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INHOMOGENEOUS EVOLUTION EQUATIONS IN BANACH SPACES WITH A BOUNDED VARIATION DATA

HUGO BEIRÃO DA VEIGA
Facoltà di Scienze, 38050 POVO (Trento), Italy

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1. INTRODUCTION

Some notations and assumptions

LET X BE a real or complex Banach space, X^* its dual space, X_w the linear space X endowed with the weak topology $\mathscr{T}(X, X^*)$ and $\mathscr{L}(X)$ the linear space of all bounded linear operators on X into X endowed with the uniform operator topology. Norms in X and in $\mathscr{L}(X)$ are denoted by the same symbol $\| \;\|$. The symbol "$\rightharpoonup$" denotes weak convergence.

When the following condition holds

$$\text{there exists a Banach space } Y \text{ such that } Y^* = X, \tag{1.1}$$

we denote by X_{w^*} the linear space X endowed with the weak* topology $\mathscr{T}(X, Y)$. The symbol "$\rightharpoonup^*$" denotes weak* convergence.

The results proved in this paper concern: (i) the case where X is reflexive; (ii) the more general case where X is non-reflexive but (1.1) holds.

Let $T > 0$ be a real number and k a non-negative integer. We denote by $C^k([0, T]; X)$ the space of k times continuously differentiable functions on the closed interval $[0, T]$ with values in X and by $BV([0, T]; X)$ the space of the bounded variation functions. The total variation of $f(t)$ in $[0, T]$ is denoted by $V(f)$. We put $|||f|||_\infty = \sup\|f(t)\|$ for $t \in [0, T]$, $f^+(t) = \lim f(t+h)$ when $h \to 0^+$, $f^-(t) = \lim f(t-h)$ when $h \to 0^+$.

Finally $\mathrm{Lip}([0, T]; X)$ is the space of the Lipschitz continuous functions on $[0, T]$ with values on X. We put

$$|||v|||_{0,1} = \sup_{t_0, t \in [0, T]} \left\| \frac{v(t) - v(t_0)}{t - t_0} \right\|.$$

In the sequel $S(t)$, $0 \leqslant t < +\infty$, is a strongly continuous semigroup of operators on X with infinitesimal generator $-A$ (for definitions and properties see [1] and [2]).

When X is non-reflexive but (1.1) holds we assume that the operators A verifies the following property:

$$\text{if } x_n \to x \quad \text{and} \quad Ax_n \rightharpoonup^* y \in X \tag{1.2}$$

then $x \in D(A)$ and $Ax = y$.

If A is a differential operator the property (1.2) holds for the usual functional spaces since Ax_n converges to Ax in the distribution sense.

The results obtained in this paper when X is reflexive are a particular case of the results obtained when X is non-reflexive but (1.1) and (1.2) hold. In fact if X is reflexive it verifies (1.1) with

$Y = X^*$ and furthermore the weak* topology $\mathscr{T}(X, Y)$ is nothing but the usual weak topology $\mathscr{T}(X, X^*)$; moreover the condition (1.2) holds since A has a closed graph in $X \times X_w$. However we give first the results and proofs for the reflexive case and afterwards the modifications for the more general case (1.1), (1.2).

Results

This paper is concerned with the non-homogeneous Cauchy problem

$$\begin{aligned} u'(t) + Au(t) &= f(t) \quad \text{on } [0, T], \\ u(0) &= u_0, \end{aligned} \tag{1.3}$$

where $u_0 \in D(A)$ and $f: [0, T] \to X$.

Definition 1.1. One says that $u: [0, T] \to X$ is a *solution* of the Cauchy problem (1.3) if $u(t)$ is absolutely continuous (a.c.) and almost everywhere differentiable on $[0, T]$†, (1.4)

$$u(t) \in D(A) \quad \text{for all } t \in [0, T], \tag{1.5}$$

$$u'(t) + Au(t) = f(t), \quad \text{a.e. in } [0, T], \tag{1.6}$$

$$u(0) = u_0. \tag{1.7}$$

We recall the following result due to Phillips [3]:
If $f \in C^1([0, T]; X)$ and $u_0 \in D(A)$ then the function

$$u(t) = S(t)\, u_0 + v(t), \tag{1.8}$$

where

$$v(t) = \int_0^t S(t - s)\, f(s)\, ds, \tag{1.9}$$

is continuously differentiable, belongs to $D(A)$ for all $t \in [0, T]$ and is a solution (the unique) of problem (1.3). Moreover

$$\begin{cases} -Av(t) = S(t)\, f(0) - f(t) \\ \qquad\qquad + \displaystyle\int_0^t S(t - s) f'(s)\, ds. \end{cases} \tag{1.10}$$

This result holds again if f is assumed to be only a.c. and a.e. differentiable in $[0, T]$ (see Brezis [4] and Pazy [5]).

Our aim is to prove the existence of a strong solution (in the sense of Definition 1.1) when $f \in BV([0, T]; X)$; in the general Banach space case this is not possible without further assumptions since counterexamples (even with a Lipschitz continuous data) are well known. Since the initial data u_0 belongs to $D(A)$ it is obvious that to solve the problem (1.3) it suffices to restrict our attention to the problem

$$\begin{cases} v'(t) + Av(t) = f(t), \quad \text{a.e. on } [0, T], \\ v(0) = 0. \end{cases} \tag{1.11}$$

We prove in this paper the following results:

†If X is reflexive the last property is implied by the first one.

THEOREM 1.1. Let X be reflexive and $f \in BV([0, T]; X)$. Then for all $t \in [0, T]$ the integral in the right hand side of (1.9) exists in the Riemann sense and the function $v(t)$ satisfies the following properties:

(a) For every $t \in [0, T]$ one has $v(t) \in D(A)$ and

$$\begin{cases} -Av(t) = \text{w-}\int_0^t [dS(s)] f(t-s) \\ \qquad\quad = S(t) f(0) - f(t) + \text{w-}\int_0^t S(t-s) \, df(s). \end{cases} \tag{1.12}$$

Moreover $Av(t)$ is weakly continuous on $[0, T]$ and right continuous on $[0, T[$.

(b) The function $v(t)$ is Lipschitz continuous on $[0, T]$ and

$$|||v|||_{0,1} \leqslant (|||f|||_\infty + V(f)) \sup_{[0,T]} \|S(t)\|.$$

(c) The function $v(t)$ is a solution of the problem (1.11) in the sense of Definition 1.1.

(d) Equations

$$\frac{d^+ v(t)}{dt} + Av(t) = f^+(t) \quad \text{for all } t \in [0, T[\tag{1.13}$$

and

$$\text{w-}\frac{d^- v(t)}{dt} + Av(t) = f^-(t) \quad \text{for all } t \in]0, T] \tag{1.14}$$

hold. In particular $d^+v(t)/dt$ is right continuous on $[0, T[$ and w-$(d^-v(t)/dt)$ is left weakly continuous on $]0, T]$.

Moreover if $Av(t)$ is left continuous at a point t_0 the strong derivative $d^-v(t)/dt$ exists (and hence verifies (1.14)) at t_0.

The integrals in (1.12) are weak Riemann-Stieltjes integrals. Note that if $f(t)$ is regular the known formula (1.10) follows from (1.12).

Remark 1.1. If $-A$ is the infinitesimal generator of a strongly continuous *group* of operators then $Av(t)$ is continuous on $[0, T]$ and (1.14) holds strongly,

Remark 1.2. Assume that $f \in BV([0, T]; X) \cap C^0([0, T]; X)$. Then the weak derivative w-(dv/dt) exists for all $t \in [0, T]$, is a weakly continuous function on $[0, T]$ and

$$\text{w-}\frac{dv}{dt} + Av(t) = f(t) \quad \text{for all } t \in [0, T].$$

Remark 1.3. In the Appendix it is shown that if $f \in L^1(\varepsilon, T; X)$, $\forall \varepsilon > 0$, and $u(t)$ is a "solution" of (1.3) then $u(t)$ must be given by (1.8), (1.9).

Remark 1.4. Under the assumption that X is a real Hilbert space and A is monotone (but not necessarily linear) it was proved by Brezis (see [6], chap. III, proposition 3.3) that if $u_0 \in D(A)$ and $f \in BV([0, T]; X)$ the equation (1.3) has a (unique) Lipschitz continuous solution $u(t)$ such that

$u(t) \in D(A)$ for all $t \in [0, T]$ and (1.13) and (1.14) hold. In this context (X real Hilbert space and A monotone) if A is linear then the existence of the solution $v(t)$ of the problem (1.11), guaranteed in this case by the result of [6], implies that $v(t)$ is given by formula (1.9). Formula (1.12) is new, even in this case.

In the non-reflexive case the following statement holds:

THEOREM 1.2. If X is non-reflexive but (1.1) and (1.2) hold then the results stated in Theorem (1.1), in proposition (1.1) and in the remarks (1.1) and (1.2) hold if we replace everywhere "weak" by "weak*".

Remark 1.5. In [7] Webb has proved some results which we summarize as follows:
Theorem 1.1 holds in a general Banach space if the semigroup $S(t)$ verifies

$$S(t)\,X \subset D(A), \qquad \forall t > 0, \tag{1.15}$$

and hence if $S(t)$ is an analytic semigroup; moreover one can replace everywhere in Theorem 1.1 "weak" by "strong".

In connection with this last statement we remark that in our paper the left continuity of $Av(t)$ on $]0, T]$ (and the existence of a strong left derivative $d^- v(t)/dt$ in all of $]0, T]$; see the last statement in Theorem 1.1) remains open.

However the right strong continuity of $Av(t)$, which was not proved in a first version of our paper, is now proved by simplifying a device used in the Lemma 3.2 of [7].

The author is indebted to H. Brezis who called his attention to the Webb's paper [7] and has made useful remarks.

2. PROOFS OF THE BASIC RESULT

We recall that a function $g: [a, b] \to X$ is said to be of *bounded variation* if

$$\sup \sum_{i=0}^{n-1} \|g(t_{i+1}) - g(t_i)\| < +\infty,$$

where all possible finite partitions $t_0 = a \leqslant t_1 \leqslant \ldots \leqslant t_n = b$ are allowed for. The suprema, known as the *total variation* of g on $[a, b]$ is denoted by $V(g; [a, b])$. We put $V(g; [0, T]) = V(g)$. If g is of bounded variation on $[a, b]$ we write $g \in BV([a, b]; X)$.

The following results are well known (see for instance [6], Appendix 2): if g is of bounded variation in $[a, b]$ the discontinuity points to g are (at most) a denumerable set $N(g)$. Furthermore the limits

$$\lim_{\tau \to t^+} g(\tau) = g^+(t), \qquad t \in [a, b[\tag{2.1}$$

and

$$\lim_{\tau \to t^-} g(\tau) = g^-(t), \qquad t \in]a, b] \tag{2.2}$$

exist. The function $g^+(t)$ is right continuous on $[a, b[$ and its total variation verifies $V(g^+; [a, b[) \leqslant V(g; [a, b])$. Corresponding results hold for $g^-(t)$ on $]a, b]$.

We denote a finite partition $a = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n = b$ together with the points $\tau_i \in [t_i, t_{i+1}]$,

$0 \leqslant i \leqslant n-1$, by π and we set

$$|\pi| = \max_{0 \leqslant i \leqslant n-1} |t_{i+1} - t_i|.$$

The set of all such partitions, together with the partial ordering "$\pi < \pi'$ if $|\pi| \geqslant |\pi'|$" is a directed set. Let now $T: [a, b] \to \mathscr{L}(X)$, $g: [a, b] \to X$ and consider the Riemann–Stieltjes sums

$$S_\pi(T, g) = \sum_{i=0}^{n-1} T(\tau_i)[g(t_{i+1}) - g(t_i)]. \tag{2.3}$$

If the weak limit $\text{w-}\lim_{|\pi| \to 0} S_\pi(T, g)$ exists in X this limit is (by definition) the weak Riemann–Stieltjes integral

$$\text{w-}\int_a^b T(t)\, \mathrm{d}g(t).$$

In a similar way one defines the weak Riemann–Stieltjes integral

$$\text{w-}\int_a^b [\mathrm{d}T(t)]\, g(t)$$

as the weak limit of the sums

$$S_\pi(g, T) = \sum_{i=0}^{n-1} [T(t_{i+1}) - T(t_i)]\, g(\tau_i) \tag{2.4}$$

as $|\pi| \to 0$. The existence of one of the integrals implies the existence of the other and

$$\text{w-}\int_a^b [\mathrm{d}T(t)]\, g(t) = T(t)\, g(t)\Big|_a^b - \text{w-}\int_a^b T(t)\, \mathrm{d}g(t). \tag{2.5}$$

If the condition (1.1) holds one has similar definitions for the weak* Riemann–Stieltjes integrals.

Let now $g: [a, b] \to X$ where $0 \leqslant a < b < +\infty$ and X is a generical real or complex Banach space and let $S(t)$ be a strongly continuous semigroup on X. Put

$$R_\pi(S; g) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} S(\tau)\, g(\tau_i)\, \mathrm{d}\tau. \tag{2.6}$$

One has the following result:

LEMMA 2.1. Assume that $g(t)$ is bounded and continuous a.e. in $[a, b]$. Then

$$\lim_{|\pi| \to 0} R_\pi(S; g) = \int_a^b S(\tau)\, g(\tau)\, \mathrm{d}\tau \tag{2.7}$$

where the last integral exists in the Riemann sense.

Proof. Denote by E the set of points where g is discontinuous. Since $S(t)\, g(t)$ is continuous in points $t \notin E$ and is bounded on the bounded interval $[a, b]$ it follows that the integral in the equation (2.7) exists in the Riemann sense. Put now

$$F_\pi(\tau) = S(\tau)\, g(\tau_i), \qquad \text{if } \tau \in [t_i, t_{i+1}[. \tag{2.8}$$

One sees easily that

$$\lim_{|\pi| \to 0} F_\pi(\tau) = S(\tau)\, g(\tau) \quad \text{a.e. in } [a, b]. \tag{2.9}$$

Moreover $\|F_\pi(\tau)\|$ is bounded on $[a, b]$ uniformly with respect to π. Consequently by the Lebesgue dominated convergence theorem

$$\lim_{|\pi| \to 0} \int_a^b F_\pi(\tau)\, \mathrm{d}\tau = \int_a^b S(\tau)\, g(\tau)\, \mathrm{d}\tau. \tag{2.10}$$

We are now able to prove the following result:

THEOREM 2.1. Let X be a (real or complex) reflexive Banach space and $S(t)$ a strongly continuous semigroup in X with generator $-A$. Let $g \in BV([a, b]; X)$, $0 \leqslant a < b < +\infty$. Then

$$\int_a^b S(\tau)\, g(\tau)\, \mathrm{d}\tau \in D(A) \tag{2.11}$$

and

$$\begin{aligned} -A \int_a^b S(\tau)\, g(\tau)\, \mathrm{d}\tau &= \text{w-}\int_a^b [\mathrm{d}S(\tau)]\, g(\tau) \\ &= S(t)\, g(t)\Big|_a^b - \text{w-}\int_a^b S(\tau)\, \mathrm{d}g(\tau). \end{aligned} \tag{2.12}$$

If X is non-reflexive but the conditions (1.1) and (1.2) are verified the statements of the theorem hold again with "weak" replaced by "weak*".

Proof. Let $R_\pi(s, g)$ be defined by (2.6). Since

$$-A \int_{t_i}^{t_{i+1}} S(\tau)\, g(\tau_i)\, \mathrm{d}\tau = S(t_{i+1})\, g(\tau_i) - S(t_i) g(\tau_i)$$

it follows that $R_\pi(s; g) \in D(A)$ and

$$-AR_\pi(S; g) = S(t)\, g(t)\Big|_a^b - \sum_{i=0}^{n-1} S(t_{i+1})[g(\tau_{i+1}) - g(\tau_i)] \tag{2.13}$$

where $\tau_{-1} = a$ and $\tau_n = b$. In particular

$$\|-AR_\pi(S, g)\| \leqslant \|S(b)\, g(b) - S(a)\, g(a)\| + \sup_{[a,b]} \|S(t)\| \cdot V(g; [a, b]) \tag{2.14}$$

and consequently $\|-AR_\pi(S: g)\|$ is bounded uniformly with respect to π. Since X is reflexive it follows that the set of all the elements $-AR_\pi(S; g)$ is conditionally compact in X_w. On the other hand the graph of the operator A is closed in $X \times X_w$ since it is a linear closed subspace of $X \times X$. It follows from these results and from Lemma 2.1 that (2.11) holds and

$$-A \int_a^b S(\tau)\, g(\tau)\, \mathrm{d}\tau = \text{w-} \lim_{|\pi| \to 0} \sum_{i=0}^{n-1} [S(t_{i+1} - S(t_i)]\, g(\tau_i)$$

$$= S(t)\,g(t)\Big|_a^b - \text{w-}\lim_{|\pi|\to 0}\sum_{i=0}^{n-1} S(t_{i+1})[g(\tau_{i+1}) - g(\tau_i)].$$

Thus Theorem 2.1 is proved in the reflexive case. If X is not reflexive but (1.1) and (1.2) hold, the proof is analogous; the Banach–Alaoglu theorem guarantees the conditionally compactness of $-AR_\pi(S; g)$ in X_{w*} and the condition (1.2) guarantees that the operator A (i.e. the graph of A) is sequentially closed in $X \times X_{w*}$ (sequentially closedness of A is sufficient since the convergence of a net u_π to u is implied by the convergence to u of all the sequences u_{π_n} for which $\lim_{n\to+\infty} |\pi_n| = 0$).

3. PROOF OF THEOREM 1.1 AND 1.2

In this section we prove the results stated in the introduction. The statements corresponding to the non-reflexive case shall be proved together with the corresponding statements for the reflexive one.

Proof of statement (*a*). By putting $\tau = t - s$ and $g(\tau) = f(t - \tau)$, $s \in [0, t]$, (1.9) becomes

$$v(t) = \int_0^t S(\tau)\,g(\tau)\,d\tau. \tag{3.1}$$

By using Theorem 2.1 it follows that $v(t) \in D(A)$, $\forall t \in [0, T]$, and that (1.12) holds. From (1.12) one gets easily the boundedness of $Av(t)$ on $[0, T]$ and

$$|||Av|||_\infty \leqslant |||f|||_\infty + (\|f(0)\| + V(f)) \sup_{[0,T]} \|S(t)\|. \tag{3.2}$$

Since $v(t)$ is continuous, A is closed and X is reflexive it follows that $Av(t)$ is weakly continuous on $[0, T]$. If X is non-reflexive but (1.1) and (1.2) hold we use the Banach–Alaoglu theorem and (1.2) to prove the weak* continuity of $Av(t)$.

Now we prove that $Av(t)$ is right continuous on $[0, T[$. Let $t, t+h \in [0, T[$, $h > 0$; one has

$$Av(t+h) - Av(t) = A\int_t^{t+h} S(t+h-s)\,f(s)\,ds + (S(h) - I)\,Av(t)$$

and by Theorem 2.1 one gets

$$Av(t+h) - Av(t) = \text{w-}\int_t^{t+h} S(t+h-s)\,df(s)$$
$$- S(h)[f^+(t) - f(t)] + [S(h)f^+(t) - f(t+h)] + (S(h) - I)\,Av(t). \tag{3.3}$$

The two last terms in (3.3) converges to zero when $h \downarrow 0$. On the other hand†

$$\text{w-}\int_t^{t+h} S(t+h-s)\,df(s) = S(h)[f^+(t) - f(t)] + \text{w-}\int_{]t,t+h]} S(t+h-s)\,df(s)$$

and the last integral converges to zero when $h \downarrow 0$ since it is bounded in norm by

$$V(f;]t, t+h]) \sup_{\tau\in[0,h]} \|S(\tau)\|$$

as follows from the lower semicontinuity of the norm in the weak (or in the weak*) convergence.

† The Riemann–Stieltjes integral in a left open interval $]a, b]$ is defined as the limit of the sums (2.3) with $\sum_{i=0}^{n-1}$ replaced by $\sum_{i=1}^{n-1}$; a corresponding definition hold for the total variation of f in $]a, b]$.

Proof of statement (*b*): Let $t, t + h \in [0, T]$ with $h > 0$. Then

$$v(t + h) - v(t) = \int_0^{t+h} S(t + h - s) f(s) \, ds - \int_0^t S(t - \tau) f(\tau) \, d\tau$$

and by putting in the first integral $s - h = \tau$ one gets

$$v(t + h) - v(t) = \int_{-h}^0 S(t - \tau) f(\tau + h) \, d\tau + \int_0^t S(t - \tau)[f(\tau + h) - f(\tau)] \, d\tau.$$

Therefore

$$\|v(t + h) - v(t)\| \leqslant \sup_{[0, T]} \|S(t)\| \left(|||f|||_\infty h + \int_0^t \|f(\tau + h) - f(\tau)\| \, d\tau \right). \tag{3.4}$$

On the other hand

$$\|f(\tau + h) - f(\tau)\| \leqslant V(f; [\tau, \tau + h]) = V(f; [0, \tau + h]) - V(f; [0, \tau]).$$

Putting $V(s) = V(f; [0, s])$ one has

$$\int_0^t \|f(\tau + h) - f(\tau)\| \, d\tau \leqslant \int_0^t (V(\tau + h) - V(\tau)) \, d\tau$$
$$= \int_h^{t+h} V(\tau) \, d\tau - \int_0^t V(\tau) \, d\tau \leqslant \int_t^{t+h} V(\tau) \, d\tau$$

since $V(\tau)$ is non-negative. Hence

$$\int_0^t \|f(\tau + h) - f(\tau)\| \, d\tau \leqslant hV(f). \tag{3.5}$$

Statement (b) follows now from (3.4) and (3.5).

Proof of (1.13): Let $t \in [0, T[$ and $h > 0$. One has

$$(v(t + h) - v(t))/h = \int_0^t ((S(h) - I)/h)S(t - s) f(s) \, ds + (1/h) \int_t^{t+h} S(t + h - s) f(s) \, ds;$$

consequently

$$(v(t + h) - v(t))/h = ((S(h) - I)/h) v(t) + (1/h) \int_t^{t+h} S(t + h - s)[f(s) - f^+(t)] \, ds$$
$$+ (1/h) \int_t^{t+h} S(t + h - s) f^+(t) \, ds.$$

Since $v(t) \in D(A)$ one sees easily that

$$\lim_{h \to 0^+} (v(t + h) - v(t))/h = -Av(t) + f^+(t).$$

Proof of statement (c): Since $v(t)$ is Lipschitz continuous and X is reflexive $v'(t)$ exists a.e. in $[0, T]$. The statement (c) follows now from (1.13) since $f(t) = f^+(t)$ excluding at most a denumerable set of values of t. In the non-reflexive case the Lipschitz continuity of $v(t)$ and the existence of a bounded and measurable (hence integrable) $d^+v(t)/dt$ implies that

$$v(t) = \int_0^t \frac{d^+v(\tau)}{d\tau}\, d\tau.$$

Hence $v'(t)$ exists and is a.e. equal to $d^+v(t)/dt$ in the interval $[0, T]$.

Proof of (1.14): Let $t \in]0, T]$. One has

$$v(t) = \int_0^t v'(s)\, ds = -\int_0^t Av(s)\, ds + \int_0^t f^-(s)\, ds$$

where the integrals here are in the Bochner sense. Thus, if $h > 0$,

$$(v(t-h) - v(t))/-h = -(1/h)\int_{t-h}^t Av(s)\, ds + (1/h)\int_{t-h}^t f^-(s)\, ds. \tag{3.6}$$

Since $Av(s)$ is weakly continuous (w^*-continuous in the non-reflexive case with (1.1), (1.2)) and $f^-(s)$ is left continuous the result follows easily from (3.6).

Proof of the other assertions: The last statement of Proposition 1.1 follows from (3.6). The Remark 1.1 is obvious and the remark 1.2 follows directly from the other results.

4. APPENDIX

In this section X is an *arbitrary* real or complex Banach space and A^* is the adjoint of A. We denote by X_0^* the closure of $D(A^*)$ in X^* and we put

$$D_0(A^*) = \{x^* \in D(A^*) : A^*x^* \in X_0^*\}.$$

It is well known that the restriction of A^* to $D_0(A^*)$ is the infinitesimal generator of the strongly continuous semi-group $S^*(t)_{|X_0^*}$. Moreover the w^*-closure of $D_0(A^*)$ is X^*.

The space of the Bochner integrable functions on $]a, b[$ with values in X is denoted by $L^1(a, b; X)$.

PROPOSITION 4.1. Let $f \in L^1(\varepsilon, T; X)$ for all $\varepsilon > 0$ and assume that the function $u:]0, T] \to X$ satisfies the following conditions:

$$u(t) \in D(A) \quad \text{a.e. in }]0, T], \tag{4.1}$$

$$u(t) \text{ is absolutely continuous and a.e. differentiable on } [\varepsilon, T] \text{ for all } \varepsilon > 0. \tag{4.2}$$

$$u'(t) + Au(t) = f(t) \quad \text{a.e. in }]0, T[. \tag{4.3}$$

Then

(a) If $f \in L^1(0, T; X)$ and if

$$\text{w-}\lim_{t \to 0^+} u(t) = u_0 \in X$$

one has for every $t \in [0, T]$

$$u(t) = S(t)u_0 + \int_0^t S(t-s) f(s)\, ds. \tag{4.4}$$

(b) If $\lim_{t \to 0^+} u(t) = u_0 \in X$ one has for every $t \in [0, T]$

$$u(t) = S(t)u_0 + \lim_{\varepsilon \to 0^+} \int_\varepsilon^t S(t-s) f(s)\, ds. \tag{4.5}$$

(c) If $\text{w-}\lim_{t\to 0^+} u(t) = u_0 \in X$ and if X is reflexive one has for every $t \in [0, T]$

$$u(t) = S(t)\, u_0 + \text{w-}\lim_{\varepsilon\to 0^+} \int_\varepsilon^t S(t-s)\, f(s)\, ds. \tag{4.6}$$

Proof. Let $x^* \in D_0(A^*)$ and $0 < \varepsilon < t \leqslant T$. The function

$$s \to \langle S(t-s)\, u(s), x^* \rangle, \qquad s \in [\varepsilon, T], \tag{4.7}$$

is a.c. In fact by the change of variables $t - s = \tau$ (4.7) gives rise to the function

$$\tau \to \langle S(\tau)\, v(\tau), x^* \rangle$$

where $v(\tau)$ is an a.c. function in $[0, t-\varepsilon]$. If $0 \leqslant \alpha < \beta \leqslant t - \varepsilon$ one has

$$|\langle S(\beta)\, v(\beta), x^* \rangle - \langle S(\alpha)\, v(\alpha), x^* \rangle| \leqslant c(\|v(\beta) - v(\alpha)\| + \|S^*(\beta)\, x^* - S^*(\alpha)\, x^*\|)$$

and consequently the derived result follows. Thus

$$\langle u(t) - S(t-\varepsilon)\, u(\varepsilon), x^* \rangle = \int_\varepsilon^t \frac{d}{ds} \langle S(t-s)\, u(s), x^* \rangle\, ds, \qquad \forall x^* \in D_0(A^*).$$

On the other hand let $E \subset [0, T]$ be the set of zero measure where the equation (4.3) does not hold. If $s \in]0, t[$, $s \notin E$, one has

$$\begin{aligned}(1/h)[S(t-s-h)\, u(s+h) - S(t-h)\, u(s)] &= (1/h)[S(t-s-h) - S(t-s)]\, u(s) \\ &\quad + S(t-s-h)\, [(u(s+h) - u(s))/h - u'(s) + u'(s)]\end{aligned}$$

and passing to the limit when $h \to 0$ one obtains

$$\frac{d}{ds}(S(t-s)\, u(s)) = S(t-s)\, f(s)$$

a.e. in $]0, t[$. Hence for every $x^* \in D_0(A^*)$ and $\varepsilon \in]0, t[$ one has

$$\langle u(t), x^* \rangle = \langle u(\varepsilon), S^*(t-\varepsilon)\, x^* \rangle + \left\langle \int_\varepsilon^t S(t-s)\, f(s)\, ds, x^* \right\rangle. \tag{4.8}$$

Equation (4.8) holds for every $x^* \in X^*$ since $D_0(A^*)$ is w^*-dense in X^*. Consequently

$$u(t) = S(t-\varepsilon)\, u(\varepsilon) + \int_\varepsilon^t S(t-s)\, f(s)\, ds. \tag{4.9}$$

To prove the part (a) of the Proposition 4.1 we pass to the limit in (4.8) when $\varepsilon \to 0^+$ (with $x^* \in D_0(A^*)$) and we use the density of $D_0(A^*)$ in X^* in the $\mathcal{T}(X^*, X)$ topology. This yields (4.4).

Statement (b) follows from (4.9). Finally if $u(t) \rightharpoonup u_0$ and X is reflexive we obtain (4.6) by passing to the limit in (4.8) when $\varepsilon \to 0^+$ (with $x^* \in X^*$).

REFERENCES

1. HILLE E. & PHILLIPS R. S., *Functional Analysis and Semigroups*, Am. Math. Soc., Providence, Rhode Island (1957).
2. KATO T., *Perturbation Theory for Linear Operators*. Springer Verlag, Berlin (1966).
3. PHILLIPS R. S., Perturbation theory for semi-groups of linear operators, *Trans. Am. math. Soc.* **74**, 199–221 (1954).
4. BREZIS H., Cours Université Paris VI, 1976.
5. PAZY A., Semigroups of linear operators and applications, Lecture Note 10, University of Maryland (1974).

6. BREZIS H., *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, *North-Holland Maths. Stud.* Amsterdam (1973).
7. WEBB G. F., Regularity of solutions to an abstract inhomogeneous linear differential equation, *Proc. Am. math. Soc.* **62**, 271–277 (1977).