as the completion of $C_0^\infty(\overline{\boldsymbol{R}_+^3})$ with respect to the norm $[\cdot]_k$. It is well known (see [DL], [L]) that

$$\tilde{H}^k = \{u \in L^6 : D^\alpha u \in L^2, \ \text{ for all } \alpha \text{ such that } 1 \leq |\alpha| \leq k\} ,$$

since $|u|_6 \leq c\|Du\|$, by a Sobolev's embedding theorem.

In the sequel we use freely some well known Sobolev's embedding theorems. In particular the following continuous embeddings and the corresponding inequalities for norms will be used very often

$$H^1 \subset \tilde{H}^1 \subset L^6; \quad H^2 \subset \tilde{H}^2 \subset C^{0,1/2} \subset L^\infty, \tag{0.1}$$

$$H^2 \subset L^2 \cap L^\infty \subset L^p, \quad \text{if } p \in [2, \infty], \tag{0.2}$$

$$H^1 \subset L^2 \cap L^6 \subset L^p, \quad \text{if } p \in [2, 6]. \tag{0.3}$$

In particular,

$$\|uv\| \leqslant c[u]_1 \|v\|_1 .$$

$C^{0,1/2}$ denotes the Banach space of bounded and uniformly Hölder continuous functions with exponent $1/2$, endowed with the canonical norm.

Since our results are local with respect to the time variable we assume, for convenience, that $T \in]0,1]$.

Let X denote a Banach space consisting on functions defined on $\boldsymbol{R}^3_+$. We set

$$C_T(X) = C([0, T]; X), \qquad L^p_T(X) = L^p(0, T; X),$$

and so on. The canonical norm in the space $L^\infty_T(X)$ is denoted by adding the symbol T to the symbol denoting the X-norm. For instance

$$\|u\|_{k,T} \equiv \|u\|_{L^\infty(0,T;H^k)}$$

and

$$[u]_{k,T} \equiv \|u\|_{L^\infty(0,T;\tilde{H}^k)} .$$

Furthermore, we set

$$|||u|||_{k,T} \equiv \|u\|_{L^2(0,T;H^k)},$$

and

$$[u]_{k,T} \equiv \|u\|_{L^2(0,T;\tilde{H}^k)} .$$

We do not distinguish between notations for scalar fields and corresponding notations for vector fields. For instance, if $v = (v_1, v_2, v_3)$ and $v_i \in H^k$ for $i = 1,2,3$, we write $v \in H^k$ and $\|v\|_k^2 = \|v_1\|_k^2 + \|v_2\|_k^2 + \|v_3\|_k^2$.

Given a function $f(t, z)$ we denote by $f(t)$ (for each fixed t) the function $f(t, \cdot)$ of the z variable.

1. – The existence theorem.

The barotropic motion of a compressible inviscid fluid obeys to the following equations (see for instance [Se] C.I and E.I,II; [Sd] IV § 1; [LL] § 1 and § 2)

$$(1.1)\qquad \begin{cases} \rho(\partial_t v + (v\cdot\nabla)v) + \nabla p(\rho) = 0\,, \\ \partial_t\rho + \nabla\cdot(\rho v) = 0\,, \qquad \text{in } Q_T\,, \\ v\cdot\nu = 0\,, \qquad \text{on } \Sigma_T\,, \\ v(0) = v_0\,, \qquad \rho(0) = \rho_0\,, \end{cases}$$

where v is the velocity field, ρ the density, and p the pressure. The function $p\colon \boldsymbol{R}^+ \to \boldsymbol{R}$ is a *given* C^4 function, moreover $p'(s) > 0$ for all $s \in \boldsymbol{R}^+$. We denote by $\bar\rho$ a *fixed* positive constant, the value of the density ρ at infinity. We define

$$H^k_{\bar\rho} = \{\rho\colon \rho - \bar\rho \in H^k\}$$

and set $m(\rho) = \inf \rho(z)$ for $z \in \boldsymbol{R}^3_+$. We assume that the initial data satisfy the assumptions

$$(1.2)\qquad v_0 \in \tilde H^3\,,\quad v_0\cdot\nu = 0 \text{ on } \Gamma; \qquad \rho_0 \in H^3_{\bar\rho}\,,\quad m(\rho_0) > 0\,,$$

and (compatibility conditions)

$$(1.3)\qquad \partial_x\rho_0 = 0\,, \qquad \partial_x[(v_0\cdot\nabla\rho_0)/\rho_0 + \nabla\cdot v_0] = 0 \text{ on } \Gamma\,.$$

These conditions are also necessary in order to get the existence results proved in the sequel.

Finally, we note that the uniqueness of our solutions is trivially proved since it holds in much larger functional classes than those considered here. See, for instance, [Se2].

The first existence result for the mixed problem was proved by EBIN [E1] by assuming that the initial velocity is sub-sonic and the initial density is not too large. The existence of solution for arbitrarily large initial data was first proved by us [BV1] and, in an independent paper, by AGEMI [A]. In reference [BV1] a main role is played by the couple of operators curl and divergence. These operators play again a main role in Schochet's paper [S1], where the authors proves the existence of the solution in the general case $p = p(\rho, s)$ and studies the incompressible limit. These results were extended to a class of first order hyperbolic systems in Schochet's paper [S2] (for the Cauchy problem see KATO [K1], and KLAINERMAN and MAJDA [KM1], [KM2]).

In reference [BV1] we have considered fluids filling bounded domains Ω. Here, by following similar ideas we consider the case $\Omega = \boldsymbol{R}^3_+$. One has the following result.

THEOREM 1.1. – *Under the above hypothesis there exist two positive constants T and c, that depend only on $[v_0]_3$, $\|\rho_0 - \bar\rho\|_3$, $m(\rho_0)$, $\bar\rho$, and on the particular function* p(·), *such that there is a solution* (v, ρ) *of problem* (1.1) *in* Q_T. *Moreover,*

$$\partial_t^j(v, \rho - \bar{\rho}) \in C_T(\tilde{H}^{3-j} \times H^{3-j}), \qquad \text{for } j = 0, 1, 2$$

and

$$\sum_{j=0}^{2} ([\partial_t^j v]_{3-j,T} + \|\partial_t^j(\rho - \bar{\rho})\|_{3-j,T}] \le c. \tag{1.4}$$

Furthermore, if $v_0 \in H^3$,

$$\sum_{j=0}^{3} (\|\partial_t^j v\|_{3-j,T} + \|\partial_t^j(\rho - \bar{\rho})\|_{3-j,T}) \le c. \tag{1.5}$$

Actually, the equation (1.5) holds under the weaker assumption $v_0 \in \tilde{H}^3$, provided that the term $\|v\|_{3,T}$ is replaced by $[v]_{3,T}$.

It is convenient to study the problem (1.1) by making the change of variables

$$g = \log(\rho/\bar{\rho}) \tag{1.6}$$

and by introducing the function

$$h(s) = p'(\bar{\rho}e^s), \qquad s \in \boldsymbol{R}. \tag{1.7}$$

Clearly, $h \in C^3(\boldsymbol{R}; \boldsymbol{R}^+)$. The system of equations (1.1) turn out to be equivalent to the system

$$\begin{cases} \partial_t v + (v \cdot \nabla) v + h(g) \nabla g = 0, & \\ \partial_t g + v \cdot \nabla g + \nabla \cdot v = 0, & \text{in } Q_T, \\ v \cdot \nu = 0, \quad \text{on } \Sigma_T, & \\ v(0) = v_0, \quad g(0) = g_0, & \end{cases} \tag{1.8}$$

where, by definition, $g_0(z) = \log(\rho_0(z)/\bar{\rho})$. The assumptions (1.2) and (1.3) turn into

$$v_0 \in \tilde{H}^3, \quad v_0 \cdot \nu = 0 \text{ on } \Gamma; \quad g_0 \in H^3 \tag{1.9}$$

and

$$\partial_x g_0 = 0, \qquad \partial_x [v_0 \cdot \nabla g_0 + \nabla \cdot v_0] = 0 \text{ on } \Gamma. \tag{1.10}$$

It is immediate to verify that the Theorem 3.1 is equivalent to the following theorem and corollary.

THEOREM 1.2. – *Let* v_0 *and* g_0 *satisfy the assumptions* (1.9) *and* (1.10). *There exist positive constants c and T such that there is a solution* (v, g) *of problem* (1.8). *More-*

over, $\partial_t^j(v, g) \in C_T(\tilde{H}^{3-j} \times H^{3-j})$, $j = 0,1,2,$ *and*

$$\sum_{j=0}^{2} ([\partial_t^j v]_{3-j,T} + \|\partial_t^j g\|_{3-j,T}) \leq c. \tag{1.11}$$

The constants T and c are universal with respect to bounded sets of initial data (see definition below).

COROLLARY. – *If, moreover,* $v_0 \in H^3$ *then* $\partial_t^j(v, g) \in C_T(H^{3-j} \times H^{3-j})$, $j = 0,1,2,3,$ *and*

$$\sum_{j=0}^{3} (\|\partial_t^j v\|_{3-j,T} + \|\partial_t^j g\|_{3-j,T}) \leq c. \tag{1.12}$$

Actually, this last result holds under the weaker assumptions $v_0 \in \tilde{H}^3$, provided that the term $\|v\|_{3,T}$ is replaced by $[v]_{3,T}$ and $v \in C_T(H^3)$ by $v \in C_T(\tilde{H}^3)$.

Positive constants c and T are said to be *universal* with respect to bounded sets of initial data if a (positive) lower bound for T and an upper bound for c depend only on upper bounds for the norms $\|g_0\|_3$ and $[v_0]_3$ [resp. $\|v_0\|_3$, if $v_0 \in H^3$].

We will write the equations (1.8) in the equivalent form (1.15) below, which is very appropriate for our purposes. Set

$$\zeta_0 = \nabla \times v_0, \qquad g_1 = -(v_0 \cdot \nabla g_0 + \nabla \cdot v_0). \tag{1.13}$$

From equation (1.10) it follows that $\zeta_0 \in H^2$, $g_1 \in H^2$, and

$$\partial_\nu g_0 = 0, \qquad \partial_\nu g_1 = 0 \text{ on } \Gamma. \tag{1.14}$$

We consider the following system of equations:

$$\begin{cases} \partial_t \zeta + (v \cdot \nabla)\zeta - (\zeta \cdot \nabla) v + (\nabla \cdot v)\zeta = 0, \\ (\partial_t + v \cdot \nabla)^2 g - \nabla \cdot (h(g)\nabla g) = \sum_{i,j=1}^{3} (\partial_i v_j)(\partial_j v_i), \\ -\nabla \cdot v = \partial_t g + v \cdot \nabla g, \\ \nabla \times v = \zeta, \qquad \text{in } Q_T, \\ v \cdot \nu = 0, \qquad \partial_\nu g = 0 \qquad \text{on } \Sigma_T, \\ \zeta(0) = \zeta_0, \qquad g(0) = g_0, \qquad (\partial_t g)(0) = g_1, \qquad \nabla \cdot v(0) = \nabla \cdot v_0. \end{cases} \tag{1.15}$$

A couple (v, g) is a solution of (1.8) if and only if (v, g, ζ) is a solution of (1.15), where $\zeta = \nabla \times v$. The quite immediate proof will be done after stating the following existence theorem for problem (1.15).

THEOREM 1.3. – *Let* v_0 *and* g_0 *be as in Theorem 1.2, and define* ζ_0 *and* g_1 *as above. There are positive constants c and T, universal with respect to bounded sets of initial data, such that there is a solution* (v, g, ζ) *of problem* (1.15) *in* Q_T. *Moreover,* $\partial_t^j(v, g, \zeta) \in C_T(\tilde{H}^{3-j} \times H^{3-j} \times H^{2-j})$, *for* $j = 0,1,2,$ *and* (1.11) *holds.*

PROOF OF THE EQUIVALENCE BETWEEN THEOREMS 1.2 AND 1.3. – Assume that v, g, ζ is a solution of (1.15). Define, for each fixed t, the vector field $V = \partial_t v + (v\cdot\nabla)v + h(g)\nabla g$. Since $\zeta = \nabla\times v$ and since $h(g)\nabla g$ is a gradient field on $\boldsymbol{R}^3_+$, it readily follows, from eq. $(1.15)_1$, that $\nabla\times V = 0$ on $\boldsymbol{R}^3_+$. Recall the vector analysis formulae $\nabla\times[(v\cdot\nabla)v] = (v\cdot\nabla)\zeta - (\zeta\cdot\nabla)v + (\nabla\cdot v)\zeta$. On the other hand, $\nabla\cdot V = \partial_t\delta + (v\cdot\nabla)\delta + \Sigma(\partial_i v_j)(\partial_j v_i) + \nabla\cdot(h(g)\nabla g)$, where $\delta = \nabla\cdot v$. Since $\delta = -(\partial_t g + v\cdot\nabla g)$, it follows from $(1.15)_2$ that $\nabla\cdot V = 0$. The orthogonality of the vector fields v and $\nabla(v\cdot\nu)$ on Γ shows that

$$0 = \sum_{i,j} v_i\,\partial_i(v_j\nu_j) = [(v\cdot\nabla)v]\cdot\nu + \sum_{i,j}(\partial_i\nu_j)\,v_i v_j\,.$$

Hence, $[(v\cdot\nabla)v]\cdot\nu = 0$ on Γ. On the other hand, $\partial_t(v\cdot\nu) = 0$ and $h(g)\,\partial_x g = 0$ on Γ. Consequently, $V\cdot\nu = 0$ on Γ.

From $\nabla\times V = 0$ and $\nabla\cdot V = 0$ on $\boldsymbol{R}^3_+$, and from $V\cdot\nu = 0$ on Γ, it follows that $V = 0$ on $\boldsymbol{R}^3_+$ (since $V\in\tilde{H}^2$). Hence, $(1.8)_1$ holds. Moreover, $v(0) = v_0$, since both the vector fields have in $\boldsymbol{R}^3_+$ the same divergence and the same curl, and both are tangential to Γ.

Reciprocally, if (v, g) is a solution of (1.8), set $\zeta = \nabla\times v$ and apply the operators curl, divergence, and $\nu\cdot$ to the equation $(1.8)_1$ and to the identity $v(0) = v_0$. This device yields (1.15). ■

A large part of this paper (the Section 3) is dedicated to the study of system (3.1), the linear counterpart of equation $(1.15)_2$. The system (3.1) plays a main rule in the proof of Theorem 1.3. This proof will be carried out in Section 2, by assuming Theorems 3.1 and 3.2. The proofs of these two last theorems are postponed to Section 3.

2. – Proof of Theorem 1.3.

We start by reducing the proof of Theorem 1.3 to the existence of a suitable fixed point. We assume that v_0 and g_0 (*fixed once for all*) verify (1.9), (1.10), and we define ζ_0 and g_1 by (1.3). Consider functions ϕ, q, ϑ. enjoying the following properties:

$$\partial_t^j\phi\in L_T^\infty(H^{2-j}),\ \ j=0,1,2;\qquad \nabla\cdot\phi = 0\ \text{ in } Q_T;\qquad \phi(0)=\zeta_0, \tag{2.1}$$

$$\partial_t^j q\in L_T^\infty(H^{3-j}),\ \ j=0,1;\qquad q(0)=g_0, \tag{2.2}$$

$$\partial_t^j\vartheta\in L_T^\infty(H^{2-j}),\ \ j=0,1,2;\qquad \vartheta(0)=\delta_0\equiv\nabla\cdot v_0, \tag{2.3}$$

$$\|q\|_{2,T} + \sum_{j=0}^{1}(\|\partial_t^j\phi\|_{1-j,T} + \|\partial_t^j\vartheta\|_{1-j,T}) \leq A, \tag{2.4}$$

$$\sum_{j=0}^{1}(\|\partial_t^j\phi\|_{2-j,T} + \|\partial_t^j\vartheta\|_{2-j,T} + \|\partial_t^j q\|_{3-j,T}) \leq B, \tag{2.5}$$

$$\|\partial_t^j\phi\|_{0,T} + \|\partial_t^j\vartheta\|_{0,T} \leq C. \tag{2.6}$$

The values of the positive constants A, B, C, and T, will be fixed later on. For the time being we only assume that A and B are larger than $\|\zeta_0\|_2 + \|g_0\|_3 + \|\delta_0\|_2$. This assumption guarantees that the set $\boldsymbol{K}$ (see below) is not empty. Set, for convenience, $X = H^0 \times \tilde{H}^1 \times H^0$, $\mathcal{X} = C_T(X)$, and define

$$\boldsymbol{K} \equiv \boldsymbol{K}(A, B, C, T) = \{(\phi, q, \vartheta) \in \mathcal{X}: (2.1) \text{ to } (2.6) \text{ hold}\}. \tag{2.7}$$

LEMMA 2.1. – *$\boldsymbol{K}$ is a non-empty, convex, closed subset of $\mathcal{X}$.*

PROOF. – The two first assertions are obvious. Let us show that $\boldsymbol{K}$ is closed. For, assume that $(\phi_n, q_n, \vartheta_n) \to (\phi, q, \vartheta)$ in $\mathcal{X}$, where $(\phi_n, q_n, \vartheta_n) \in \boldsymbol{K}$ for $n \in \boldsymbol{N}$. In particular, $\phi_n \to \phi$ in $C_T(H^0)$, hence, $\partial_t^2 \phi_n \to \partial_t^2 \phi$ in $\mathcal{D}'_T(H^0)$, as $n \to \infty$. Since

$$L_T^\infty(H^0) = [L^1((H^0)')]' \quad (Y' \text{ denotes the strong dual of } Y),$$

a well known theorem (see, for instance [T] Theorem 4.61-A) guarantees that $\partial_t^j \phi_n \rightharpoonup \partial_t^j \phi$ weakly-* in $L_T^\infty(H^0)$. Moreover, $\|\partial_t^j \phi\|_{0,t} \leq \liminf_{n\to\infty} \|\partial_t^j \phi_n\|_{0,T}$. Similarly, $\partial_t^j \vartheta_n \rightharpoonup \partial_t^j \vartheta$ weakly-* in $L_T^\infty(H^0)$ and $\|\partial_t^j \vartheta\|_{0,T} \leq \liminf_{n\to\infty} \|\partial_t^j \vartheta_n\|_{0,T}$. It readily follows that (2.6) holds. A similar argument applies in connection with (2.4) and (2.5). ∎

We consider the following linear problems

$$\begin{cases} \nabla \times v = \phi, \quad \nabla \cdot v = \vartheta, & \text{in } Q_T, \\ v \cdot \nu = 0, \quad \text{on } \Sigma_T, \end{cases} \tag{2.8}$$

$$\begin{cases} \partial_t \zeta + (v \cdot \nabla)\zeta - (\zeta \cdot \nabla) v + \vartheta \zeta = 0, & \text{in } Q_T, \\ \zeta(0) = \zeta_0, \end{cases} \tag{2.9}$$

$$\begin{cases} (\partial_t + v \cdot \nabla)^2 g - \nabla(h(q)\nabla g) = \Sigma(D_i v_j)(D_j v_i), & \text{in } Q_T, \\ \partial_\nu g = 0, \quad \text{on } \Sigma_T, \\ g(0) = g_0, \quad (\partial_t g)(0) = g_1, \end{cases} \tag{2.10}$$

$$\delta = -(\partial_t g + v \cdot \nabla g), \tag{2.11}$$

and we define on $\boldsymbol{K}$ a map S as follows. For each $(\phi, q, \vartheta) \in \boldsymbol{K}(A, B, C, T)$ we solve the problem (2.8) (which gives v), the problem (2.9) (which gives ζ), the problem (2.10) (which gives g), and (2.11) (which gives δ).

We define the map S by setting

$$S(\phi, q, \vartheta) = (\zeta, g, \delta).$$

The following two lemmas will be proved in the sequel.

LEMMA 2.2. – *There are positive constants A, B, C, and T, such that $S(\boldsymbol{K}) \subset \boldsymbol{K}$. These constants depend only on d,*

$$d = \|g_0\|_3 + [v_0]_3 . \tag{2.12}$$

More precisely, A, B, and C depend non-decreasingly on d, and T non-increasingly on d.

LEMMA 2.3. – *In Lemma 2.2, T can be chosen in such a way that S turns to be a strict contraction of $\boldsymbol{K}$ into itself.*

PROOF OF THEOREM 1.3 (by assuming Lemmas 2.2 and 2.3). – By the contraction mapping principle, S has a fixed point $(\phi, q, \vartheta) = (\zeta, g, \delta) \in \boldsymbol{K}$. By seting $\phi = \zeta$, $q = g$, $\vartheta = \delta$, in equations (2.8) to (2.11), we get a solution of (1.15). Since $(\zeta, g, \delta) \in \boldsymbol{K}$, the functions g and $\partial_t g$ satisfy (1.11) and (due also to the Lemma 2.4 below) v, $\partial_t v$, and $\partial_t^2 v$ satisfy (1.11). The Lemma 2.6 below shows that $\partial_t^2 g$ satisfies (1.11).

The Lemmas 2.5, 2.6, and 2.7, and the corollary to the Lemma 2.4 guarantee that $\partial_t^j(\zeta, g, \delta) \in C_T(H^{2-j} \times H^{3-j} \times H^{2-j})$ and that $\partial_t^j v \in C_T(\tilde{H}^{3-j})$, for $j = 0, 1, 2$ (note C_T instead of L_T^∞). ■

In order to prove the Lemmas 2.2 and 2.3 we need some results on the solutions of the linear problems (2.8) to (2.11). These results will be established below (see Lemmas 2.4 to 2.7). We start by recalling the following well known result (see, for instance [BV2]). For the reader's convenience we sketch briefly the proof of the a priori estimate (2.14). Here, c denote numerical positive constants.

PROPOSITION 2.1. – *Assume that $v \in L_T^\infty(\tilde{H}^3) \cap C_T(\tilde{H}^2)$, $v \cdot \nu = 0$ on Σ_T, $f \in L_T^2(H^k)$, $\zeta_0 \in H^k$, where $k = 1$, 2, or 3, Then, there is a solution $\zeta \in C_T(H^k)$ of*

$$\begin{cases} \partial_t \zeta + (v \cdot \nabla) \zeta = f, & \text{in } Q_T, \\ \zeta(0) = \zeta . \end{cases} \tag{2.13}$$

Moreover,

$$\|\zeta\|_{k,T} \leq e^{c[v]_{3,T}T} \left(\|\zeta_0\|_k + \int_0^T \|F(t)\|_k \, dt \right). \tag{2.14}$$

PROOF. – The technique to get the a priori estimate (2.14) is standard. Let ∂^α denote a fixed derivative with respect to space variables, $|\alpha| \leq k$. By applying the operator ∂^α to both side of equation (2.13), by multiplying by $\partial^\alpha \zeta$, by integrating on $\boldsymbol{R}_+^3$, and by doing straightforward calculations, we show that

$$\frac{1}{2} \frac{d}{dt} \|\partial^\alpha \zeta\|^2 \leq c[v]_{3,T} \|\zeta\|_{|\alpha|}^2 + \|\partial^\alpha \zeta\| \, \|\partial^\alpha f\| .$$

By adding side by side all the above estimates for the multi-indices α such that $|\alpha| \leq k$, one gets

$$\frac{1}{2}\frac{d}{dt}\|\zeta\|_k^2 \leq c[v]_{3,T}\|\zeta\|_k^2 + \|\zeta\|_k\|f\|_k. \tag{2.15}$$

The estimate (2.14) follows now easily. We remark that if a solution ζ belongs to $L_T^\infty(H^k)$, the estimate (2.14) together with $\zeta \in C_T(H^{k-1})$ yields $\zeta \in C_T(H^k)$. This is proved by a standard argument.

It is convenient to adopt here the following convention. We denote by $\Psi(\cdot, \dots, \cdot)$ generic real, positive, non-increasing functions of each of the arguments. The arguments are some, or all, of the quantities d, A, B, C. For convenience we denote by $\overline{\Psi}$ functions Ψ that may depend on the four arguments, i.e., $\overline{\Psi} = \Psi(d, A, B, C)$. Distinct functions Ψ (or $\overline{\Psi}$) can be denoted by the same symbol. In particular, we can write $\overline{\Psi} + \overline{\Psi} = \overline{\Psi}$, $\Psi\overline{\Psi} = \overline{\Psi}$, and so on.

LEMMA 2.4. – *Let $(\phi, q, \vartheta) \in \boldsymbol{K}(A, B, C, T)$. There is a (unique) solution v of the linear elliptic problem (2.8) satisfying $\partial_t^j v \in L_T^\infty(\tilde{H}^{3-j})$, $j = 0,1,2$. One has $v(0) = v_0$. Furthermore, there are functions of type Ψ such that*

$$[v]_{2,T} + [\partial_t v]_{1,T} \leq \Psi(A), \tag{2.16}$$

$$\begin{cases} [v]_{3,T} + [\partial_t v]_{2,T} \leq \Psi(B), \\ [\partial_t^j v]_{1,T} \leq \Psi(C). \end{cases} \tag{2.17}$$

COROLLARY. – *If, moreover, $\partial_t^j(\phi, \vartheta) \in C_T(H^{2-j} \times H^{2-j})$, for $j = 0,1,2$, then $\partial_t^j v \in C_T(\tilde{H}^{3-j})$.*

The proof of the above results for the function v is easily done by applying well known results to the elliptic problem (2.8), for each fixed $t \in [0, T]$. By differentiating both sides of equations (2.8) with respect to t we get similar equations for $\partial_t v$ and for $\partial_t^2 v$, from which we prove the desired results for $\partial_t v$ and for $\partial_t^j v$.

LEMMA 2.5. – *Let ϕ, q, ϑ, and v be as in Lemma 2.4, and let ζ be the solution of problem (2.9). Then $\partial_t^j \zeta \in C_T(H^{2-j})$ for $j = 0,1,2$, morevoer, there are functions of types ψ and $\overline{\psi}$ such that*

$$\|\zeta\|_{1,T} + \|\partial_t \zeta\|_{0,T} \leq \psi(d)\, e^{T\overline{\Psi}}, \tag{2.18}$$

$$\begin{cases} \|\zeta\|_{2,T} + \|\partial_t \zeta\|_{1,T} \leq \Psi(d, A)\, e^{T\overline{\Psi}}, \\ \|\partial_t^j \zeta\|_{0,T} \leq \Psi(d, A, B). \end{cases} \tag{2.19}$$

Finally, $\nabla \cdot \zeta = 0$ on Q_T.

PROOF. – We start by proving that $\nabla\cdot\zeta = 0$ on Q_T. A direct computation show that

$$(v\cdot\nabla)\zeta - (\zeta\cdot\nabla)v = (\nabla\cdot\zeta)v - (\nabla\cdot v)\zeta - \nabla\times(v\times\zeta).$$

This formula, together with (2.8) and (2.9), shows that

$$\partial_t\zeta + (\nabla\cdot\zeta)v = \nabla\times(v\times\zeta).$$

Applying the divergence to both sides of this equation and using the vector calculus formulae

$$\nabla\cdot[(\nabla\cdot\zeta)v] = v\cdot\nabla(\nabla\cdot\zeta) + (\nabla\cdot v)(\nabla\cdot\zeta),$$

one gets

$$\begin{cases}\partial_t(\nabla\cdot\zeta) + v\cdot\nabla(\nabla\cdot\zeta) + \vartheta(\nabla\cdot\zeta) = 0, & \text{in } Q_T,\\ \nabla\cdot\zeta(0) = \nabla\cdot\zeta_0 = 0.\end{cases}$$

This transport equation shows that $\nabla\cdot\zeta = 0$ on Q_T.

The remaining assertions are proved by arguing as in Proposition 2.1. Since here $\|f\|_2 \le c[v]_{3,T}\|\zeta\|_2$, we can drop the last term on the right hand side of (2.15). Consequently, we obtain the equation (2.14) without the integral term. In particular

$$\|\zeta\|_{2,T} \le \Psi(d)\,e^{T\Psi(B)}. \tag{2.20}$$

From equations $(2.9)_1$ and from (2.20) it follows that $\|\partial_t\zeta\|_1 \le c[v]_2\|\zeta\|_2$, for each $t\in[0,T]$. Hence

$$\|\partial_t\zeta\|_{1,T} \le \Psi(d,A)\,e^{T\Psi(B)}. \tag{2.21}$$

From (2.20) and (2.21) we get, in particular, $(2.19)_1$.

By differentiation with respect to t of the equation $(2.9)_1$ and by using (2.20) and (2.21), one shows that $\|\partial_t^2\zeta\|_{0,T} \le \Psi(d,A,B)$. Hence, $(2.19)_2$ holds.

Let us prove (2.18). Take the derivative with respect to t of both sides of equation $(2.9)_1$, multiply by $\partial_t\zeta$ and integrate over $\boldsymbol{R}^3_+$. It readily follows that

$$\frac{1}{2}\frac{d}{dt}\|\partial_t\zeta\|^2 \le \Psi(d,A,B)\|\partial_t\zeta\|^2,$$

for each $t\in[0,T]$. Use, in particular, the estimate $(2.19)_2$. Since $\|\partial_t\zeta(0)\| \le \Psi(d)$, one has $\|\partial_t\zeta\|_{0,T} \le \Psi(d)\exp[T\Psi(d,A,B)]$. This estimate, together with (2.20), proves (2.18). The continuity of $\partial_t\zeta$ with values in H^1 and that of $\partial_t^2\zeta$ with values in H^0 follow easily from the expression of these derivatives (obtained from $(2.9)_1$), since $\zeta\in C_T(H^2)$, $v\in C_T(\tilde{H}^2)$, and $\partial_t v\in C_T(\tilde{H}_1)$.

LEMMA 2.6. – *Let ϕ, p, q, and v be as in Lemma 2.4. Then, the problem (2.10) has a (unique) solution g satisfying $\partial_t^j g\in C_T(H^{3-j})$, $j=0,1,2$. Moreover there is a function*

$\overline{\Psi}_0$ *such that if*

$$T\overline{\Psi}_0 \leqslant 1, \tag{2.22}$$

then

$$\|g\|_{2,T} \leqslant \Psi(d) + T\overline{\Psi}, \tag{2.23}$$

and

$$\sum_{j=0}^{2} \|\partial_t^j g\|_{3-j,T} \leqslant \Psi(d, A) + T^{1/2}\overline{\Psi}. \tag{2.24}$$

The proof of Lemma 2.6 is done below, together with that of Lemma 2.7.

LEMMA 2.7. – *Let* ϕ, p, q, *and* v *be as in Lemma* 2.4 *and* g *be as in Lemma* 2.6. *Then, the function* δ *defined by equation* (2.11) *satisfies* $\partial_t^j \delta \in C_T(H^{2-j})$, $j = 0, 1, 2$. *Moreover, under the hypothesis* (2.22), *and for suitable functions* Ψ *and* $\overline{\Psi}$,

$$\|\delta\|_{1,T} + \|\partial_t \delta\|_{0,T} \leqslant \Psi(d) + T\overline{\Psi}, \tag{2.25}$$

and

$$\begin{cases} \|\delta\|_{2,T} + \|\partial_t \delta\|_{1,T} \leqslant \Psi(d, A) + T^{1/2}\overline{\Psi}, \\ \|\partial_t^2 \delta\|_{0,T} \leqslant \Psi(d, A, B). \end{cases} \tag{2.26}$$

PROOF OF LEMMAS 2.6 AND 2.7. – We start by proving (2.24). Equation (2.10) has the form (3.1) by setting in (3.1) $l = h(q)$ and $f = \Sigma(\partial_i v_j)(\partial_j v_i)$. Hence, the estimate (3.18) applies to the solution of (2.10). In this case, the equations (1.13) and (2.12) show that $\|g_1\|_2 \leqslant \Psi(d)$ and that $\|f(0)\|_1 \leqslant \Psi(d)$, for suitable functions $\Psi(\cdot)$.

Moreover, under the assumptions of Lemma 2.6, polynomials of type P turn out to be functions of type $\Psi(d, A)$. It is sufficient to verify that the arguments of P are functions of type $\Psi(A)$. Let us start by m. By (2.4), $|q|_{\infty,T} \leqslant c\|q\|_{2,T} \leqslant cA$. Hence, $l(t, x) = h(q(t, x)) \geqslant \inf\{h(s): |s| \leqslant cA\} \equiv [\Psi(A)]^{-1}$. Since $h(s) > 0$ for all $s \in \boldsymbol{R}$, the last equality defines, in fact, $\Psi(A)$ as a positive nondecreasing function of A. Hence, the first inequality in equation (3.4) holds for $l = h(q)$ and $m = \Psi(A)$. The assumption $\|q\|_{2,T} \leqslant A$ (cf. (2.4)) and equation (2.16) easily show that the remaining arguments of the polynomials P are, in fact, functions of type $\Psi(A)$.

On the other hand, the particular form of f together with (2.17) show that the integral on the right hand side of (3.18) is bounded by $T\Psi(B)$, for a suitable $\Psi(\cdot)$. The facts proved until now show that, under the assumptions of Lemma 2.6, the estimate (3.18) turns into (2.24). Finally, we must show that the condition (3.19) turns into (2.22). In other words, we must prove that $R \leqslant \Psi(d, A, B, C)$. This is easily done by showing that all the argument of R are, at most, of type $\Psi(d, A, B, C)$. This follows immediately from the result proved above for P, from (2.17), and from the assumptions (2.5) on q (together with the C^3 regularity of $h(\cdot)$). ■

In the sequel it is well understood that the assumption (2.22) holds. Let us prove (2.26). From (2.11), by taking into account (2.24) and (2.16), it readily follows that $\|\delta\|_{2,T}$ satisfies $(2.26)_1$. Moreover, by taking the derivatives of both sides of (2.11) with respect to t it readily follows that

$$\|\partial_t \delta\|_{1,T} \leq \|\partial_t^2 g\|_{1,T} + [\partial_t v]_{1,T}\|g\|_{3,T} + [v]_{2,T}\|\partial_t g\|_{2,T}.$$

By using (2.24) and (2.16) one proves that $\|\partial_t \delta\|_{1,T}$ satisfies $(2.26)_1$. Now, we prove $(2.26)_2$. From $(2.10)_1$ and (2.11) one gets

$$\partial_t \delta = -v\cdot\nabla\delta + \nabla\cdot(h(q)\nabla g) + \Sigma(\partial_i v_j)(\partial_j v_i). \tag{2.27}$$

Consequently,

$$\partial_t(\partial_t\delta) = -\partial_t v\cdot\nabla\delta - v\cdot\partial_t\nabla\delta + \partial_t[h'(q)\nabla q\cdot\nabla g + h(q)\nabla g] + 2\Sigma(\partial_t\partial_i v_j)(\partial_j v_i). \tag{2.28}$$

By taking into account (2.17), (2.24), $(2.26)_1$, (2.2), and the estimates $|h'(q)|_{\infty,T} \leq \Psi(A)$, $|h''(q)|_{\infty,T} \leq \Psi(A)$, it readily follows that the $\|\cdot\|_{0,T}$ norm of the right hand side of (2.28) is bounded by $\overline{\Psi}$. Hence $(2.26)_2$ holds.

Since $\partial_t g + v\cdot\nabla g = -\delta$, one gets

$$\frac{d}{dt}\|g(t)\|_2 \leq c([v]_{3,T}\|g(t)\|_2 + c\|\delta\|_{2,T}),$$

which yields (2.23). Finally, we prove (2.25). From the above estimates it follows that the $\|\cdot\|_{1,T}$ norm of the right hand side of equation (2.27) is bounded by $\overline{\Psi}$. Since $\|\delta(0)\|_1 \leq \Psi(d)$, $\|\delta\|_{1,T}$ satisfies (2.25). We have already shown that the $\|\cdot\|_{0,T}$ norm of the right hand side of equation (2.28) is bounded by $\overline{\Psi}$. Moreover, the initial data $\partial_t\delta(0)$ (obtained by setting $t=0$ in equation (2.27)) satisfies $\|\partial_t\delta(0)\| \leq \Psi(d)$. A standard argument applied to the transport equation (2.28) shows that $\|\partial_t\delta\|_{0,T}$ satisfies (2.25). ■

Proof of Lemma 2.2. – From Lemmas 2.5, 2.6, and 2.7 it follows that, for suitable function of types Ψ and $\overline{\Psi}$,

$$\|g\|_{2,T} + \sum_{j=0}^{1}(\|\partial_t^j\zeta\|_{1-j,T} + \|\partial_t^j\delta\|_{1-j,T}) \leq \Psi_1(d) + T\overline{\Psi}_1, \tag{2.29}$$

$$\sum_{j=0}^{1}(\|\partial_t^j\zeta\|_{2-j,T} + \|\partial_t^j g\|_{3-j,T} + \|\partial_t^j\delta\|_{2-j,T}) \leq \Psi_2(d, A) + T^{1/2}\overline{\Psi}_2, \tag{2.30}$$

and

$$\|\partial_t^2\zeta\|_{0,T} + \|\partial_t^2\delta\|_{0,T} \leq \Psi_3(d, A, B), \tag{2.31}$$

if

$$T\overline{\Psi}_0 \leq 1. \tag{2.32}$$

We dropped from the above formulas the terms $e^{T\Psi}$ since, by (2.32), $T\Psi \leq 1$. Note that we can replace $\overline{\Psi}_0 + \Psi$ by $\overline{\Psi}_0$, since both are functions of type $\overline{\Psi}$.

The lemma is proved if we show that A, B, C, T, can be chosen in such a way that $\Psi_1(d) + T\overline{\Psi}_1 \leq A$; $\Psi_2(d, A) + T^{1/2}\overline{\Psi}_2 \leq B$; $\Psi_3(d, A, B) \leq C$; and $T\overline{\Psi}_0 \leq 1$. We start by fixing $A = 1 + \Psi_1(d)$, and we impose to T the condition $T\overline{\Psi}_1 \leq 1$. This condition is included in (2.32), by replacing $\overline{\Psi}_0$ by $\overline{\Psi}_0 + \overline{\Psi}_1$. Hence, $\Psi_1(d) + T\overline{\Psi}_1 \leq A$. Now, we set $B = 1 + \Psi_2(d, A)$, and we impose to T the condition $T^{1/2}\overline{\Psi}_2 \leq 1$ (included in (2.32)). Hence, $\Psi_2(d, A) + T^{1/2}\overline{\Psi}_2 \leq B$. Then, we set $C = \Psi_3(d, A, B)$. Finally, we chose T such that $T\overline{\Psi}_0(d, A, B, C) \leq 1$. ∎

PROOF OF LEMMA 2.3. – The values of the constants A, B, C, are that fixed one the proof of Lemma 2.2. Hence, for suitable functions Ψ, one has $A = \Psi(d)$, $B = \Psi(d)$, $C = \Psi(d)$. Consequently, every constant of type $\Psi(d, A, B, C)$ is now simply of type $\Psi(d)$. In the sequel we assume that T obeys to condition (2.32) (which can be written in the form $\Psi_0(d)\,T \leq 1$) and we show that there is a function $\Psi(d)$ such that if $\Psi(d)\,T \leq 1$, then S is a contraction.

For convenience, constants of type $\Psi(d)$ will be denoted simply by c. Also numerical positive constants will be denoted by c. The symbol c may denote distinct constants, even in the same equation.

Let (ϕ, q, ϑ) and $(\overline{\phi}, \overline{q}, \overline{\vartheta})$ belong to $\boldsymbol{K}$ and set $(\zeta, g, \delta) = S(\phi, q, \vartheta)$, $(\overline{\zeta}, \overline{g}, \overline{\delta}) = S(\overline{\phi}, \overline{q}, \overline{\vartheta})$. Since these elements belong to $\boldsymbol{K}$, all the norms considered below are bounded by constants $c = \Psi(d)$. In the calculations that follow the reader also should take into account the embeddings (0.1), (0.2), (0.3). Note, in particular, that $\|fg\| \leq |f|_6\,|g|_3 \leq c[f]_1\,\|g\|_1$.

From $(2.10)_1$ and (2.11) one gets $\partial_t\delta + v\cdot\nabla\delta - \nabla\cdot(h(q)\nabla g) = \Sigma(\partial_i v_j)(\partial_j v_i)$. By taking the difference (side by side) between this equation and the corresponding equation for $\overline{\delta}, \overline{q}, \overline{g}, \overline{v}$, by multiplying both sides of the equation obtained by $\delta - \overline{\delta}$, and by integrating over $\boldsymbol{R}^3_+$ it readily follows that

$$\frac{1}{2}\frac{d}{dt}\|\delta - \overline{\delta}\|^2 + \int h(q)\,\nabla(g - \overline{g})\cdot\nabla(\delta - \overline{\delta}) \leq \tag{2.33}$$

$$\leq c\|\delta - \overline{\delta}\|(\|\delta - \overline{\delta}\| + [v - \overline{v}]_1 + [g - \overline{g}]_1 + [q - \overline{q}]_1)\,,$$

for each $t \in [0, T]$. On the other hand, by taking into account that $\delta - \overline{\delta} = \partial_t(g - \overline{g}) + v\cdot\nabla(g - \overline{g}) + (v - \overline{v})\cdot\nabla\overline{g}$, and that $\nabla(g - \overline{g})\cdot\nabla[v\cdot\nabla(g - \overline{g})] = \Sigma(\partial_j v_i)\,\partial_i(g - \overline{g})\,\partial_j(g - \overline{g}) + (1/2)\,v\cdot\nabla|\nabla(g - \overline{g})|^2$, it readily follows that

$$\int h(q)\,\nabla(g - \overline{g})\cdot\nabla(\delta - \overline{\delta}) \geq \tag{2.34}$$

$$\geq \frac{1}{2}\frac{d}{dt}\int (h(q)\,|\nabla(q - \overline{q})|^2 - c[g - \overline{g}]_1^2 - c[g - \overline{g}]_1\,[v - \overline{v}]_1)\,.$$

Since $|q|_\infty \leq c_1 A$ (by (2.4)), where c_1 is a Sobolev's embedding constant, one gets

$h[q(t, z)] \geq \min_{|s| \leq c_1 A} h(s) \equiv \Psi(A)^{-1}$ (definition of $\Psi(A)$). This shows that

$$[g - \bar{g}]_1^2 \leq c \int h(q) |\nabla(g - \bar{g})|^2 .$$

Consequently, (2.33) and (2.34) yield

$$\frac{1}{2}\frac{d}{dt}(\|\delta - \bar{\delta}\|^2 + \|h(q)^{1/2}\nabla(g - \bar{g})\|^2) \leq$$

$$\leq c(\|\delta - \bar{\delta}\| + \|h(q)^{1/2}\nabla(g - \bar{g})\|)\cdot(\|\delta - \bar{\delta}\| + \|h(q)^{1/2}\nabla(g - \bar{g})\| + [v - \bar{v}]_1 + [q - \bar{q}]_1).$$

Since $\delta(0) - \bar{\delta}(0) = g(0) - \bar{g}(0) = 0$, it readily follows that

$$\|\delta - \bar{\delta}\|_{0,T} + [g - \bar{g}]_{1,T} \leq cT([q - \bar{q}]_{1,T} + [v - \bar{v}]_{1,T}). \tag{2.35}$$

Now, we take the difference, side by side, between the equation $(2.9)_1$ and the corresponding equation for $\bar{\zeta}$, $\bar{v}$, $\bar{\vartheta}$. Then, we multiply by $\zeta - \bar{\zeta}$ both sides of the equation obtained and we integrate over $\boldsymbol{R}_+^3$. Straighforward calculations show that

$$\|\zeta - \bar{\zeta}_0\|_{0,T} \leq cT([v - \bar{v}]_1 + \|\vartheta - \bar{\vartheta}\|). \tag{2.36}$$

By using (2.35) and (2.36), and by taking into account that $[v - \bar{v}]_1 \leq c(\|\phi - \bar{\phi}\| + \|\vartheta - \bar{\vartheta}\|)$, one gets

$$\|\zeta - \bar{\zeta}\|_{0,T} + [g - \bar{g}]_{1,T} + \|\delta - \bar{\delta}\|_{0,T} \leq cT(\|\phi - \bar{\phi}\|_{0,T} + [q - \bar{q}]_{1,T} + \|\vartheta - \bar{\vartheta}\|_{0,T}),$$

where $c = \Psi(d)$. The map S is a strict contraction if T satisfies the inequality $2cT \leq 1$. ∎

PROOF OF THE COROLLARY TO THEOREM 1.2. – Let us go on assuming that $v_0 \in \tilde{H}_3$. Equation $(1.8)_1$ immediately shows that $\partial_t v \in C_T(H^0)$. Moreover, by (1.11), it readily follow that $\|\partial_t v\|_{0,T} \leq c$. Corresponding results hold for $\partial_t^2 v$ and $\partial_t^3 v$, by differentiation of equation $(1.8)_1$ with respect to t. By differentiation of equation $(1.8)_2$ with respect to t we show that $\partial_t^3 g \in C_T(H^0)$ and we get the estimate $\|\partial_t^3 g\|_{0,T} \leq c$.

Assume now, in addition, that $v_0 \in H^0$. Since $v(t) = v_0 + \int_0^t (\partial_t v)(\tau)\, d\tau$, and $\partial_t v \in C_T(H^0)$, it follows that $v \in C_T(H^3)$ and that $\|v\|_{0,T} \leq c$. ∎

3. – On a linear second order hyperbolic equation.

This section is dedicated to the study of the linear hyperbolic mixed problem

$$\begin{cases} (\partial_t + v\cdot\nabla)^2 g - \nabla\cdot(l\nabla g) = f, & \text{in } Q_T, \\ \partial_x g = 0, \quad \text{on } \Sigma_T, \\ g(0) = g_0, \quad \partial_t g(0) = g_1, \end{cases} \tag{3.1}$$

which plays a main rule in the study of the nonlinear problem (1.8). The main results are Theorems 3.1 and 3.2 below. We are interested on proving sharp estimates for the solution g in terms of the data and, specially, in terms of the coefficients v and l. Methods to construct the solution, departing from the a priori estimates, are well know. See, for instance [M1], [M2], and references therein.

We will mention, together with the estimates, the conditions on the data which are sufficient to prove the existence of the solution. In order to simplify the notations and to make fluent the reading we do not write the estimates in the sharper form, but just in the form needed to apply them to our nonlinear problem.

The estimates are proved under the assumption (3.19). We can get ride of this assumption by using weighted L^2-norms $e^{-\gamma t}\|\cdot\|$. However this gives no advantage in the application to our nonlinear problem. For convenience, we assume that $T \leqslant 1$.

In this section the coefficients v and l are as follows

$$\partial_t^j v \in L_T^\infty(\bar{H}^{3-j}), \qquad j = 0, 1, 2; \qquad v \cdot \nu = 0 \text{ on } \Sigma_T, \tag{3.2}$$

$$\nabla l \in L_T^\infty(H^2), \qquad \partial_t l \in L_T^\infty(H^2), \tag{3.3}$$

$$m^{-1} \leqslant l(t, x) \leqslant M, \qquad \text{on } Q_T. \tag{3.4}$$

We use the following notation. P, Q, and R, denote *polynomials with non-negative, costant coefficients* that depend *at most* on the arguments indicated below

$$P = P(m, M, [v]_{2,T}, [\partial_t v]_{1,T}, \|\nabla l\|_{1,T}),$$

$$Q = Q(m, M, [v]_{3,T}, [\partial_t v]_{2,T}, \|\nabla l\|_{2,T}, \|\partial_t l\|_{2,T}),$$

$$R = R(m, M, [v]_{3,T}, [\partial_t v]_{2,T}, [\partial_t^2 v]_{1,T}, \|\nabla l\|_{2,T}, \|\partial_t l\|_{2,T}).$$

We use the same symbol P (or Q, or R) to denote distinct polynomials, even in the same equation. Hence, we are allowed to write $P + P = P$, $PQ = Q$, and so on.

We start by proving a priori estimates of order zero for the Cauchy-Neumann problem (3.1) and for the Cauchy-Dirichlet problem

$$\begin{cases} (\partial_t + v \cdot \nabla)^2 g - \nabla \cdot (l \nabla g) = f, & \text{in } Q_T, \\ g = 0, & \text{on } \Sigma_T, \\ g(0) = g_0, & (\partial_t g)(0) = g_1. \end{cases} \tag{3.5}$$

We set

$$-\delta = \partial_t g + v \cdot \nabla g. \tag{3.6}$$

LEMMA 3.1. – *Let $g_0 \in H^1$, $g_1 \in H^0$, $f \in L_T^2(H^0)$, and let g satisfy* (3.1). *Then*

$$\int (\delta^2 + lg^2 + l|\nabla g|^2)(t) \le e^{QT} \int (\delta^2 + lg^2 + l|\nabla g|^2)(0) + e^{QT} \int_0^t \|f(s)\|^2 ds, \tag{3.7}$$

in $[0, T]$. *This estimate is also verified by the solution g of* (3.5). *In this case, we assume that $g_0 \in H_0^1$.*

PROOF. – By multiplying both sides of $(3.1)_1$ by δ and by integrating on $\boldsymbol{R}_+^3$, one easily shows that the estimate

$$\frac{1}{2}\frac{d}{dt}\int \delta^2 - \int l\nabla g \cdot \nabla \delta = -\int f\delta, \tag{3.8}$$

holds in both the cases. Note that, for the Dirichlet problem, $\delta = 0$ on Σ_T. By using (3.6) it readily follows that

$$-\int l\nabla g \cdot \nabla \delta = \frac{1}{2}\frac{d}{dt}\int l|\nabla g|^2 - \frac{1}{2}\int (\partial_t l)|\nabla g|^2 - \frac{1}{2}\int [\nabla \cdot (lv)]|\nabla g|^2 + \int l\Sigma(\partial_i v_j)(\partial_i g)(\partial_j g).$$

Hence, from (3.8), we get

$$\frac{1}{2}\frac{d}{dt}\int (\delta^2 + l|\nabla g|^2) \le Q\int l|\nabla g|^2 - \int f\delta.$$

On the other hand,

$$\frac{1}{2}\frac{d}{dt}\int lg^2 = \frac{1}{2}\int (\partial_t l)g^2 - \int l\delta g - \frac{1}{2}\int lv\nabla g^2.$$

Consequently,

$$\frac{1}{2}\frac{d}{dt}\int (\delta^2 + lg^2 + l|\nabla g|^2) \le Q\int (\delta^2 + lg^2 + l|\nabla g|^2) + \int f^2. \quad \blacksquare$$

THEOREM 3.1. – *Assume that the conditions* (3.2), (3.3), *and* (3.4) *hold and that $f \in L_T^2(H^1)$, $g_0 \in H^2$, $\partial_x g_0 = 0$ on Γ, and $g_1 \in H^1$. Let g be the solution of* (3.1). *Then there are polynomials P and Q such that*

$$\|g\|_{2,T}^2 + \|\partial_t g\|_{1,T}^2 \le P(\|g_0\|_2^2 + \|g_1\|_1^2) + c\int_0^T \|f(s)\|_1^2 dS, \tag{3.9}$$

if

$$QT \le 1. \tag{3.10}$$

PROOF. – We use the notation $F_x = \partial_x F$, $F_{y_i} = \partial_{y_i} F$, and so on. We denote by y any of the variables y_1 or y_2, and by z any of the variables y_1, y_2, or x. By taking the derivative of both sides of equation $(3.1)_1$ with respect to a space variable, we get

$$(3.11)\qquad \begin{cases} (\partial_t + v\cdot\nabla)^2 g_z - \nabla\cdot(l\nabla g_z) = F_{(z)}, & \text{in } Q_T,\\ \partial_x g_y = 0 \quad \text{on } \Sigma_T \text{ if } z = y \text{ [resp., } g_x = 0 \text{ on } \Sigma_T \text{ if } z = x], \\ g_z(0) = g_{0,z}, \quad \partial_t g_z(0) = g_{1,z}, \end{cases}$$

where

$$(3.12)\qquad F_{(z)} = f_z - (\partial_t + v\cdot\nabla)(v_z\cdot\nabla g) - (v_z\cdot\nabla)(\partial_t g + v\cdot\nabla g) + \nabla\cdot(l_z\nabla g).$$

Note that $g_{0,x}\in H_0^1$. The function g_y [resp. g_x] is the solution of the Cauchy-Neumann [resp. Cauchy-Dirichlet] problem (3.11). Set

$$(3.13)\qquad -\delta_{(z)} = \partial_t g_z + v\cdot\nabla g_z.$$

Equation (3.7) yields

$$(3.14)\qquad \int(\delta_{(z)}^2 + lg_z^2 + l|\nabla g_z|^2)(t) \leqslant e^{QT}\int(\delta_{(z)}^2 + lg_z^2 + l|\nabla g_z|^2)(0) + e^{QT}\int_0^T \|F_{(z)}(s)\|^2\,ds,$$

on $[0,T]$. Straightforward calculations show that

$$\|F_{(z)}(t)\| \leqslant \|f(t)\|_1 + Q(\|g(t)\|_2 + \|\partial_t g(t)\|_1).$$

Hence, from (3.14) we get the estimate

$$(3.15)\qquad \int(\delta_{(z)}^2 + lg_z^2 + l|\nabla g_z|^2)(t) \leqslant$$

$$\leqslant e^{QT}\int(\delta_{(z)}^2 + lg_z^2 + l|\nabla g_z|^2)(0) + e^{QT}\int_0^t \|f(s)\|_1^2\,ds + Qe^{QT}\int_0^t (\|g(s)\|_2^2 + \|\partial_t g(s)\|_1^2)\,ds.$$

Now, we want to obtain estimates for $\|g\|_{2,T}$ and $\|\partial_t g\|_{1,T}$. Since

$$\delta^2 + l|\nabla g|^2 = (\partial_t g)^2 + 2(v\cdot\nabla g)(\partial_t g) + |v\cdot\nabla g|^2 + l|\nabla g|^2,$$

it readily follows that, for every $\varepsilon\in\,]0,1[$,

$$\delta^2 + l|\nabla g|^2 \geqslant (1-\varepsilon)(\partial_t g)^2 + \left[m^{-1} - \frac{1-\varepsilon}{\varepsilon}|v|_\infty\right]|\nabla g|^2.$$

By setting $\varepsilon = 2|v|_\infty/(2|v|_\infty + m^{-1})$, one obtains, in particular,

$$|\partial_t g|^2 + |\nabla g|^2 \leqslant P(\delta^2 + l|\nabla g|^2).$$

Hence, from (3.7), it follows that

$$\|g\|_{1,T}^2 + \|\partial_t g\|_{0,T}^2 \leq Pe^{QT}\|g_0\|_2^2 + \|g_1\|_1^2 + Pe^{QT}\int_0^T \|f(s)\|^2 ds . \tag{3.16}$$

On the other hand, by replacing $\delta^2 + l|\nabla g|^2$ by $\delta_{(z)}^2 + l|\nabla g_z|^2$ and by using (3.15) instead of (3.7), one gets

$$\|g_z\|_{1,T}^2 + \|\partial_t g_z\|_{0,T}^2 \leq$$
$$\leq Pe^{QT}\left(\|g_0\|_2^2 + \|g_1\|_1^2 + \int_0^T \|f(s)\|_1^2 ds\right) + Qe^{QT}\int_0^T (\|g(s)\|_2^2 + \|\partial_t g(s)\|_1^2)\, ds , \tag{3.17}$$

either for $z = x$ and for $z = y_i$, $i = 1, 2$. In particular (3.9) follows if T satisfies (3.10), for a suitable polynomial Q. ■

We point out that the above estimates allows us to prove that a L_T^∞ solution of (3.1) must belong to C_T.

LEMMA 3.2. – *Assume that the hypothesis stated in Theorem* 3.1 *hold and let g be a solution of problem* (3.1) *in the class $g \in L_T^\infty(H^2)$, $\partial_t g \in L_T^\infty(H^1)$. Then, $g \in C_T(H^2)$ and $\partial_t g \in C_T(H^1)$.*

The proof follows a well known technique. In order to avoid a deviation from the main lines, we postoponed the discussion of the above result to Appendix A.

Now, we prove our main result.

THEOREM 3.2. – *Assume that the conditions* (3.2), (3.3), *and* (3.4) *hold, and that $f \in L_T^2(H^2)$, $\partial_t f \in L_T^2(H^1)$, $g_0 \in H^3$, $g_1 \in H^2$, $\partial_x g_0 = \partial_x g_1 = 0$ on Σ_T. Let g be the solution of* (3.1). *There are polynomials P and R such that*

$$\sum_{j=0}^{2} \|\partial_t^j g\|_{3-j,T}^2 \leq P(\|g_0\|_3^2 + \|g_1\|_2^2 + \|f(0)\|_1^2) + P\int_0^T (\|f(s)\|_2^2 + \|\partial_t f(s)\|_1^2)\, ds , \tag{3.18}$$

provided that

$$RT \leq 1 . \tag{3.19}$$

PROOF. – By taking the derivative with respect to t of both sides of equation $(3.1)_1$ and by setting $g_t = \partial_t g$, one gets

$$\begin{cases} (\partial_t + v\cdot\nabla)^2 g_t - \nabla\cdot(l\nabla g_t) = F_{(t)}, & \text{in } Q_T, \\ \partial_x g_t = 0, \quad \text{on } \Sigma_T, \\ g_t(0) = g_1, \quad \partial_t g_t(0) = g_2, \end{cases} \tag{3.20}$$

where the expression of $F_{(t)}$ is obtained by replacing z by t in equation (3.12). Tedious but straightforward calculations show that

$$\|F_{(t)}\|_1 \leq \|\partial_t f\|_1 + R(\|g\|_3 + \|\partial_t g\|_2). \tag{3.21}$$

almost everywhere in $[0, T]$, for a suitable R. In particular, $F_{(t)} \in L^2_T(H^1)$. On the other hand, by setting $t = 0$ in equation $(3.1)_1$, one easily prove that

$$\|g_2\|_1 \leq \|f(0)\|_1 + P(\|g_0\|_3 + \|g_1\|_2). \tag{3.22}$$

By applying Theorem 3.1 to the solution g_t of the Cauchy-Neumann problem (3.20) we prove that

$$\|g_t\|^2_{2,T} + \|\partial_t g_t\|^2_{1,T} \leq \tag{3.23}$$

$$\leq P(\|g_0\|^2_3 + \|g_1\|^2_2 + \|f(0)\|^2_1) + c\int_0^T \|\partial_t f(s)\|^2_1 ds + RT(\|g\|^2_{3,T} + \|\partial_t g\|^2_{2,T}),$$

if (3.10) holds. Lemma 3.2 shows that

$$g_t \in C_T(H^2), \qquad \partial_t g_t \in C_T(H^1). \tag{3.24}$$

On the other hand g_{y_1} and g_{y_2} (both denoted here by g_y) are solutions of the Cauchy-Neumann problem (3.11). Note that $\partial_x(\partial_y g_0) = 0$ on Σ_T. Straightforward calculations show that

$$\|F_{(y)}\|_1 \leq \|f\|_2 + Q(\|g\|_3 + \|\partial_t g\|_2), \tag{3.25}$$

a.e. in $[0, T]$. Again by Theorem 3.1, one gets

$$\|g_y\|^2_{2,T} + \|\partial_t g_y\|^2_{1,T} \leq P(\|g_0\|^2_3 + \|g_1\|^2_2) + c\int_0^T \|f(s)\|^2_2 ds + QT(\|g\|^2_{3,T} + \|\partial_t g\|^2_{2,T}), \tag{3.26}$$

if (3.10) holds. Moreover, by Lemma 3.2,

$$g_y \in C_T(H^2), \qquad \partial_t g_y \in C_T(H^1). \quad \blacksquare \tag{3.27}$$

It remains to estimate the $L^\infty_T(H^0)$ norm of $\partial^3_x g$. If it were no boundary, the estimate (3.26) would hold also for g_x and (3.23) would be superfluous. Nevertheless, in the presence of a boundary, the argument developed above for g_y can be applied to g_x only in order to prove an interior estimate. In fact, by using a suitable cut-off function $\vartheta(x)$, we will obtain the desired estimate on $\boldsymbol{R}^3_+ \setminus S$, where S is a neighbourhood of the boundary Γ. The width r of S will depend on the particular coefficients v and l (a crucial point for the application to our nonlinear problem is that $r \leq P^{-1}$, for a suitable polynomial of type P). Finally, by solving (algebraically) the equation $(3.1)_1$ for $\partial^2_x g$ we are able to estimate $\partial^3_3 g$ on the slab S.

Let us start from the interior estimate. We define r by (3.35) below and we fix a real nonnegative function $\Theta \in C^\infty(\boldsymbol{R}^+)$ such that $\Theta(x) = 0$ if $x \in [0, 1/2]$ and $\Theta(x) = 1$ if

$x \geqslant 1$. We set $\vartheta(x) = \Theta(x/r)$. From $(3.1)_1$ we get

$$(\partial_t + v \cdot \nabla)^2 (\vartheta g) - \nabla \cdot [l \nabla (\vartheta g)] = \vartheta f + H[\vartheta, l, v, g] \tag{3.28}$$

in Q_T, where H is the «commutator»

$$H = (\partial_t + v \cdot \nabla)^2 (\vartheta g) - \vartheta (\partial_t + v \cdot \nabla)^2 g - \nabla \cdot [l \nabla (\vartheta g)] + \vartheta \nabla \cdot (l \nabla g).$$

We left to reader the straightforward calculations leading to a more explicit expression for H.

Now, we take the derivatives of both sides of (3.28) with respect to x, and we associate boundary and initial conditions to the equation obtained in this way. One gets

$$\begin{cases} (\partial_t + v \cdot \nabla)^2 (\vartheta g)_x - \nabla \cdot [l \nabla (\vartheta g)_x] = \partial_x (\vartheta f) + \partial_x H + G, & \text{on } Q_T, \\ \partial_x (\vartheta g)_x = 0, \quad \text{on } \Sigma_T, \\ (\vartheta g)_x (0) = (\vartheta g_0)_x, \quad \partial_t (\vartheta g)_x (0) = (\vartheta g_1)_x, \end{cases} \tag{3.29}$$

where G is the «commutator»

$$G[\vartheta, l, v, g] = \left(\frac{\partial}{\partial t} + v \cdot \nabla \right)^2 \partial_x (\vartheta g) - \partial_x (\partial_t + v \cdot \nabla)^2 (\vartheta g) - \nabla \cdot [l \nabla \partial_x (\vartheta g)] + \partial_x \nabla \cdot [l \nabla (\vartheta g)].$$

A more explicit expression for G is left to the reader. By applying the Theorem 3.1 to the solution $(\vartheta g)_x$ of the above Cauchy-Neumann problem we show that

$$\|(\vartheta g)_x\|_{2,T}^2 + \|\partial_t (\vartheta g)_x\|_{1,T}^2 \leqslant \tag{3.30}$$

$$\leqslant P(\|(\vartheta g_0)_x\|_2^2 + \|(\vartheta g_1)_x\|_1^2) + c \int_0^T \|\partial_x (\vartheta f)\|_1^2 \, dt + c \int_0^T \|\partial_x H + G\|_1^2 \, dt.$$

The last integral on the right hand side of (3.30) is easily estimated, by doing straightforward calculations. Use, in particular, (0.1), (0.2) and (0.3). The only difference with respect to the situations considered above consists on the presence of ϑ and of its derivatives. However, the corresponding terms are trivially estimated, since (3.35) shows that

$$\left| \frac{d^k \vartheta}{dx^k} \right| \leqslant c_k r^{-k} \leqslant (2c_0^2)^k c_k m^k [v]_{2,T}^{2k} \leqslant P, \tag{3.31}$$

for each $k \in N_0$. Moreover, for $k \geqslant 1$, these derivatives vanishes if $0 \leqslant x \leqslant r/2$ or if $x \geqslant r$. After some calculations one gets

$$\|\partial_x H + G\|_1 \leqslant Q(\|g\|_3 + \|\partial_t g\|_2),$$

for each $t \in [0, T]$. It readily follows from (3.30) that

$$\|g_x\|^2_{2, T(\mathbf{R}^3_+ \setminus S)} + \|\partial_t g_x\|^2_{1, T(\mathbf{R}^3_+ \setminus S)} \leqslant \leqslant P(\|g_0\|^2_3 + \|g_1\|^2_2) + P\int_0^T \|f\|^2_2\,dt + QT(\|g\|^2_{3,T} + \|\partial_t g\|^2_{2,T}), \tag{3.32}$$

where the left hand side of the above inequality has been restricted to the interior domain $\mathbf{R}^3_+ \setminus S$. Furthermore, Lemma 3.2 shows that

$$g_x \in C_T(H^2(\mathbf{R}^3_+ \setminus S)), \qquad \partial_t g_x \in C_T(H^1(\mathbf{R}^3_+ \setminus S)), \tag{3.33}$$

since $(\vartheta g)_x \in C_T(H^2)$ and $\partial_t(\vartheta g)_x \in C_T(H^1)$. ■

Finally, we estimate $\partial^3_x g$ near the boundary Γ. In equation $(3.1)_1$ the coefficient of $\partial^2_x g$ is given by $-(l - v_3^2)$, where v_3 is the third component of the vector field v. Since $v_3 = 0$ on Σ_T and $v_3 \in C_T(\tilde{H}^2) \subset C_T(C^{0,1/2})$, one has

$$|v_3(t, y, x)| \leqslant c_0 [v]_{2,T} x^{1/2}, \tag{3.34}$$

where c_0 is a positive numerical constant. Set

$$r = (2mc_0^2 [v]^2_{2,T})^{-1} \tag{3.35}$$

and define

$$S = \{z \in \mathbf{R}^3_+ : 0 < x < r\}, \qquad E =]0, T[\times S.$$

The coefficient $l - v_3^2$ satisfies the estimate

$$l - v_3^2 \geqslant 1/2m, \qquad \text{on } E. \tag{3.36}$$

From equation (3.1) it follows

$$\partial^2_x g = (-f + A + B\cdot\nabla g)/(l - v_3^2), \tag{3.37}$$

on E, where

$$A = \partial^2_t g + 2v\cdot(\partial_t \nabla g) + \sum_{(i,j)\neq(3,3)} v_i v_j \partial^2_{ij} g - l(\partial^2_{y_1} + \partial^2_{y_2}) g$$

and $B = \partial_t v + (v\cdot\nabla)v - \nabla l$. From (3.37) it readily follows that, for each $t \in [0, T]$,

$$\|\partial^3_x g\|_{L^2(S)} \leqslant P[-f + A + B\cdot\nabla g]_1. \tag{3.38}$$

for a suitable polynomial P. Moreover, straightforward calculations show that $[B]_1 \leqslant \leqslant P$ and that $[A]_1 \leqslant P\|g\|^*_3$. Here

$$\|g\|^*_3 = \sum_{(i,j)\neq(3,3)} \|\partial^2_{ij} g\|_1 + \|\partial_t g\|_2 + \|\partial^2_t g\|_1 .$$

Denote by $\|\cdot\|_{k,p}$ the canonical norm in the Sobolev space $W^{k,p}$. The Cauchy-

Schwarz inequality yields $|\cdot|_4 \leqslant |\cdot|_2^{1/4}|\cdot|_6^{3/4}$. Hence $\|\cdot\|_{1,4} \leqslant c\|\cdot\|_1^{1/4}\|\cdot\|_2^{3/4} \leqslant$ $\leqslant c\|\cdot\|_1^{1/4}\|\cdot\|_2^{3/4}$. On the other hand, straightforward calculations show that $[B\cdot\nabla g]_1 \leqslant$ $\leqslant c[B]_1\|\nabla g\|_{1,4} \leqslant P\|\nabla g\|_{1,4}$. It readily follows that

$$[B\cdot\nabla g]_1 \leqslant \varepsilon^{-3}P\|g\|_2 + \varepsilon P\|g\|_3,$$

for every $\varepsilon > 0$. In particular, (3.38) yields

$$\|\partial_x^3 g\|_{L^2(S)} \leqslant P\|f\|_1 + P\|g\|_3^* + \varepsilon P\|g\|_3 + \varepsilon^{-3}P\|g\|_3^*.$$

Hence, for $\varepsilon \in]0,1[$,

$$\|\partial_x^3 g\|^2_{L_T^\infty(L^2(S))} \leqslant P\left(\|f(0)\|_1^2 + T\int_0^T \|\partial_t f\|_1^2\,dt\right) + \varepsilon P\|g\|^2_{3,T} + \varepsilon^{-3}P\|g\|^{*2}_{3,T}. \tag{3.39}$$

By adding, side by side, (3.32) and (3.39) one proves that

$$\|\partial_x^3 g\|^2_{0,T} \leqslant P(\|g_0\|_3^2 + \|g_1\|_2^2 + \|f(0)\|_1^2) + \tag{3.40}$$

$$+P\int_0^T(\|f\|_2^2 + \|\partial_t f\|_1^2) + QT\|g\|^{*2}_{3,T} + \varepsilon^{-3}P\|g\|^{*2}_{3,T} + \varepsilon P\|g\|^2_{3,T},$$

provided that (3.10) holds. ■

Finally, we prove (3.18). Equations (3.23) and (3.26) show that

$$\|g\|^{*2}_{3,T} \leqslant P(\|g_0\|_3^2 + \|g_1\|_2^2 + \|f(0)\|_1^2) + \tag{3.41}$$

$$+c\int_0^T(\|f\|_2^2 + \|\partial_t f\|_1^2)\,dt + RT(\|g\|^2_{3,T} + \|\partial_t g\|^2_{2,T}).$$

By adding (3.40) and (3.41) we get (recall that $P \leqslant Q \leqslant R$)

$$\sum_{j=0}^{2}\|\partial_t^j g\|^2_{3-j,T} \leqslant P(\|g_0\|_3^2 + \|g_1\|_2^2 + \|f(0)\|_1^2) + \tag{3.42}$$

$$+P\int_0^T(\|f\|_2^2 + \|\partial_t f\|_1^2)\,dt + RT\sum_{j=0}^{2}\|\partial_t^j g\|^2_{3-j,T} + \varepsilon^{-3}P\|g\|^{*2}_{3,T} + \varepsilon P\|g\|^2_{3,T}.$$

Now we drop the last term on the right hand side of (3.42), by choosing ε such that $\varepsilon P = 1/4$. Since this value of ε is of type P^{-1}, it follows that the coefficient $\varepsilon^{-3}P$ is a polynomial of type P. Hence, the penultimate term on the right hand side of (3.42) is bounded by $P\|g\|^{*2}_{3,T}$. We estimate this term by using (3.41). After the above simplifications, the inequality (3.42) turns out to (3.18), except for an additional term $RT\sum_{j=0}^{2}\|\partial_t^j g\|^2_{3-j,T}$ on the right hand side of the inequality. By using the assumption (3.19) we drop this undesired term. The proof of (3.18) is accomplished. ■

In order to verify that L_T^∞ solutions must belong to C_T, it only remains to show that

$$\partial_x^2 g \in C_T(H^1(S)). \tag{3.43}$$

In fact, (3.43) together with (3.24), (3.27), and (3.33), shows that $g \in C_T(H^3)$, $\partial_t g \in C_T(H^2)$, $\partial_t^2 g \in C_T(H^1)$. Equation (3.43) follows easily from (3.37) by taking into account that $A \in C_T(H^1)$, that $B \cdot \nabla g \in C_T(H^1)$ (use (3.24) and (3.27)), and that $(l - v_3^2)^{-1} \in C_T(L^\infty(S))$, $\nabla[(l - v_3^2)^{-1}] \in C_T(H^1(S))$.

4. – Appendix A.

Here, we prove the Lemma 3.2. If Y is a Banach space, we denote by $w - C_T(Y)$ the linear space consisting of weakly continuous functions on $[0, T]$ with values in Y.

Equation (3.1) shows that $\partial_t^2 g \in L_T^2(H^0)$. Hence $\partial_t g \in C_T(H^0) \cap L_T^\infty(H^1) \subset w - C_T(H^1)$. Similarly, $g \in C_T(H^1) \cap L_T^\infty(H^2) \subset w - C_T(H^2)$. On the other hand, the assumption (3.2) show that $v \in C_T(\tilde{H}^2) \subset C_T(L^\infty)$. Hence, $-\partial_{(z)} = \partial_t g_z + v \cdot \nabla g_z$ belongs to $w - C_T(H^0)$. In particular

$$\|\partial_{(z)}(0)\|^2 \leq \liminf_{t \to 0^+} \|\partial_{(z)}(t)\|^2. \tag{4.1}$$

On the other hand, by using (3.14) and by taking into account that $l \in C_T(H^2) \subset C_T(L^\infty)$, it readily follows that

$$\limsup_{t \to 0^+} \int [\partial_{(z)}^2(t) + l(0)(|g_z(t)|^2 + |\nabla g_z(t)|^2)] \leq \tag{4.2}$$

$$\leq \int [\partial_{(z)}^2(0) + l(0)(|g_z(0)|^2 + |\nabla g_z(0)|^2)].$$

Since the norm

$$|||g_z|||_1 \equiv \left\{ \int (g_z^2 + |\nabla g_z|^2) l(0) \right\}^{1/2},$$

is equivalent to the canonical H^1-norm, and since $g \in w - C_T(H^1)$, it follows that

$$|||g_z(0)|||_1^2 \leq \liminf_{t \to 0^+} |||g_z(t)|||_1^2. \tag{4.3}$$

From (4.1), (4.2), and (4.3), it follows that $\lim \|\partial_{(z)}(t)\|^2 = \|\partial_{(z)}(0)\|^2$ and that $\lim |||g_z(t)|||_1^2 = |||g_z(0)|||_1^2$, as $t \to 0^+$. A well known result in Functional Analysis guarantees that $\partial_{(z)}(t) \to \partial_z(0)$ strongly in H^0 and that $g_z(t) \to g_z(0)$ strongly in H^1, as $t \to 0^+$. Taking into account (3.13), one proves that $\partial_t g_z(t) \to \partial_t g_z(0)$ strongly in H^0, as $t \to 0^+$. It readily follows that $g(t)$ is right continuous in H^2 at $t = 0$ and that $\partial_t g(t)$ is right continuous in H^1 at $t = 0$. The uniqueness of the solution g shows that the above properties of right continuity hold at every t. Finally, since the equations satisfies by g are reversible with respect to the time, the right continuity properties turns into left continuity properties. ∎

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