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L^p-Stability for the Strong Solutions of the Navier-Stokes Equations in the Whole Space

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Introduction

Consider the initial value problem for the non-stationary Navier-Stokes equations in the whole space R^3

$$\begin{aligned} v' + (v \cdot \nabla) v - \Delta v + \nabla \pi = 0 \quad & \text{in } Q_T \equiv]0, T[\times R^3, \\ \operatorname{div} v = 0 \quad & \text{in } Q_T, \\ v_{|t=0} = a \quad & \text{in } R^3, \\ \lim_{|x| \to \infty} v(t, x) = 0 \quad & \text{for } t \in]0, T[, \end{aligned} \tag{1}$$

where

$$T \in]0, \infty], \quad v' = \frac{\partial v}{\partial t} \quad \text{and} \quad (v \cdot \nabla) v = \sum_{i=1}^{3} v_i \frac{\partial v}{\partial x_i}.$$

The given initial velocity $a(x)$ satisfies $\operatorname{div} a = 0$ in R^3. Moreover, the pressure π is determined by the condition $\lim_{|x| \to \infty} \pi(t, x) = 0$ for $t \in]0, T[$. By a solution of problem (1), we mean a divergence free vector $v(t, x) \in L^q(0, T; L^r)$ for some q, r with $q, r \geqq 2$, such that

$$\int_0^T \int [v \cdot \varphi' + (v \cdot \nabla) \varphi \cdot v + v \cdot \Delta \varphi] \, dx \, dt = - \int a \varphi_{|t=0} \, dx,$$

for every regular divergence free vector field $\varphi(t, x)$, with compact support with respect to the space variables and such that $\varphi(T, x) = 0$. We set

$$L^p = L^p(R^3), \quad |\ |_p = \|\ \|_{L^p(R^3)},$$
$$N_p(v) = \int_{R^3} |\nabla v|^2 \, |v|^{p-2} \, dx.$$

Other notations are standard, or will be introduced in the sequel. Let $p > 3$ and $a_1 \in L^1 \cap L^{p+2}$, $\operatorname{div} a_1 = 0$. Assume that there exists a global solution

$v_1 \in L^\infty(0, +\infty; L^{p+2})$ of problem (1), with initial velocity a_1 and pressure π_1. This solution is strong and unique[1]. We prove the following stability result:

Theorem A. *Assume that the above conditions hold and let* $a_2 \in L^1 \cap L^p$, *be such that* $\operatorname{div} a_2 = 0$. *Then, there exists a positive constant* y_0 *such that, if*

$$|a_1 - a_2|_p < y_0, \tag{2}$$

there exists a unique solution $v_2 \in C([0, +\infty); L^p)$ *of* (1) *with initial data* a_2. *This solution satisfies the estimate*

$$|v_1(t) - v_2(t)|_p \leqq C(1 + t)^{-\frac{3}{4}}. \tag{3}$$

The constants y_0 and C depend on p, on the L^1 and L^2 norms of the initial data a_1 and a_2, and on the $L^\infty(0, +\infty; L^{p+2})$ norm of v_1. In particular, by considering initial data a_2 such that

$$|a_2|_1 \leqq |a_1|_1 + k_1, \qquad |a_2|_2 \leqq |a_1|_2 + k_2,$$

where k_1 and k_2 are any positive constants, y_0 and C depend only on k_1, k_2 and on the norms $|a_1|_1$, $|a_1|_2$ and $\|v_1\|_{L^\infty(0, +\infty; L^{p+2})}$ of the unperturbed solution v_1. The local existence and uniqueness of a strongly continuous solution $v_2(t)$ with values in $L^2 \cap L^p$ is well known. The bound (3) guarantees the global existence of $v_2(t)$. See the note added in proof.

The proof of theorem A follows the method introduced in reference [1] in order to study the asymptotic behavior of the solutions of system (1).

Proof of Theorem A. The difference $w = v_2 - v_1$, satisfies the following system:

$$w' + (v_2 \cdot \nabla) w + (w \cdot \nabla) v_1 - \Delta w + \nabla P = 0 \quad \text{in } Q_T, \tag{4}$$

$$\operatorname{div} w = 0 \quad \text{in } Q_T,$$

$$w_{|t=0} = \alpha \quad \text{in } R^3,$$

where $P = \pi_2 - \pi_1$ and $\alpha = a_2 - a_1$. Multiply both sides of equation (4) by $|w|^{p-2} w$ and integrate over R^3. After suitable integrations by parts we obtain

$$\frac{1}{p} \frac{d}{dt} |w|_p^p + N_p(w) + 4 \frac{p-2}{p^2} \int |\nabla| w|^{p/2}|^2$$
$$= - \int (v_2 \cdot \nabla) w \cdot |w|^{p-2} w - \int (w \cdot \nabla) v_1 \cdot |w|^{p-2} w - \int \nabla P \cdot |w|^{p-2} w.$$

The first term on the right-hand side is zero since v_2 is divergence free. By integrating by parts the other two terms we get

$$\frac{1}{p} \frac{d}{dt} |w|_p^p + N_p(w) \leqq (p-1) \int |w|^{p-1} |\nabla w| |v_1| + (p-2) \int |P| |w|^{p-2} |\nabla w|. \tag{5}$$

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[1] We refer the reader to the results proved in [2]. See also [5], and references there.

Consider the first integral on the right-hand side of (5). By Hölder's and Young's inequalities one has

$$\begin{aligned}(p-1)\int |w|^{p-1}\,|\nabla w|\,|v_1| &\leqq (p-1)\,N_p(w)^{1/2}\left(\int |w|^p\,|v_1|^2\right)^{1/2}\\ &\leqq \tfrac{1}{8}N_p(w)+2(p-1)^2\int |w|^p\,|v_1|^2 \qquad (6)\\ &\leqq \tfrac{1}{8}N_p(w)+2(p-1)^2\,|w|^p_{p+2}\,|v_1|^2_{p+2}.\end{aligned}$$

On the other hand, one can prove the following inequality (see [1], equation (1.14)):

$$|w|^p_{p+2}\leqq CN_p(w)^{\frac{3}{p+2}}\,|w|_p^{\frac{p(p-1)}{p+2}}; \qquad (7)$$

hence, from (6), one obtains

$$(p-1)\int |w|^{p-1}\,|\nabla w|\,|v_1|\leqq \tfrac{1}{4}N_p(w)+C\,|w|^p_p\,|v_1|_{p+2}^{\frac{2(p+2)}{p-1}}. \qquad (8)$$

Consider now the second integral on the right-hand side of (5). By Hölder's and Young's inequalities we have

$$\begin{aligned}(p-2)\int |P|\,|w|^{p-2}\,|\nabla w| &\leqq (p-2)\left(\int |P|^2\,|w|^{p-2}\right)^{1/2}N_p(w)^{1/2}\\ &\leqq 2(p-2)^2\int |P|^2\,|w|^{p-2}+\tfrac{1}{8}N_p(w) \qquad (9)\\ &\leqq 2(p-2)^2\,|P|^2_{\frac{p+2}{2}}\,|w|^{p-2}_{p+2}+\tfrac{1}{8}N_p(w).\end{aligned}$$

To estimate P, we take the divergence of (4) and obtain

$$-\Delta P=\sum_{i,j}\frac{\partial^2}{\partial x_i\,\partial x_j}w^i(v_1^j+v_2^j)=\sum_{i,j}\frac{\partial^2}{\partial x_i\,\partial x_j}[w^i(2v_1^j+w^j)].$$

From the Calderon-Zygmund inequality one has

$$|P|_{\frac{p+2}{2}}\leqq C\sum_{i,j}|w^i(2v_1^j+w^j)|_{\frac{p+2}{2}};$$

then by Hölder's inequality

$$|P|^2_{\frac{p+2}{2}}\leqq C\,|w|^2_{p+2}\,(|v_1|^2_{p+2}+|w|^2_{p+2}).$$

By introducing this last inequality in (9) and by using inequality (7), we get

$$\begin{aligned}&(p-2)\int |P|\,|w|^{p-2}\,|\nabla w|\\ &\quad\leqq C\,|w|^p_{p+2}\,(|v_1|^2_{p+2}+|w|^2_{p+2})+\tfrac{1}{8}N_p(w) \qquad (10)\\ &\quad\leqq CN_p(w)^{\frac{3}{p+2}}\,|w|_p^{\frac{p(p-1)}{p+2}}\,|v_1|^2_{p+2}+CN_p(w)^{\frac{3}{p}}\,|w|^{p-1}_p+\tfrac{1}{8}N_p(w)\\ &\quad\leqq \tfrac{1}{4}N_p(w)+C\,|w|^p_p\,|v_1|_{p+2}^{2\frac{p+2}{p-1}}+C\,|w|_p^{\frac{p(p-1)}{p-3}}.\end{aligned}$$

Hence, by using inequalities (8) and (10), we get from (5)

$$\frac{1}{p}\frac{d}{dt}|w|^p_p+\tfrac{1}{2}N_p(w)\leqq C\,|w|^p_p\,|v_1|_{p+2}^{\frac{2(p+2)}{p-1}}+C\,|w|_p^{\frac{p(p-1)}{p-3}}. \qquad (11)$$

On the other hand, one can prove the following Sobolev type inequality (see [1], equation (3.2))

$$N_p(w) \geqq C\,|w|_{3p}^{p}.$$

By interpolation, one has

$$|w|_p \leqq |w|_2^{\frac{4}{3p-2}}\,|w|_{3p}^{\frac{3(p-2)}{3p-2}}.$$

Hence

$$N_p(w) \geqq C\,|w|_2^{-\beta}\,|w|_p^{p+\beta}, \tag{12}$$

where $\beta = 4p/3(p-2)$. On the other hand, by a result proved in reference [3] (see also [4]) on the L^2-decay of the solutions of the Navier-Stokes equations, one has

$$|w(t)|_2 \leqq |v_1(t)|_2 + |v_2(t)|_2 \leqq C(t+1)^{-3/4}, \tag{13}$$

where C depends only on the L^1 and L^2-norms of the initial velocities. Then, from (12) and (13), we obtain

$$N_p(w) \geqq C(t+1)^{3\beta/4}\,|w|_p^{p+\beta}. \tag{14}$$

Hence, from (11) and (14), we have

$$\frac{1}{p}\,\frac{d}{dt}\,|w|_p^p + C(t+1)^{3\beta/4}\,|w|_p^{p+\beta} \leqq C\,|w|_p^p\,|v_1|_{p+2}^{\frac{2(p+2)}{p-1}} + C\,|w|_p^{\frac{p(p-1)}{p-3}},$$

from which we obtain, since v_1 is bounded in L^{p+2} uniformly in time,

$$\frac{d}{dt}\,|w|_p + C_1(t+1)^{3\beta/4}\,|w|_p^{\beta+1} \leqq C_2\,|w|_p + C_3\,|w|_p^{1+\beta+\gamma}, \tag{15}$$

where $\gamma = 2p^2/3(p-2)(p-3)$. In (15), C_1 depends on p and on the L^1 and L^2-norms of the initial velocities, C_2 depends on p and $\|v_1\|_{L^\infty(0,T;L^{p+2})}$, C_3 depends only on p. Consider now the corresponding ordinary differential equation

$$\begin{aligned} y'(t) + C_1(t+1)^{3\beta/4}\,(y(t))^{\beta+1} &= C_2 y(t) + C_3 (y(t))^{1+\beta+\gamma},\\ y(0) &= |\alpha|_p. \end{aligned} \tag{16}$$

We prove now that, if $|\alpha|_p$ is sufficiently small, then $y(t) \leqq C(t+1)^{-3/4}$. By comparison theorems for ordinary differential equations it will follow that $|w(t)|_p \leqq y(t)$; hence (3) holds. Let $t_0 \in\,]0, +\infty[$ be such that

$$t_0 > \left(\frac{C_2 + C_3}{C_1}\right)^{4/3\beta} - 1. \tag{17}$$

By the continuous dependence on the initial data of the solution of (16), one can find $y_0 > 0$ (depending on p and on C_i, $i = 1, 2, 3$) sufficiently small that, if $|\alpha|_p \leqq y_0$, then $y(t) < 1$ for each $t \in [0, t_0]$. Moreover, if there exists $t \geqq t_0$ such that $y(t) = 1$, then from (16) and (17) one has

$$\begin{aligned} y'(t) &= -C_1(t+1)^{3\beta/4} + C_2 + C_3\\ &\leqq -C_1(t_0+1)^{3\beta/4} + C_2 + C_3 < 0. \end{aligned}$$

This implies $y(t) \leqq 1$, for any $t \geqq 0$. From (16), we then obtain $y(t) \leqq z(t)$ for any $t \geqq 0$, where $z(t)$ is the solution of

$$z'(t) + C_1(t+1)^{3\beta/4}(z(t))^{\beta+1} = (C_2 + C_3)\, z(t), \qquad z(0) = |\alpha|_p. \tag{18}$$

Equation (18) is of Bernoulli type and its solution is given by

$$z(t) = e^{(C_2+C_3)t}\left[|\alpha|_p^{-\beta} + \int_0^t e^{\beta(C_2+C_3)s}\beta C_1(s+1)^{3\beta/4}\,ds\right]^{-\frac{1}{\beta}}.$$

Hence

$$z(t)^\beta (1+t)^{3\beta/4} \leqq (1+t)^{3\beta/4} e^{\beta(C_2+C_3)t}\left[y_0^{-\beta} + \beta C_1 \int_0^t e^{\beta(C_2+C_3)s}(1+s)^{3\beta/4}\,ds\right]^{-1}.$$

By the l'Hôpital theorem one easily shows that the right-hand side of the above inequality converges to $(C_2 + C_3)/C_1$ as $t \to +\infty$. Since it is equal to y_0^β for $t = 0$, it is bounded in the interval $[0, +\infty)$ by a constant C (which depends only on C_1, C_2, C_3 and p).

Note added in proof. We point out that both v_1 and v_2 (hence $v_1 - v_2$) must decay like $(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}$. This follows from the estimates in references [1], since v_1 and v_2 belong to $C([0, +\infty); L^p)$. However, theorem A guarantees that v_2 *exists* (in the above space), and that the constants C and y_0 depend only on the unperturbed solution v_1.

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