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On the Solutions in the Large of the Two-Dimensional Flow of a Nonviscous Incompressible Fluid

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The Euler equations (1.1) for the motion of a nonviscous imcompressible fluid in a plane domain Ω are studied. Let E be the Banach space defined in (1.4), let the initial data v_0 belong to E, and let the external forces $f(t)$ belong to $L^1_{\text{loc}}(\boldsymbol{R}; E)$. In Theorem 1.1 the strong continuity and the global boundedness of the (unique) solution $v(t)$ are proved, and in Theorem 1.2 the strong-continuous dependence of v on the data v_0 and f is proved. In particular the vorticity $\operatorname{rot} v(t)$ is a continuous function in $\bar{\Omega}$, for every $t \in \mathbf{R}$, if and only if this property holds for one value of t. In Theorem 1.3 some properties for the associated group of nonlinear operators $S(t)$ are stated. Finally, in Theorem 1.4 a quite general sufficient condition is given on the data in order to get classical solutions.

1. INTRODUCTION AND MAIN RESULTS

Let Ω be an open, connected, bounded set of the plane $\mathbf{R}^2$ with a regular boundary Γ, say, of class $C^{2,\alpha}$, $\alpha > 0$. We denote by n the outward unit normal to Γ. In this paper we study the Euler equations

$$\begin{aligned}
&\frac{\partial v}{\partial t} + (v \cdot \nabla) v = f - \nabla \pi && \text{in} \quad Q \equiv \mathbf{R} \times \Omega, \\
&\operatorname{div} v = 0 && \text{in} \quad Q, \\
&v \cdot n = 0 && \text{on} \quad \Sigma \equiv \mathbf{R} \times \Gamma, \\
&v_{|t=0} = v_0(x) && \text{in} \quad \Omega,
\end{aligned} \tag{1.1}$$

where the velocity field $v(t, x)$ and the pressure $\pi(t, x)$ are unknowns. In (1.1) the external force field $f(t, x)$ and the velocity $v_0(x)$ are given; moreover, $\operatorname{div} v_0(x) = 0$ in Ω and $v_0 \cdot n = 0$ on Γ.

Existence of local solutions of (1.1) was proved by L. Lichtenstein. Global classical solutions were studied by many authors, for instance, E. Hölder, J.

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Leray, A. C. Schaeffer [7], and W. Wolibner [8]. More recent studies are those of V. I. Judovich [3], T. Kato [4], J. C. W. Rogers [6], and C. Bardos [1].

The aim of our paper is to prove some properties for the global solutions of Eq. (1.1) by setting the problem in a very natural functional framework, the Banach space $E(\bar{\Omega})$ consisting of all divergence-free vector fields $v(x)$ which are tangential to the boundary and for which rot $v(x) \in C(\bar{\Omega})$. The properties of global solutions in this space can be summarized as follows:

(i) For every initial velocity $v_0 \in E(\bar{\Omega})$ and for every exterior force $f \in L^1_{\text{loc}}(\mathbf{R}; E(\bar{\Omega}))$ the solution $v(t)$ is strongly continuous, i.e., $v \in C(\mathbf{R}; E(\bar{\Omega}))$ (see Theorem 1.1; see also Remark 2.1).

(ii) The solution $v(t)$ depends continuously, in the *norm* topology, on the data v_0 and f. More precisely, if $v_0^{(m)} \to v_0$ in $E(\bar{\Omega})$ and if $f_m \to f$ in $L^1(I; E(\bar{\Omega}))$ for every compact time interval I, then $v_m(t) \to v(t)$ in $E(\bar{\Omega})$, the convergence being uniform on every compact time interval I (Theorem 1.2).

(iii) Estimate (1.6′) holds; in particular the solution is globally bounded in time if $f \in L^1(\mathbf{R}; E(\bar{\Omega}))$. Moreover if $f \equiv 0$ then $|||v(t)||| = |||v_0|||$, $\forall t \in \mathbf{R}$, $|||\cdot|||$ being the norm of $E(\bar{\Omega})$.

The crucial property (ii) appears not to have been proved in any Banach space. Note that continuous dependence with respect to weaker topologies can be (in many cases) trivially verified.

Property (iii) shows that $E(\bar{\Omega})$ might be a suitable space for the study of asymptotic properties; note that $E(\bar{\Omega})$ seems to be the space of the most regular functions for which property (iii) holds.

Assuming for simplicity that $f \equiv 0$, and combining the above results, one gets Theorem 1.3, which shows that the essential properties of hyperbolic groups of operators hold for Eq. (1.1) in the space $E(\bar{\Omega})$.

On the other hand, we note that Theorems 1.1 and 1.2 also prove the nonexistence of shocks for the curl of the velocity field; more precisely, rot $v(t)$ is a continuous function in $\bar{\Omega}$, for every $t \in \boldsymbol{R}$, if and only if this property holds for one (arbitrary) value of t; this statement holds even in presence of quite discontinuous (in time) external forces. Actually, rot $v(t)$ must then be a continuous function in $\bar{Q}$. In the remainder of this section we introduce notation and state the above results in complete form. For simplicity, we will assume that Ω is simply-connected; the reader should verify that the usual device (see [3, Sect. 5] and [4]) utilized to treat the general case also applies to our proofs; hence the results stated in our paper hold for non-simply-connected domains.

In the sequel $\bar{\Omega}$ denotes the closure of Ω and $C(\bar{\Omega})$ the space of continuous (scalar or vector valued) functions in $\bar{\Omega}$ normed by $\|\theta\| \equiv \sup |\theta(x)|$, $x \in \bar{\Omega}$. For simplicity we avoid in our notation any distinction between scalars and vectors. $C^k(\bar{\Omega})$ (k, a positive integer) is the space of all k times continuously differentiable functions in $\bar{\Omega}$ equipped with the usual norm $\|\cdot\|_k$. Sometimes we will write $D_x^l\theta$ to denote a generical derivative of order l. The scalar product in the Hilbert space $L^2(\Omega)$ is denoted by $(\, , \,)$.

If X is a Banach space, $L^1_{\mathrm{loc}}(\mathbf{R}; X)$ is the linear space of all X-valued strongly measurable functions $u(t)$, $t \in R$, such that $\|u(t)\|_X$ is integrable on compact intervals $[-T, T]$, $\forall t > 0$.

Some of the above definitions will be utilized also with Ω replaced by Q or by $Q_T \equiv [0, T] \times \Omega$.

If $\theta(t, x)$ is defined in Q we sometimes denote by $\theta(t)$ the function $\theta(t, \cdot)$ defined for $x \in \Omega$.

Finally, $\mathbf{N}$ denotes the set of positive integers and $c, c_0, c_1, \ldots$ denote constants depending at most on Ω. Different constants may be denoted by the same symbol c.

The following definitions are classical: for a scalar function $\psi(x)$ in Ω we define the vector $\mathrm{Rot}\,\psi = (\partial\psi/\partial x_2, -\partial\psi/\partial x_1)$ and for a vector function $v = (v_1, v_2)$ we define the scalar $\mathrm{rot}\, v = \partial v_2/\partial x_1 - \partial v_1/\partial x_2$. One has $-\Delta \equiv \mathrm{rot}\,\mathrm{Rot}$. Note that $\mathrm{Rot}\,\psi$ is the rotation of the gradient $\nabla\psi$ by $\pi/2$ in the negative direction (counterclockwise). Let now ψ be the solution of

$$\begin{aligned} -\Delta\psi &= \theta \qquad \text{in } \Omega, \\ \psi &= 0 \qquad \text{on } \Gamma. \end{aligned} \tag{1.2}$$

By the above remarks $v \equiv \mathrm{Rot}\,\psi$ is the solution of

$$\begin{aligned} \mathrm{rot}\, v &= \theta \qquad \text{in } \Omega, \\ \mathrm{div}\, v &= 0 \qquad \text{in } \Omega, \\ v \cdot n &= 0 \qquad \text{on } \Gamma. \end{aligned} \tag{1.3}$$

Let us introduce the Banach space

$$E(\bar{\Omega}) \equiv \{v \in C(\bar{\Omega}) : \mathrm{div}\, v = 0 \text{ in } \Omega,\ v \cdot n = 0 \text{ on } \Gamma,\ \mathrm{rot}\, v \in C(\bar{\Omega})\}, \tag{1.4}$$

equipped with the norm $|\!|\!| v |\!|\!| \equiv \|\mathrm{rot}\, v\| + \|v\|_{L^2(\Omega)}$. In the sequel ($\Omega$ being simply-connected) we use the equivalent norm

$$|\!|\!| v |\!|\!| \equiv \|\mathrm{rot}\, v\|. \tag{1.5}$$

Concerning the existence of solutions we prove the following statement:

THEOREM 1.1. *Let* $v_0 \in E(\bar{\Omega})$ (*or equivalently* $\mathrm{rot}\, v_0 \in C(\bar{\Omega})$, $\mathrm{div}\, v_0 = 0$

in Ω, $v_0 \cdot n = 0$ *on* Γ) *and let* $f \in L^1_{\mathrm{loc}}(\mathbf{R}; L^2(\Omega))$ *with* $\mathrm{rot}\, f \in L^1_{\mathrm{loc}}(\mathbf{R}; C(\bar{\Omega}))$.[1] *Then problem* (1.1) *is uniquely solvable in the large, the solution* v *belongs to* $C(\mathbf{R}; E(\bar{\Omega}))$ (*or equivalently* $\mathrm{rot}\, v \in C(\mathbf{R}; C(\bar{\Omega}))$) *and*

$$|||v(t)||| \leqslant |||v_0||| + \left| \int_0^t \|\mathrm{rot} f(\tau)\|\, d\tau \right|, \qquad \forall t \in \mathbf{R}. \tag{1.6}$$

If $\mathrm{rot}\, f \equiv 0$ *equality holds in* (1.6).

Remark. Instead of $f \in L^1_{\mathrm{loc}}(\mathbf{R}; L^2(\Omega))$ we could assume that f is a distribution in Q, the significant condition being only $\mathrm{rot}\, f \in L^1_{\mathrm{loc}}(\mathbf{R}; C(\bar{\Omega}))$. Furthermore, in view of the decomposition formulae (5.1), our assumption on f is equivalent to $f \in L^1_{\mathrm{loc}}(\mathbf{R}; E(\bar{\Omega}))$ and estimate (1.6) is equivalent to

$$|||v(t)||| \leqslant |||v_0||| + \left| \int_0^t |||f(\tau)|||\, d\tau \right|, \qquad \forall t \in \mathbf{R}. \tag{1.6$'$}$$

Remark. We don't consider explicitly the regularity of $\partial v/\partial t$ and $\nabla \pi$ since it follows from the regularity of ∇v and f. See Appendix 2.

THEOREM 1.2. *Let* $v_0, f, v_0^{(m)}$, *and* f_m, $m \in \mathbf{N}$, *be as in Theorem* 1.1, *and let* v *and* v_m *be the solutions of* (1.1) *with data* v_0, f *and* $v_0^{(m)}, f_m$, *respectively. If* $v_0^{(m)} \to v_0$ *in* $E(\bar{\Omega})$ *and* $\mathrm{rot}\, f_m \to \mathrm{rot}\, f$ *in* $L^1_{\mathrm{loc}}(\mathbf{R}; C(\bar{\Omega}))$[2] *then* $v_m(t) \to v(t)$ *in* $E(\bar{\Omega})$, *the convergence being uniform on every compact interval, i.e.,* $v_m \to v$ *in* $C([-T, T]; E(\bar{\Omega}))$, *for every* $T > 0$.

Now assume $\mathrm{rot}\, f \equiv 0$ and denote by $S(t)$, $t \in \mathbf{R}$, the nonlinear operator defined by $S(t)\, v_0 \equiv v(t)$, $\forall v_0 \in E(\bar{\Omega})$, where $v(t)$ is the solution of problem (1.1). Put also $Ju \equiv -u$. One has the following result:

THEOREM 1.3. *Under the above assumptions and definitions one has:*

(i) $S(t)\, S(\tau) = S(t + \tau)$, $\forall t, \tau \in \mathbf{R}$; $S(0) = I$.

(ii) $S(t)$ *is "unitary" in the sense that* $|||S(t)u||| = |||u|||$, $\forall u \in E(\bar{\Omega})$. *Moreover* $S(t)^{-1} = S(-t) = JS(t)J$, $\forall t \in \mathbf{R}$.

(iii) $S(t)$ *is a strongly-continuous group of operators, i.e., for every* $u \in E(\bar{\Omega})$ *the function* $S(t)u$ *is a strongly-continuous function in* $\mathbf{R}$ *with values in* $E(\bar{\Omega})$.

(iv) *For every* $t \in \mathbf{R}$ *the nonlinear operator* $S(t)$ *is a bicontinuous map (in the norm topology of* $E(\bar{\Omega})$) *from all of* $E(\bar{\Omega})$ *onto itself. Moreover if* $u_m \to u$ *the convergence* $S(t)u_m \to S(t)u$ *is uniform on compact t-intervals.*

[1] Plus $(f, u_k) \in L^1_{\mathrm{loc}}(\mathbf{R})$, $k = 1, \ldots, N$, if Ω is not simply-connected. For the definition of u_k see Appendix 1.

[2] If Ω is not simply-connected we also assume that $(f_m, u_k) \to (f, u_k)$ in $L^1_{\mathrm{loc}}(\mathbf{R})$, $k = 1, \ldots, N$.

We also study some questions concerning the existence of classical solutions. Our main concern will be the continuity of ∇v on $\bar{Q}$, additional conditions on f in order to get continuity for $\partial v/\partial t$ and $\nabla \pi$ will then be trivial. We want to characterize explicitly a Banach space $C_*(\bar{\Omega})$, the data space, such that $v \in C(\mathbf{R}; C^1(\bar{\Omega}))$ whenever $\operatorname{rot} v_0 \in C_*(\bar{\Omega})$ and $\operatorname{rot} f \in L^1_{\text{loc}}(\mathbf{R}; C_*(\bar{\Omega}))$.

We don't expect the above result if we just define $C_*(\bar{\Omega})$ as $C(\bar{\Omega})$. On the other hand, if we define $C_*(\bar{\Omega})$ as $C^{0,\lambda}(\bar{\Omega})$, $\lambda > 0$, the result holds easily; hence we want a larger space. We construct $C_*(\bar{\Omega})$ as follows; for every $\theta \in C(\bar{\Omega})$ let us denote by $\omega_\theta(r)$ the oscillation of θ on sets of diameter less than or equal to r:

$$\omega_\theta(r) \equiv \sup_{\substack{0<|x-y|\leqslant r \\ x,y\in\bar{\Omega}}} |\theta(x) - \theta(y)|. \tag{1.7}$$

Clearly $\omega_\theta(r) = \omega_\theta(R)$, $\forall r \geqslant R \equiv$ diameter of Ω. Let us put

$$[\theta]_* \equiv \int_0^R \omega_\theta(r) \frac{dr}{r}, \tag{1.8}$$

and define $C_*(\bar{\Omega}) \equiv \{\theta \in C(\bar{\Omega}) : [\theta]_* < +\infty\}$. Then $\|\theta\|_* \equiv [\theta]_* + \|\theta\|$ is a norm in the linear space $C_*(\bar{\Omega})$. Moreover $C_*(\bar{\Omega})$ is a Banach space. Note, by the way, that $[\theta]_* \leqslant R^\lambda \lambda^{-1} [\theta]_\lambda$ where $[\cdot]_\lambda$ is the usual λ-Hölder semi-norm.

We prove the following result:

THEOREM 1.4. *Let* $\operatorname{rot} v_0 \in C_*(\bar{\Omega})$ *and* $f \in L^1_{\text{loc}}(\mathbf{R}; L^2(\Omega))$ *with* $\operatorname{rot} f \in L^1_{\text{loc}}(\mathbf{R}; C_*(\bar{\Omega}))$.[1] *Then the solution of problem* (1.1) *belongs to* $C(\mathbf{R}; C^1(\bar{\Omega}))$, *moreover*

$$\|v(t)\|_1 \leqslant c e^{c_1 B|t|} \{\|\operatorname{rot} v_0\|_* + \|\operatorname{rot} f\|_{L^1(0,t;C_*(\bar{\Omega}))}\}, \tag{1.9}$$

where $B \equiv \|\operatorname{rot} v_0\| + \|\operatorname{rot} f\|_{L^1(0,t;C(\bar{\Omega}))}$. *If in addition f is such that the terms* $g(t)$ *and* $\nabla F(t)$, *in the canonical decomposition* (5.1), *are continuous in* $\bar{Q}$[3] *also* $\partial v/\partial t$ *and* $\nabla \pi$ *are continuous in* $\bar{Q}$ (*classical solution*).

2. PROOF OF THEOREM 1.1

In the following we consider Eq. (1.1) in the time interval $[0, T]$, $T > 0$ arbitrary. Proofs apply also to intervals $[-T, 0]$; alternatively one can reduce this case to the previous one by a change of variables. In fact the solution of

[3] It suffices that $f \in C(\mathbf{R}; W^{1,p}(\Omega))$, for some $p > 2$.

the problem $(\partial v/\partial t)+(v\cdot\nabla)v=f-\nabla\pi$, $t\in[-T,0]$, with $v_{|t=0}=v_0(x)$ is given by $v(t)=-u(-t)$ where $(\partial u/\partial t)+(u\cdot\nabla)u=g-\nabla\pi_1$, $g(t,x)\equiv f(-t,x)$, $\pi_1(t,x)\equiv\pi(-t,x)$, $t\in[0,T]$, $u_{|t=0}=-v_0(x)$.

Assume the data v_0 and f fixed as well as $T>0$. For convenience put $\zeta_0\equiv\operatorname{rot}v_0$, $\phi\equiv\operatorname{rot}f$, $B\equiv\|\zeta_0\|+\int_0^T\|\phi(\tau)\|\,d\tau$, and define

$$\mathbf{K}=\{\theta\in C(\bar{Q}_T):\|\theta\|_T\leqslant B\},\tag{2.1}$$

where $\|\cdot\|_T$ denotes the norm in $C(\bar{Q}_T)=C([0,T];C(\bar{\Omega}))$. $\mathbf{K}$ is convex, closed, and bounded in $C(\bar{Q}_T)$. From now on θ denotes an arbitrary element of $\mathbf{K}$. Now let ψ be the solution of problem (1.2)

$$\psi(x)=\int_\Omega g(x,y)\,\theta(y)\,dy,\qquad x\in\Omega,\tag{2.2}$$

and let $v=\operatorname{Rot}\psi$ be the solution of (1.3); since $v\in W^{1,p}(\Omega)$, $\forall p<+\infty$, the meaning of Eq. (1.3) is clear. It is well known that the Green's function $g(x,y)$ for the Laplace operator $-\Delta$ with zero boundary condition (see, for instance, [5]) verifies the estimates

$$|D_x g(x,y)|\leqslant c\,|x-y|^{-1},\qquad |D_x^2 g(x,y)|\leqslant c\,|x-y|^{-2}.\tag{2.3}$$

By using classical devices in potential theory one shows that $\|v\|\leqslant c_1\|\theta\|$ and that $|v(x)-v(y)|\leqslant c_1\|\theta\|\,|x-y|\,\chi(|x-y|)$ where $\chi(r)\equiv\log(eR/r)$, $\forall r>0$; see [4, Lemma 1.4]. Hence for every $t\in[0,T]$, $\|v(t)\|\leqslant c_1B$ and

$$|v(t,x)-v(t,y)|\leqslant c_1B\,|x-y|\,\chi(|x-y|),\qquad \forall x,y\in\Omega.\tag{2.4}$$

Clearly $v\in C(\bar{Q}_T)$. Let $U(s,t,x)$ be the solution of the system of ordinary differential equations

$$\begin{aligned}&\frac{d}{ds}U(s,t,x)=v(s,U(s,t,x)),\qquad \text{for}\quad s\in[0,T],\\&U(t,t,x)=x,\end{aligned}\tag{2.5}$$

where $(t,x)\in Q_T$. Let us show that

$$\begin{aligned}&|U(s,t,x)-U(s_1,t_1,x_1)|\\&\qquad\leqslant c_1B\,|s-s_1|+c_2(1+c_1B)(|x-x_1|^\delta+|t-t_1|^\delta),\end{aligned}\tag{2.6}$$

where $c_2\equiv\max\{1,eR\}$ and $\delta\equiv e^{-c_1BT}$. Put $x(s)=U(s,t,x)$, $x_1(s)=U(s,t,x_1)$, $\rho(s)=|x(s)-x_1(s)|$. One has $|\rho'(s)|\leqslant c_1B\rho(s)\,\chi(\rho(s))$ and $\rho(t)=|x-x_1|$. On the other hand the function

$$\rho_1(s)=eR\left(\frac{|x-x_1|}{eR}\right)e^{-c_1B(s-t)}$$

is the solution of $\rho_1'(s) = c_1 B \rho_1(s)\, \chi(\rho_1(s))$, $s \in [0, T]$, with $\rho_1(t) = |x - x_1|$. Hence $\rho(s) \leqslant \rho_1(s)$ for $s \geqslant t$. For $s \leqslant t$ a corresponding argument holds. Then

$$|U(s, t, x) - U(s, t, x_1)| \leqslant (eR)^{1-\delta} |x - x_1|^{\delta} \leqslant c_2 |x - x_1|^{\delta}. \tag{2.7}$$

Now one easily gets $|U(s, t, x) - U(s_1, t, x)| \leqslant c_1 B |s - s_1|$ and $|U(s, t, x) - U(s, t_1, x)| \leqslant c_2 c_1^{\delta} B^{\delta} |t - t_1|^{\delta}$ (see [4]); estimate (2.6) follows.

Define now the map $\zeta = \Phi[\theta]$ by

$$\zeta(t, x) \equiv \zeta_0(U(0, t, x)) + \int_0^t \phi(s, U(s, t, x))\, ds. \tag{2.8}$$

THEOREM 2.1. *The inclusion $\Phi(\mathbf{K}) \subset \mathbf{K}$ holds, moreover $\Phi(\mathbf{K})$ is a family of equicontinuous functions in $\bar{Q}_T$. Hence $\Phi(\mathbf{K})$ is a relatively compact set in $C(\bar{Q}_T)$.*

Proof. Obviously $|\zeta(t, x)| \leqslant B$. The equicontinuity of the family $\zeta_0(U(0, t, x))$ follows from (2.6) and from the uniform continuity of ζ_0 on $\bar{\Omega}$. Let us prove the equicontinuity of the second term on the right-hand side of (2.8); clearly

$$\left| \int_0^t \phi(s, U(s, t, x))\, ds - \int_0^{t_1} \phi(s, U(s, t_1, x_1))\, ds \right|$$
$$\leqslant \left| \int_{t_1}^t \|\phi(s)\|\, ds \right| + \int_0^t |\phi(s, U(s, t, x)) - \phi(s, U(s, t_1, x_1))|\, ds. \tag{2.9}$$

Moreover, to each $\nu > 0$ there corresponds $\lambda_1 > 0$ such that

$$|t_1 - t| < \lambda_1 \Rightarrow \left| \int_{t_1}^t \|\phi(s)\|\, ds \right| < \nu. \tag{2.10}$$

Define for every $\varepsilon > 0$

$$\omega(s, \varepsilon) \equiv \sup_{|y - y_1| < \varepsilon} |\phi(s, y) - \phi(s, y_1)|. \tag{2.11}$$

Since $\omega(s, \varepsilon) \leqslant 2 \|\phi(s)\|$ and $\lim_{\varepsilon \to 0} \omega(s, \varepsilon) = 0$ for almost all $s \in [0, T]$, it follows from the Lebesgue dominated convergence theorem that to each $\nu > 0$ there corresponds an $\varepsilon_0 > 0$ such that

$$\int_0^T \omega(s, \varepsilon_0)\, ds < \nu. \tag{2.12}$$

Furthermore to every $\varepsilon_0 > 0$ there corresponds a $\lambda_2 > 0$ such that

$$\max\{|t - t_1|, |x - x_1|\} < \lambda_2 \Rightarrow |U(s, t, x) - U(s, t_1, x_1)| < \varepsilon_0 \tag{2.13}$$

uniformly with respect to s; this follows from (2.6). Hence

$$\left| \int_0^t \phi(s, U(s,t,x))\, ds - \int_0^{t_1} \phi(s, U(s, t_1, x_1)) \right| ds < 2\nu \tag{2.14}$$

if $\max\{|t-t_1|, |x-x_1|\} < \min\{\lambda_1, \lambda_2\}$. The equicontinuity of the family $\Phi(\mathbf{K})$ is proved. The last statement follows from the Ascoli–Arzelà compactness theorem. ∎

THEOREM 2.2. *The map $\Phi: \mathbf{K} \to \mathbf{K}$ has a fixed point.*

Proof. It remains to prove the continuity of the map Φ. Let $\theta_m \in \mathbf{K}$, $\theta_m \to \theta$ uniformly on $\bar{Q}_T$. Denoting by v_m the solution of (1.3) with data θ_m it is clear that $v_m \to v$ uniformly on $\bar{Q}_T$. Let $\varepsilon > 0$ be given and N_ε be such that $\|v - v_m\|_{Q_T} < \varepsilon$ whenever $m > N_\varepsilon$. Put $x(s) = U(s,t,x)$, $x_m(s) = U_m(s,t,x)$, and $\rho(s) = |x(s) - x_m(s)|$, where U_m denotes the solution of (2.5) with v replaced by v_m. For $m > N_\varepsilon$ one has $|\rho'(s)| \leqslant |x'(s) - x_m'(s)| \leqslant \varepsilon + c_1 B\,|x(s) - x_m(s)|\,\chi(|x(s) - x_m(s)|)$. Hence $|\rho'(s)| \leqslant \varepsilon + c_1 B\varepsilon\chi(\varepsilon)$ because $r\chi(r)$ is an increasing function on $[0, R]$. Moreover $\rho(t) = 0$. Consequently $|U(s,t,x) - U_m(s,t,x)| \leqslant T(\varepsilon + c_1 B\varepsilon\chi(\varepsilon))$, $\forall s \in [0,T]$, and $U_m(s,t,x)$ is uniformly convergent to $U(s,t,x)$ on $[0,T]^2 \times \bar{\Omega}$, when $m \to +\infty$. It follows easily from (2.8) and (2.12) that $\zeta_m \to \zeta$ uniformly in Q_T, where $\zeta_m \equiv \Phi[\theta_m]$. Actually, it suffices to show the pointwise convergence of ζ_m to ζ; uniform convergence follows then from the compactness of subsets of $\Phi(\mathbf{K})$. ∎

Remark 2.1. The above method of proving strong continuity of $\zeta(t)$ in $C(\bar{\Omega})$ seems not to work in Hölder spaces, even if $f \equiv 0$. In fact if $\zeta_0 \in C^{0,\lambda}(\bar{\Omega})$ we cannot prove that $\zeta_0(U(t,x)) \in C(\mathbf{R}; C^{0,\lambda}(\bar{\Omega}))$ by using (only) regularity results for $U(t,x)$ (other arguments must eventually be added); in fact, if $\zeta_0(U) \equiv \sqrt{|U|}$ and $U(t,x) \equiv t - x$ the function $\zeta(t,x) \equiv \zeta_0(U(t,x))$ verifies

$$|\zeta(t,x) - \zeta(\tau,x) - \zeta(t,y) + \zeta(\tau,y)| = |x-y|^{1/2} \tag{2.15}$$

if $x = \tau$, $y = t$.

The situation becomes worse with respect to the strong-continuous dependence on the data.

Now we verify that the function v corresponding to the fixed point $\zeta = \theta$ is a solution of (1.1); see also [4].

We start by showing that for fixed (s,t) the map $x \to U(s,t,x)$ is measure preserving in Ω. Let $\theta \in \mathbf{K}$, $\theta_m \in C([0,T]; C^1(\bar{\Omega}))$, $\theta_m \to \theta$ uniformly on $\bar{Q}_T$. If v_m is the solution of (1.3) with data θ_m one has $v_m \in C([0,T]; C^1(\bar{\Omega}))$ and $\operatorname{div} v_m = 0$. Hence $x \to U_m(s,t,x)$ is measure preserving. On the other hand we know from the proof of Theorem 2.1 that $U_m \to U$ uniformly on

$[0, T]^2 \times \bar{\Omega}$. It follows that U is measure preserving. For, define $Tx = U(s, t, x)$, $T_m x = U_m(s, t, x)$, $x \in \Omega$, and let E be a compact subset of Ω and A an arbitrary open set verifying $T(E) \subset A \subset \Omega$. Recalling that $T_m x \to Tx$ uniformly and that $T(E)$ is compact one shows that there exists an integer m_0 such that $T_{m_0}(E) \subset A$; hence $|T_{m_0}(E)| = |E| \leqslant |A|$ consequently $|E| \leqslant |TE|$, where $|\cdot|$ denotes Lebesgue measure. An analogous property holds for the map $T^{-1}y = U(t, s, y)$, hence the measure preserving property holds.

LEMMA 2.3. *Let $\zeta = \theta$ be the fixed point constructed above. Then $\partial\zeta/\partial t = -\operatorname{div}(\zeta v) + \phi$ in the sense of distributions in Q_T.*

Proof. We show that

$$\frac{d}{dt}(\zeta, \Psi) = (\zeta v, \nabla\Psi) + (\phi, \Psi), \qquad \forall \Psi \in C_0^\infty(\Omega). \tag{2.16}$$

Denoting by $\zeta_2(t, x)$ the second term in the right-hand side of (2.8) and taking into account the measure preserving property one gets, by the change of variable $y = U(s, t, x)$,

$$(\zeta_2, \Psi) = \int_0^t ds \int_\Omega \phi(s, y)\, \Psi(U(t, s, y))\, dy.$$

Hence

$$\frac{d}{dt}(\zeta_2, \Psi) = \int_\Omega \phi(t, y)\, \Psi(y)\, dy + \int_0^t ds \int_\Omega \phi(s, y)\, v(t, U(t, s, y)) \cdot (\nabla\Psi)(U(t, s, y))\, dy,$$

and returning to the variable $x = U(t, s, y)$ in the last integral one gets (2.16) for ζ_2. One argues similarly with the first term in the right-hand side of (2.8).

LEMMA 2.4. *Let $v \in W^{1,2}(\Omega)$, $\operatorname{div} v = 0$ in Ω and $v \cdot n = 0$ on Γ. Put $\operatorname{rot} v = \zeta$. Then $\operatorname{rot}[(v \cdot \nabla)v] = \operatorname{div}(v\zeta)$ in the sense of distributions in Ω, i.e., $((v \cdot \nabla)v, \operatorname{Rot}\Psi) = (v\zeta, \nabla\Psi)$, $\forall \Psi \in C_0^\infty(\Omega)$.*

Proof. A direct computation shows that for a regular v, say, $v \in C^2(\Omega)$, the above equation holds pointwise. For a general v consider a sequence of regular ζ_m such that $\zeta_m \to \zeta$ in $L^2(\Omega)$. Denoting by ψ_m the solution of (1.2) with data ζ_m and defining $v_m = \operatorname{Rot}\psi_m$ it follows that $v_m \to v$ in $W^{1,2}(\Omega)$. This allows us to pass to the limit when $m \to +\infty$ in the above weak form. ∎

Now we verify that v is a solution of (1.1). Clearly $D_x v \in$

$C([0,T];L^p(\Omega))$, $\forall p<+\infty$. Moreover, $\zeta v\in C([0,T];L^2(\Omega))$ hence from Lemma 2.3 one gets $\partial\zeta/\partial t\in L^1(0,T;W^{-1,2}(\Omega))$. Recalling that $\theta=\zeta$ Eq. (1.2) yields $-\Delta(\partial\psi/\partial t)=\partial\zeta/\partial t$ in Ω, $\partial\psi/\partial t=0$ on Γ. Consequently $\partial\psi/\partial t\in L^1(0,T;H^{1,2}(\Omega))$ and $\partial v/\partial t=\mathrm{Rot}(\partial\psi/\partial t)\in L^1(0,T;L^2(\Omega))$. In particular $(\partial v/\partial t)+(v\cdot\nabla)v-f\in L^1(0,T;L^2\Omega))$. Moreover, $\mathrm{rot}[(\partial v/\partial t)+(v\cdot\nabla)v-f]=0$ in the distribution sense, by Lemmas 2.4 and 2.3. Consequently there exists $\pi\in L^1(0,T;W^{1,2}(\Omega))$ such that $(1.1)_1$ holds. On the other hand $\zeta_{|t=0}=\zeta_0$, i.e., $\mathrm{rot}\,v_{|t=0}=\mathrm{rot}\,v_0$ in Ω; $\mathrm{div}\,v_{|t=0}=\mathrm{div}\,v_0=0$ in Ω; and $v_{|t=0}\cdot n=v_0\cdot n=0$ on Γ. Hence $v_{|t=0}v_0$. Finally, the uniqueness of the solution v follows as in Bardos [1, Theorem 2].

3. Proof of Theorem 1.2

In this section we write $\zeta=\Phi_1(\theta,\zeta_0,\phi)$ instead of $\zeta=\Phi(\theta)$ since ζ_0 and ϕ are variable. For convenience we denote by ψ_1, ψ_2, ψ_3, respectively, the maps $v=\psi_1(\theta)$ defined by (1.3), $U=\psi_2(v)$ defined by (2.5), and $\zeta=\psi_3(U,\zeta_0,\phi)$ defined by (2.8). Hence $\Phi_1(\theta,\zeta_0,\phi)=\psi_3(\psi_2(\psi_1(\theta)),\zeta_0,\phi)$. The map Φ_1 is defined for every $(\theta,\zeta_0,\phi)\in C(\bar{Q}_T)\times C(\bar{\Omega})\times L^1(0,T;C(\bar{\Omega}))$. Note that v is the solution of problem (1.1) if and only if $v=\psi_1(\zeta)$ for a ζ verifying $\zeta=\Phi_1(\zeta,\zeta_0,\phi)$.

THEOREM 3.1. *Let $\mathbf{K}_1$ be a relatively compact set in $C(\bar{\Omega})$, $\mathbf{K}_2$ a relatively compact set in $L^1(0,T;C(\bar{\Omega}))$, and $\mathbf{K}$ a bounded set in $C(\bar{Q}_T)$. Then the set $\Phi_1(K\times K_1\times K_2)$ is relatively compact in $C(\bar{Q}_T)$.*

Proof. Let $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}$ be contained in balls with center in the origin and radius k_1, k_2, and B_1, respectively. The set of functions $\zeta_0(U(0,t,x))$, for $\theta\in\mathbf{K}$ and $\zeta_0\in\mathbf{K}_1$, is bounded in $C(\bar{Q}_T)$ by k_1. By the necessary condition of the Ascoli–Arzelà theorem the functions $\zeta_0\in\mathbf{K}_1$ are equicontinuous in $\bar{\Omega}$. By (2.6) the functions $U(0,t,x)$ are equicontinuous in $\bar{Q}_T$. Hence the family $\zeta_0(U(0,t,x))$ is equicontinuous in $\bar{Q}_T$ and by the Ascoli–Arzelà theorem constitutes a relatively compact set in $C(\bar{Q}_T)$.

Analogously the family

$$\zeta_2(t,x)=\int_0^t\phi(s,U(s,t,x))\,dx,\qquad\theta\in\mathbf{K},\quad\phi\in\mathbf{K}_2,\tag{3.1}$$

is bounded by k_2 in $C(\bar{Q}_T)$. We want to prove that every sequence $\zeta_2^{(m)}(t,x)$ contains a convergent subsequence in $C(\bar{Q}_T)$. This proves compactness for the family (3.1).

Let $\theta_m\in\mathbf{K}$ and $\phi_m\in\mathbf{K}_2$ be arbitrary sequences and consider

$$\zeta_2^{(m)}(t,x)=\int_0^t\phi_m(s,U_m(s,t,x))\,ds.\tag{3.2}$$

By the compactness of $\mathbf{K}_2$ there exists a subsequence of ϕ_m and a function $\phi \in L^1(0, T; C(\bar{\Omega}))$ such that $\phi_m \to \phi$ in $L^1(0, T; C(\bar{\Omega}))$.[4] Moreover a well-known theorem ensures the existence of a subsequence such that

$$\phi_m(s, \cdot) \to \phi(s, \cdot) \qquad \text{in } C(\bar{\Omega}), \text{ for almost all } s \in [0, T]. \tag{3.3}$$

Denote by $\omega_m(s, \varepsilon)$ the modulus of continuity of $\phi_m(s, \cdot)$ in $\bar{\Omega}$ (see (2.11)) and define $\bar{\omega}(s, \varepsilon) \equiv \sup_{m \in \mathbf{N}} \omega_m(s, \varepsilon)$. From (3.4) and from the Ascoli–Arzelà theorem it follows that

$$\lim_{\varepsilon \to 0} \bar{\omega}(s, \varepsilon) = 0, \qquad \text{for almost all} \quad s \in [0, T]. \tag{3.4}$$

Now let $\{a_k\}_{k \in N}$ be a sequence of real positive numbers such that $\sum_{k=1}^{+\infty} a_k < +\infty$. Since $\phi_m \to \phi$ in $L^1(0, T; C(\bar{\Omega}))$ there exists a subsequence ϕ_k such that

$$\int_0^T \|\phi(s) - \phi_k(s)\| \, ds \leqslant a_k, \qquad \forall k \in \mathbf{N}.$$

Define $b_0(s) \equiv \sum_{k=1}^{+\infty} \|\phi(s) - \phi_k(s)\|$, $s \in [0, T]$; clearly b_0 is integrable over $[0, T]$. Moreover $\omega_k(s, \varepsilon) \leqslant 2 \|\phi_k(s)\| \leqslant 2 \|\phi(s)\| + 2b_0(s) \equiv 2b(s)$ hence $\bar{\omega}(s, \varepsilon) \leqslant 2b(s)$ where $\bar{\omega}$ is defined with respect to the subsequence ω_k and $b(s)$ is integrable. By using (3.4) and Lebesgue's dominated convergence theorem it follows that to every $\nu > 0$ there corresponds an $\varepsilon_0 > 0$ such that

$$\int_0^T \omega_k(s, \varepsilon_0) \, ds < \nu, \qquad \forall k \in \mathbf{N}. \tag{3.5}$$

Equation (3.5) generalizes (2.12) in the proof of Theorem 2.1.

On the other hand, by the boundedness of $\mathbf{K}$, the functions v_k and U_k verify (2.4) and (2.6) uniformly with respect to k. Hence (2.13) holds for every U_k with $\lambda_2 = \lambda_2(\varepsilon_0)$ independent of k. We now proceed as in the proof of Theorem 2.1 and we show the equicontinuity of the set of functions $\zeta_2^{(k)}(t, x)$ in $\bar{Q}_T$ (note that (2.10) holds uniformly with respect to k, since $\|\phi_k(s)\| \leqslant b(s)$). From the equicontinuity the existence of a subsequence convergent in $C(\bar{Q}_T)$ follows. ∎

THEOREM 3.2. *The map $\Phi_1 : C(\bar{Q}_T) \times C(\bar{\Omega}) \times L^1(0, T; C(\bar{\Omega})) \to C(\bar{Q}_T)$ is continuous.*

Proof. Let $(\theta_m, \zeta_0^{(m)}, \phi_m) \to (\theta, \zeta_0, \phi)$. Arguing as in the proof of the continuity of the map Φ in Theorem 2.2 one shows that $v_m \equiv \psi_1(\theta_m) \to v \equiv \psi_1(\theta)$ uniformly in $\bar{Q}_T$, consequently $U_m \equiv \psi_2(v_m) \to U \equiv \psi_2(v)$ uniformly in

[4] For convenience we use the same index m for sequences and for subsequences.

$[0, T]^2 \times \bar{\Omega}$. Now one easily verifies that $\zeta_m \equiv \psi_3(U_m, \zeta_0^{(m)}, \phi_m) \to \zeta \equiv \psi_3(U, \zeta_0, \phi)$ pointwise in Q_T since

$$\int_0^t |\phi_m(s, U_m(x, t, x)) - \phi(s, U(s, t, x))|\, ds$$

$$\leqslant \int_0^t \|\phi_m(s) - \phi(s)\|\, ds + \int_0^t |\phi(s, U_m(s, t, x)) - \phi(s, U(s, t, x))|\, ds.$$

Now by using Theorem 3.1 with $\mathbf{K} = \{\theta_m\}$, $\mathbf{K}_1 = \{\zeta_0^{(m)}\}$, and $\mathbf{K}_2 = \{\phi_m\}$ it follows that the convergence of ζ_m to ζ is uniform in $\bar{Q}_T$ (this can be shown without resort to Theorem 3.1). ∎

Proof of Theorem 1.2. Assume the hypothesis of Theorem 1.2 and put $\zeta_0 \equiv \operatorname{rot} v_0$, $\phi \equiv \operatorname{rot} f$, $\zeta \equiv \operatorname{rot} v$, $\zeta_0^{(m)} \equiv \operatorname{rot} v_0^{(m)}$, $\phi_m \equiv \operatorname{rot} f_m$, $\zeta_m \equiv \operatorname{rot} v_m$, $\forall m \in \mathbf{N}$. By the assumptions $\zeta_m = \Phi_1(\zeta_m, \zeta_0^{(m)}, \phi_m)$, $\forall m \in \mathbf{N}$. Further, $\zeta_0^{(m)} \to \zeta_0$ in $C(\bar{\Omega})$ and $\phi_m \to \phi$ in $L^1(0, T; C(\bar{\Omega}))$.

Define $\mathbf{K} = \{\zeta_m\}$, $\mathbf{K}_1 = \{\zeta_0^{(m)}\}$, $\mathbf{K}_2 = \{\phi_m\}$. From (2.8) it follows that a set $\psi_3(S, S_1, S_2)$ is bounded whenever S_1 and S_2 are bounded, independently of the particular set S. Consequently $\mathbf{K}$ is bounded because $\zeta_m = \psi_3(\psi_2(\psi_1(\zeta_m)), \zeta_0^{(m)}, \phi_m)$, $\forall m \in \mathbf{N}$. Now Theorem 3.1 shows that $\Phi_1(\mathbf{K}, \mathbf{K}_1, \mathbf{K}_2)$ is a relatively compact set in $C(\bar{Q}_T)$ hence $\mathbf{K} \subset \Phi_1(\mathbf{K}, \mathbf{K}_1, \mathbf{K}_2)$ verifies the same property.

Let ζ_ν be any convergent subsequence of ζ_m and put for convenience $\bar{\zeta} \equiv \lim_{\nu \to +\infty} \zeta_\nu$. From the identity $\zeta_\nu = \Phi_1(\zeta_\nu, \zeta_0^{(\nu)}, \phi_\nu)$ and from Theorem 3.2 it follows that $\bar{\zeta} = \Phi_1(\bar{\zeta}, \zeta_0, \phi)$. Consequently $\bar{v} = \psi_1(\bar{\zeta})$ is a solution of (1.1) hence $\bar{v} = v$ and $\bar{\zeta} = \zeta$. It follows that all the sequence ζ_m converges to ζ uniformly in $C(\bar{Q}_T)$, i.e., $v_m \to v$ in $C([0, T]; E(\bar{\Omega}))$. ∎

Remark 3.1. In Theorem 1.2 convergence of $f^{(m)}$ to f is not requested since v is determined by system (4.2). Convergence of $f^{(m)}$ to f in $L^1_{\mathrm{loc}}(\mathbf{R}; L^2(\Omega))$ would imply the additional convergence $\nabla \pi_m \to \nabla \pi$ in $L^1_{\mathrm{loc}}(\mathbf{R}; L^2(\Omega))$.

4. Proof of Theorem 1.4

We start by proving that composition of C_*-functions with Hölder continuous functions yields C_*-functions.

LEMMA 4.1. *Let* $a \in C_*(\bar{\Omega})$ *and* $U \in C^{0,\delta}(\bar{\Omega}; \bar{\Omega})$, $0 < \delta \leqslant 1$. *Then* $a \circ U \in C_*(\bar{\Omega})$, *moreover*

$$[a \circ U]_* \leqslant \frac{1}{\delta} \int_0^{[U]_\delta R^\delta} \omega_a(r) \frac{dr}{r}; \tag{4.1}$$

in particular

$$[\alpha \circ U]_* \leqslant \frac{1}{\delta} [\alpha]_* + \frac{2}{\delta} \left(\log \frac{[U]_\delta R^\delta}{R} \right) \|\alpha\|, \tag{4.2}$$

where $R \equiv$ diameter Ω and the second term in the right-hand side of (4.2) *is dropped if* $([U]_\delta R^{(\delta)})/R \leqslant 1$.

Proof. Put $\zeta \equiv \alpha \circ U$, $[U]_\delta \equiv K$. One easily verifies that

$$\omega_\zeta(r) \leqslant \omega_\alpha(Kr^\delta), \qquad \forall r > 0,$$

consequently

$$[\zeta]_* \leqslant \int_0^R \omega_\alpha(Kr^\delta) \frac{dr}{r}.$$

By using the change of variables $\rho = Kr^\delta$ one has $d\rho/\rho = \delta\, dr/r$ hence

$$[\zeta]_* \leqslant \frac{1}{\delta} \int_0^{KR^\delta} \omega_\alpha(\rho) \frac{d\rho}{\rho} \leqslant \frac{1}{\delta} \int_0^R \omega_\alpha(\rho) \frac{d\rho}{\rho} + \frac{\omega_\alpha(R)}{\delta} \int_R^{KR^\delta} \frac{d\rho}{\rho}. \qquad \blacksquare$$

LEMMA 4.2. *Let U: $[0, T]^2 \times \bar{\Omega} \to \bar{\Omega}$ be a continuous map verifying*

$$|U(s, t, x) - U(s, t, y)| \leqslant K_1 |x - y|^\delta, \qquad \forall (s, t, x) \in [0, T]^2 \times \bar{\Omega}, \tag{4.3}$$

where $0 < \delta \leqslant 1$. Let $\phi \in L^1(0, T; C_*(\bar{\Omega}))$ and define

$$\zeta_2(t, x) \equiv \int_0^t \phi(s, U(s, t, x))\, ds. \tag{4.4}$$

Then $\zeta_2 \in C([0, T]; C_*(\bar{\Omega}))$ moreover

$$[\zeta_2(t)]_* \leqslant \frac{1}{\delta} \left\{ [\phi]_{L^1(0,t;C_*(\bar{\Omega}))} + 2 \log \frac{K_1 R^\delta}{R} \|\phi\|_{L^1(0,t;C(\bar{\Omega}))} \right\}, \tag{4.5}$$

where $[\phi]_{L^1(0,t;C_*(\bar{\Omega}))} \equiv \int_0^t [\phi(\tau)]_{C^*(\bar{\Omega})}\, d\tau$.

Proof. With straightforward calculations one shows that

$$[\zeta(t)]_* \leqslant \int_0^t ds \int_0^R \sup_{\substack{0 < |x-y| \leqslant r \\ x, y \in \bar{\Omega}}} |\phi(s, U(s, t, x)) - \phi(s, U(s, t, y))| \frac{dr}{r}, \tag{4.6}$$

hence

$$[\zeta(t)]_* \leqslant \int_0^t [\phi(s) \circ U_{s,t}]_*\, ds,$$

where $\phi(s) \equiv \phi(s, \cdot)$ and $U_{s,t} \equiv U(s, t, \cdot)$. By using (4.2) one gets

$$[\zeta(t)]_* \leqslant \int_0^t \left\{ \frac{1}{\delta} [\phi(s)]_* + \frac{2}{\delta} \log \left(\frac{K_1 R^\delta}{R} \right) \|\phi(s)\| \right\} ds, \tag{4.7}$$

i.e., Eq. (4.5).

We now prove the continuity statement. Assume, for instance, $t_0 < t$. From definition (4.4) one gets

$$\begin{aligned} &[\zeta(t) - \zeta(t_0)]_* \\ &\leqslant \int_{t_0}^t ds \int_0^R \sup_{\substack{0<|x-y|\leqslant r \\ x,y\in\bar{\Omega}}} |\phi(s, U(s,t,x)) - \phi(s, U(s,t,y))| \frac{dr}{r} \\ &\quad + \int_0^T ds \int_0^R \sup_{\substack{0<|x-y|\leqslant r \\ x,y\in\bar{\Omega}}} |\phi(s, U(s,t,x)) - \phi(s, U(s,t_0,x)) \\ &\quad - \phi(s, U(s,t,y)) + \phi(s, U(s,t_0,y))| \frac{dr}{r} \equiv B_1 + B_2. \end{aligned} \tag{4.8}$$

As for (4.6) we show that B_1 is bounded by the right-hand side of (4.7) with the interval $(0, t)$ replaced by (t_0, t); hence B_1 goes to zero when $|t - t_0|$ goes to zero. We now prove that $B_2 \to 0$ when $t \to t_0$. Assumption (4.3) yields $F(t_0, t, s, r) \leqslant 2r^{-1}\omega_{\phi(s)}(K_1 r^\delta)$ where $F(t_0, t, s, r)$ is the integrand in B_2. The above function is integrable over $[0, T] \times [0, R]$ since for almost all $s \in [0, T]$ one has

$$\int_0^R \omega_{\phi(s)}(K_1 r^\delta) \frac{dr}{r} \leqslant \frac{1}{\delta} \left\{ [\phi(s)]_* + 2 \log \left(\frac{K_1 R^\delta}{R} \right) \|\phi(s)\| \right\},$$

as one shows by arguing as in the proof of Lemma 4.1. Moreover for every $s \in [0, T]$ for which $\phi(s, \cdot) \in C(\bar{\Omega})$, and for every $r \in]0, R]$, one has $\lim_{t\to t_0} F(t_0, t, s, r) = 0$. An application of Lebesgue's dominated convergence theorem proves that $B_2 \to 0$ if $t \to t_0$. ∎

LEMMA 4.3. *Let U verify the assumptions of the preceding lemma, let $\zeta_0 \in C_*(\bar{\Omega})$, and define $\zeta_1(t, x) \equiv \zeta_0(U(0, t, x))$. Then $\zeta_1 \in C([0, T]; C_*(\bar{\Omega}))$, moreover*

$$[\zeta_1(t)]_* \leqslant \frac{1}{\delta} [\zeta_0]_* + \frac{2}{\delta} \log \left(\frac{K_1 R^\delta}{R} \right) \|\zeta_0\|, \qquad \forall t \in [0, T]. \tag{4.9}$$

Proof. Estimate (4.9) follows from Lemma 4.1. The continuity statement follows as in the preceding lemma (with many simplications).

Equations (2.6), (2.7), (2.8), definition of δ, and the two preceding lemmas give the following result:

LEMMA 4.4. *Assume that hypothesis of Theorem* 1.4 *holds and let* $\zeta \equiv \operatorname{rot} v$, $\phi \equiv \operatorname{rot} f$, $\zeta_0 \equiv \operatorname{rot} v_0$. *Then* $\zeta \in C(\mathbf{R}; C_*(\bar{\Omega}))$, *moreover for every* $t \in \mathbf{R}$

$$\|\zeta(t)\|_* \leqslant e^{c_1 B|t|}\{[\zeta_0]_* + [\phi]_{L^1(0,t;C_*(\bar{\Omega}))} + 3\|\zeta_0\| + 3\|\phi\|_{L^1(0,t;C(\bar{\Omega}))}\}, \tag{4.10}$$

where $B \equiv \|\zeta_0\| + \|\phi\|_{L^1(0,t;C(\bar{\Omega}))}$.

The following theorem is crucial for our proof.

THEOREM 4.5. *Let* $\theta \in C_*(\bar{\Omega})$ *and let* ψ *be the solution of problem* (1.2). *Then* $\psi \in C^2(\bar{\Omega})$, *moreover*

$$\|\psi\|_2 \leqslant c_0\|\theta\|_*, \qquad \forall \theta \in C_*(\bar{\Omega}). \tag{4.11}$$

This result seems well known even if an exact reference is not available to us (see [2, Chap. 4, problem 4.2]); we are able to prove it for a uniformly elliptic second-order equation $L\psi = \theta$ in Ω, $Bu = 0$ on Γ, at least if L has smooth coefficients and the boundary operator B is regular (for instance, Dirichlet or Neumann boundary value problem). This result doesn't depend on the dimension $n \geqslant 2$.

The main statement in Theorem 1.4 follows immediately from $v \equiv \operatorname{Rot}\psi$ and from (4.10), (4.11); recall that $\theta = \zeta$. Moreover if g and ∇F are continuous in $\bar{Q}_T$ it follows from (5.3) that $\nabla\pi_1$ is continuous, from $\nabla\pi = \nabla\pi_1 + \nabla F$ that $\nabla\pi$ is continuous, and from $(1.1)_1$ or $(5.2)_1$ that $\partial v/\partial t$ is continuous. ∎

APPENDIX 1

We recall some well-known facts about vector fields defined in non-simply-connected domains. Let Ω be an $(N+1)$-times connected bounded region, the boundary of which consists of simple closed curves $\Gamma_0, \Gamma_1, ..., \Gamma_N$, the curve Γ_0 containing the others. In that case the kernel of the linear system $\operatorname{rot} v = 0$ in Ω, $\operatorname{div} v = 0$ in Ω, $v \cdot n = 0$ on Γ has finite dimension N. Let us fix a base $u_1, ..., u_N$ and assume for convenience that $(u_i, u_k) = \delta_{ik}$, $i, k = 1, ..., N$. Any tangential flow (vector field verifying $\operatorname{div} v = 0$ in Ω, $v \cdot n = 0$ on Γ) is uniquely determined by the field $\operatorname{rot} v$ in Ω and by the real numbers (v, u_k), $k = 1, ..., N$. The quantity $|||v||| = \|\operatorname{rot} v\| + \sum_{k=1}^{N} |(v, u_k)|$ is a norm in $E(\bar{\Omega})$, equivalent to the norm $\|\operatorname{rot} v\| + \|v\|_{L^2(\Omega)}$.

Let now f be an arbitrary vector field in Ω. Solve the problem $-\Delta\psi_0 = \operatorname{rot} f$ in Ω, $\psi_0 = 0$ on Γ and put $g_0 \equiv \operatorname{Rot}\psi_0$. Clearly $\operatorname{rot} g_0 = \operatorname{rot} f$, $\operatorname{div} g_0 = 0$, and $g_0 \cdot n = 0$ on Γ. If $g \equiv g_0 + \sum_k \lambda_k u_k$, where $\lambda_k \equiv (f, u_k)$, it follows that g is a tangential flow, moreover $\operatorname{rot}(f - g) = 0$ in Ω, $(f - g, u_k) = 0$, $k = 1,..., N$. Hence there exists a scalar field F such that $f - g = \nabla F$ in Ω, i.e., the vector field g is the tangential flow in the canonical decomposition

$$f = g + \nabla F. \tag{5.1}$$

Note that g depends only on $\operatorname{rot} f$ and on the N real numbers (f, u_k).

Appendix 2

Let us decompose the external force f in Eq. $(1.1)_1$ as indicated in (5.1) and let us consider the auxiliary problem

$$\begin{aligned}
&\frac{\partial v}{\partial t} + (v \cdot \nabla)v = g - \nabla\pi_1 && \text{in } Q,\\
&\operatorname{div} v = 0 && \text{in } Q,\\
&v \cdot n = 0 && \text{on } \Sigma,\\
&v_{|t=0} = v_0 && \text{in } \Omega.
\end{aligned} \tag{5.2}$$

The solution of (1.1) consists on the same velocity field v as in (5.2) and on the pressure term $\nabla\pi = \nabla\pi_1 + \nabla F$. Moreover, from (5.2) it follows that

$$\begin{aligned}
-\Delta\pi_1 &= \sum_{i,j=1}^{2} \frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j},\\
\frac{\partial\pi_1}{\partial n} &= \sum_{i,j=1}^{2} \frac{\partial n_i}{\partial x_j} v_i v_j.
\end{aligned} \tag{5.3}$$

Assume that the regularity of $\nabla v(t)$ is known. Then the elliptic boundary value problem (5.3) gives the regularity of $\nabla\pi_1$ and (5.2) gives the regularity of $\partial v/\partial t$. In particular various regularity results for $\partial v/\partial t$ (and for $\nabla\pi$) are trivially obtained by assuming different conditions on f. Hence the regularity of $\nabla v(t)$ is the basic one. Note by the way that $\nabla\pi$ is the only term depending fully on f. The other terms considered above depend only on $\operatorname{rot} f$ and on (f, u_k), $k = 1,..., N$.

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