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ON THE $W^{2,p}$-REGULARITY FOR SOLUTIONS OF MIXED PROBLEMS (*)

By Hugo BEIRÃO DA VEIGA

SUMMARY. — In this paper we study regularity results for local or global solutions of mixed elliptic problems. In particular it is shown that, if the data are sufficiently smooth, all $W^{1,s}$ solution $(s > 4/3)$ belong to $W^{2,p}$, for all $p < 4/3$.

0. INTRODUCTION. — In this paper we prove some regularity results for solutions of mixed second-order elliptic problems (*see* th. A and B). These results are described in this section.

In the next section notations and useful known results are given. In section 2 the local regularity is proved, [*cf.* (2.16)]; this is the main result of the paper. Finally, for the sake of completeness we prove in section 3 (with the usual method) the global regularity.

Let Ω be an open and bounded set in the n-dimensional Euclidean space $\mathbf{R}^n$ and let Γ be the boundary of Ω. We give two disjoint subsets Γ^+ and Γ^- of Γ with the same boundary on Γ, say, γ. Moreover $\Gamma = \Gamma^+ \cup \Gamma^- \cup \gamma$. We suppose that for any point $x_0 \in \Gamma$ there exists an open neighbourhood U of x_0 and a homeomorphism T of $\overline{U}$ (closure of U) onto $\overline{C}$ such that T and T^{-1} are twice continuously differentiable (we write T, $T^{-1} \in C^2$) and

$$T(\Omega \cap U) = Q, \qquad T(\Gamma \cap U) = \Lambda.$$

If $x_0 \in \gamma$ we assume that $T(\Gamma^- \cap U) = \Lambda_0^-$ and consequently $T(\Gamma^+ \cap U) = \Lambda_0^+$, $T(\gamma \cap U) = S$. If $x_0 \notin \gamma$ we take U such that $U \cap \gamma = \varnothing$. For the definitions of C, Q, Λ, $\Lambda_0^\pm$ and S *see* (1.1).

Let $a_{ij}(x)$, $b_i(x)$, $c_0(x)$ and $\sigma(x)$ $(i, j = 1, \ldots, n)$ be real coefficients and assume that

$$(0.1)\qquad \begin{cases} a_{ij}(x) \in C^1(\overline{\Omega}), \quad b_i,\ c_0 \in C^0(\overline{\Omega}) \quad \sigma \in C^{0,1}(\overline{\Gamma^+}), \\ a_{ij}(x)\xi_i\xi_j \geq \mu |\xi|^2, \quad \forall \xi \in \mathbf{R}^n \quad (\mu > 0)\ (^1); \end{cases}$$

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(1) We use throughout the paper the usual convention about the sum of repeated indices.

moreover let $u(x)$ be a solution of the mixed problem

$$
(0.2)\qquad \begin{cases} u \in W^{1,1}(\Omega), \qquad u = \varphi \quad \text{on } \Gamma^- \quad (^2), \\ \displaystyle\int_\Omega \{ a_{ij} D_i u\, D_j v + b_i D_i u\, v + c_0\, u\, v \}\, dx \\ \displaystyle\qquad = \int_\Omega f v\, dx + \int_{\Gamma^+} (\psi - \sigma u)\, v\, d\Gamma, \qquad \forall v \in C^1(\overline{\Omega}), \qquad v = 0 \quad \text{on } \Gamma^-, \end{cases}
$$

where $f \in L^1(\Omega)$, $\varphi \in L^1(\Gamma^-)$ and $\psi \in L^1(\Gamma^+)$, i. e. u is a solution of the mixed problem

$$
(0.3)\qquad \begin{cases} -D_j(a_{ij} D_i u) + b_i D_i u + c_0\, u = f \quad \text{in } \Omega, \\ u = \varphi \quad \text{on } \Gamma^-, \\ D_\nu u + \sigma u = \psi \quad \text{on } \Gamma^+, \qquad D_\nu u = a_{ij} n_j D_i u, \end{cases}
$$

where n is the unitary exterior normal to Γ.

We have the following theorem :

THEOREM A. — *Let u be a solution of* (0.2) *and assume that*

$$
f \in L^p(\Omega), \qquad \varphi \in W^{2-(1/p),p}(\Gamma^-), \qquad \psi \in W^{1-(1/p),p}(\Gamma^+) \qquad \textit{with} \quad 1 < p < 2.
$$

Put $\lambda = 2(p-1)/p$ *and* $q = 2p/(2-p)$. *If* $u \in C^{0,\lambda}(\overline{\Omega}) \cap W^{1,q}(\Omega)$ *then* $u \in W^{2,p}(\Omega)$ *and*

$$
(0.4)\qquad \| u \|_{2,p,\Omega} \leqq c(\| f \|_{p,\Omega} + \| \varphi \|_{2-(1/p),p,\Gamma^-} + \| \psi \|_{1-(1/p),p,\Gamma^+} + \| u \|_{0,\lambda,\Omega} + \| u \|_{1,q,\Omega}).
$$

If we assume that $\varphi \in L^\infty(\Gamma^-)$ then theorem A is true for the value $p = 2$; obviously we must add the term $\| \varphi \|_{\infty,\Gamma^-}$ to the second member of (0.4). This was proved in [3]. We remark that the condition $u \in C^{0,\lambda} \cap W^{1,q}$ becomes " u is a Lipschitz function in $\overline{\Omega}$ ".

We prove theorem A, in the local version, by approximating the solution u with a sequence of functions u_n each of which is the sum of one solution of a Dirichlet problem and one solution of a Neumann problem. Then we apply to these partial solutions some results of Agmon, Douglis and Nirenberg contained in [1] in order to verify that the L^p-norms of the second derivatives of these partial solutions are uniformly bounded.

We recall that the $C^{0,\lambda}(\overline{\Omega})$ regularity for $W^{1,2}$ solutions of mixed problems, even with discontinuous coefficients, was proved by Stampacchia in [11]; a variant of this method applies also to mixed-problems for a class of second-order non-linear elliptic operators, as proved in [2].

Other important results on the regularity of mixed second-order elliptic problems are given by Shamir in [10] to which we refer. We can combine these results with theorem A to obtain regularity results for second derivatives of solutions of mixed problems. In particular, from lemma 5.1 and corollary 5.4 of [10] and theorem A we get the following

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(2) i. e. φ is the trace of u on Γ^-.

THEOREM B. — *Assume that* Γ *and* γ *are* C^3 *manifolds and* $a_{ij} \in C^3(\overline{\Omega})$, $b_i \in C^2(\overline{\Omega})$, $c_0 \in C^1(\overline{\Omega})$, $\sigma \in C^1(\overline{\Gamma^+})$. *Let* $u \in W^{1,s}(\Omega)$, $s > 4/3$, *be a solution of* (0.2) *with*

$$f \in L^r(\Omega), \qquad \varphi \in W^{1-(1/r),r}(\Gamma^-) \cap W^{2-(1/p),p}(\Gamma^-), \qquad \psi \in W^{-1/r,r}(\Gamma^+) \cap W^{1-(1/p),p}(\Gamma^+)$$

for all $r < +\infty$ *and all* $p < 4/3$. *Then* $u \in W^{2,p}(\Omega)$ *for all* $p < 4/3$.

We don't expect a better result due to a counter example of Shamir in [10]; Shamir constructs an harmonic solution of the mixed Dirichlet-Neumann problem in the half plane, with identically zero data, which doesn't belong (locally) to $W^{2,4/3}$.

Remarks. — Theorem A (and consequently theorem B) is valid for local solutions as proved in the sequel.

— The method used in this paper applies also if we replace the conormal derivative by another directional derivative which covers our elliptic operator on the boundary.

1. SOME DEFINITIONS AND KNOWN RESULTS. — Let $y' = (y_1, \ldots, y_{n-2})$, $\dot{y} = (y', y_{n-1})$, $y = (\dot{y}, y_n)$ and put

$$(1.1) \qquad \begin{cases} C = \{ y \in \mathbf{R}^n : |y_j| < 1, j = 1, \ldots, n \}, \\ Q = \{ y \in C : 0 < y_n \}, \\ \Lambda = \{ y \in C : y_n = 0 \}, \\ \hat{Q} = Q \cup \Lambda, \\ S = \{ y \in \Lambda : y_{n-1} = 0 \}, \\ \Lambda_\delta^- = \{ y \in \Lambda : y_{n-1} < \delta \}, \\ \Lambda_\delta^+ = \{ y \in \Lambda : y_{n-1} > \delta \}, \qquad \delta \in [-1, 1]. \end{cases}$$

If $v(x)$ is a real function (or distribution) defined in Q we denote by $D_i v$ and $D_{ij}^2 v$ the derivatives, in the sense of distributions, $\partial v/\partial x_i$ and $\partial^2 v/\partial x_i \partial x_j$ respectively. If $\|\ \|$ is a norm in some function or distribution space we set

$$\| D v \| = \sum_{i=1}^{n} \| D_i v \| \qquad \text{and} \qquad \| D^2 v \| = \sum_{i,j=1}^{n} \| D_{ij}^2 v \|.$$

If v is defined on $\overline{Q}$ we denote by $v|_\Lambda$ the restriction of v to Λ and so on.

The following definitions will be useful in the sequel :

$C^k(\overline{Q})$ is the space of all real functions k times continuously differentiable in $\overline{Q}$ and

$$C^\infty(\overline{Q}) = \bigcap_{k \geq 1} C^k(\overline{Q}).$$

$C^{k,\lambda}(\overline{Q})$, $(0 < \lambda \leq 1)$, is the space of all $v \in C^k(\overline{Q})$ such that the derivatives of order k satisfy the Hölder condition

$$[f]_{0,\lambda} \equiv \sup_{x, y \in \overline{Q}} \frac{|f(x) - f(y)|}{|x - y|^\lambda} < +\infty.$$

We put

$$\| \quad \|_{0,\lambda} = [\quad]_{0,\lambda} + \| \quad \|_{\infty}, \qquad \text{where} \qquad \| f \|_{\infty} = \sup_{x \in \overline{Q}} | f(x) |.$$

We denote by $L^p(Q)$, $1 \leq p < +\infty$, the space of all real functions (equivalence classes of functions) such that

$$\| v \|_p = \left(\int_Q | v(x) |^p \, dx \right)^{1/p} < +\infty$$

and by $W^{2,p}(Q)$ the space of all $v \in L^p(Q)$ such that $D_i v$, $D_{ij} v \in L^p(Q)$. $W^{2,p}(Q)$ is normalized with the usual norm $\| v \|_{2,p} = \| v \|_p + \| D v \|_p + \| D^2 v \|_p$. Analogously we define $W^{k,p}(Q)$ for any positive integer k; it is known that $C^k(\overline{Q})$ is dense in $W^{k,p}(Q)$.

We give the same definitions if the domain Q is replaced by Ω. If the domain is Λ we give analogous definitions (with the obvious changes).

If the domain of definition is not made explicit it is understood that this domain is Q.

Let us define

$$W_\delta^{1,p} = \{ v \in W^{1,p} : v = 0 \text{ on } \Lambda_\delta^- \},$$
$$\hat{W}_\delta^{1,p} = \{ v \in W_\delta^{1,p} : \operatorname{supp} v \subset \hat{Q} \},$$
$$C_\delta^1 = \{ v \in C^1 : v = 0 \text{ on } \Lambda_\delta^- \},$$
$$\hat{C}^1 = \{ v \in C^1 : \operatorname{supp} v \subset \hat{Q} \}, \qquad \hat{C}_\delta^1 = C_\delta^1 \cap \hat{C}^1,$$

where supp v is the closure of the support of v and $v = 0$ on Λ_δ^- means that the trace of v on Λ_δ^- is zero ($\gamma_0 v = 0$ on Λ_δ^-; *cf.* the sequel).

Finally if $0 < s < 1$ and $1 < p < +\infty$ we denote by $W^{s,p}(\Lambda)$ the space of all $v \in L^p(\Lambda)$ such that (*cf.* [5]) :

$$(1.2) \qquad \| v \|_{p,\Lambda} + \int_\Lambda \int_\Lambda \frac{| v(\dot{y}) - v(\dot{z}) |^p}{| \dot{y} - \dot{z} |^{(n-1)+ps}} \, d\dot{y} \, d\dot{z} < +\infty$$

and we normalize v by the power $1/p$ of the first member of (1.2). If $1 < s < 2$, $W^{s,p}(\Lambda)$ is the space of all $v \in W^{1,p}(\Lambda)$ such that $D_i v \in W^{s-1,p}(\Lambda)$, $1 \leq i \leq n-1$, normalized in the natural way.

With the aid of local charts we can define $W^{s,p}$ spaces on smooth manifolds, in particular on Γ.

We shall need some known results about traces and extensions of functions in the framework of $W^{s,p}$ spaces. The literature on this subject is very extensive; *see* for instance [5], [6], [7]; in the sequel we refer the reader to [9].

We recall the following results :

I. *There exists a bounded linear map* $\gamma_0 : W^{1,p}(Q) \to W^{1-(1/p),p}(\Lambda)$ *such that* $\gamma_0 u = u|_\Lambda$ *if* $u \in C^1(\overline{Q})$. *This map is unique. Cf.* [5] theorem 1.I.

II. *There exists a bounded linear map*

$$(\gamma_0, \gamma^1, \ldots, \gamma^n) \quad : \quad W^{2,p}(Q) \to W^{2-(1/p),p}(\Lambda) \times [W^{1-(1/p),p}(\Lambda)]^n$$

such that

$$(\gamma_0 u, \gamma^1 u, \ldots, \gamma^n u) = (u|_\Lambda, (D_1 u)|_\Lambda, \ldots, (D_n u)|_\Lambda) \qquad \text{if} \quad u \in C^2(\overline{Q}).$$

This map is unique.

These results have some kind of inverse :

III. *There exists a bounded linear map* $R_1 : W^{1-(1/p),p}(\Lambda) \to W^{1,p}(Q)$ *such that* $\gamma_0 R u = u$. *Cf.* [5] theorem 1.I.

IV. *There exists a bounded linear map* $R : W^{2-(1/p),p}(\Lambda) \times W^{1-(1/p),p}(\Lambda) \to W^{2,p}(Q)$ *such that* $\gamma_0 R(u, v) = u$ *and* $\gamma^n R(u, v) = v$.

We shall need also the following extension theorem :

V. *There exists a bounded linear map* $E : W^{1-(1/p),p}(\Lambda_0^+) \to W^{1-(1/p),p}(\Lambda)$ *such that* $(E v)|_{\Lambda_0^+} = v$.

For the proofs *see* also [9] (§ 2, th. 5.4, 5.5, 5.6, 5.8 et consequence 5.3).

Finally we recall that

VI. *There exists a bounded linear map* $\gamma_0 : W^{1,1}(Q) \to L^1(\Lambda)$ *such that* $\gamma_0 u = u|_\Lambda$ *if* $u \in C^1(\overline{Q})$. *This map is unique.*

For the proof *see* [5] theorem 1.II.

For the sake of simplicity we write u and $D_j u$ instead of $\gamma_0 u$ and $\gamma^j u$.

2. LOCAL REGULARITY. — Let $a_{ij}(y)$, $i, j = 1, \ldots, n$, satisfy the following conditions

$$(2.1) \qquad \begin{cases} a_{ij}(y) \in C^1(\overline{Q}), \\ a_{ij}(y)\xi_i\xi_j \geq \nu|\xi|^2, \quad \forall \xi \in \mathbf{R}^n \quad (\nu > 0), \end{cases}$$

let $f(y) \in L^1(Q)$, $\psi(\dot{y}) \in L^1(\Lambda_0^+)$ and let $u(y)$ be a solution of

$$(2.2) \qquad \begin{cases} u \in \hat{W}_0^{1,1}, \\ \displaystyle\int_Q a_{ij} D_i u \, D_j v \, dy = \int_Q f v \, dy + \int_{\Lambda_0^+} \psi v \, d\dot{y}, \quad \forall v \in \hat{C}_0^1. \end{cases}$$

Furthermore let $\Phi : \mathbf{R}^+ \to \mathbf{R}^+$ (resp. $\Theta : \mathbf{R} \to \mathbf{R}^+$) be a nondecreasing C^∞ function such that $\Phi(r) = 0$ on $[0, 1]$, $\Phi(r) = 1$ on $[2, +\infty[$ (resp. $\Theta(r) = 0$ on $]-\infty, -1/2]$, $\Theta(r) = 1$ on $[-1/4, +\infty[$). Put $\rho = (y_{n-1}^2 + y_n^2)^{1/2}$ and define on Q, for any $\varepsilon \in]0, 1[$, the functions $\varphi_\varepsilon(y) = \Phi(\rho/\varepsilon)$ and $\theta_\varepsilon^\pm(y) = \Theta(\pm y_{n-1}/\varepsilon)$. Finally put $\varphi_\varepsilon^\pm(y) = \varphi_\varepsilon(y)\theta_\varepsilon^\pm(y)$.

It is easy to see that for $1 \leqq i, j \leqq n$ we have

$$(2.3)\qquad \begin{cases} |D_i \varphi_\varepsilon^\pm| \leqq c\,\varepsilon^{-1} & \text{if } \varepsilon \leqq \rho \leqq 2\varepsilon, \\ |D_i \varphi_\varepsilon^\pm| = 0 & \text{otherwise;} \\ |D_{ij}^2 \varphi_\varepsilon^\pm| \leqq c\,\varepsilon^{-2} & \text{if } \varepsilon \leqq \rho \leqq 2\varepsilon, \\ |D_{ij}^2 \varphi_\varepsilon^\pm| = 0 & \text{otherwise.} \end{cases}$$

(2.4) LEMMA. — *Let* $u_\varepsilon^\pm = u\,\varphi_\varepsilon^\pm$, *where* u *is a solution of* (2.2). *Then*

$$(2.4')\qquad \int_Q a_{ij} D_i u_\varepsilon^\pm D_j v\,dy = \int_Q f_\varepsilon^\pm v\,dy + \int_\Lambda \psi_\varepsilon^\pm v\,dy, \qquad \forall v \in \hat{C}^1_{-\varepsilon},$$

where

$$(2.4'')\qquad \begin{cases} f_\varepsilon^\pm = f\varphi_\varepsilon^\pm - a_{ij} u\, D_{ij}^2 \varphi_\varepsilon^\pm - a_{ij}(D_j u\, D_i \varphi_\varepsilon^\pm + D_i u\, D_j \varphi_\varepsilon^\pm) - D_j a_{ij} D_i \varphi_\varepsilon^\pm u, \\ \psi_\varepsilon^\pm = \tilde{\psi}\varphi_\varepsilon^\pm - a_{n-1,n} D_{n-1} \varphi_\varepsilon^\pm u \end{cases}$$

and $\tilde{\psi} \in L^1(\Lambda)$ *is any extension of* ψ *to all of* Λ *(we will make a more specific choice later).*

The proof is an easy computation.

(2.5) LEMMA. — *If* $f \in L^p$, $1 < p < 2$, *and* $u \in C^{0,\lambda} \cap W^{1,q}$ *with* $q = 2p/(2-p)$, $\lambda = 2(p-1)/p$ *then* $f_\varepsilon \in L^p$ *and*

$$(2.5')\qquad \| f_\varepsilon \|_p \leqq \| f \|_p + c(\| D u \|_q + [u]_{0,\lambda}),$$

with c *independent of* ε.

Proof. — Since $u(y', 0, 0) = 0$ we have, for any $y \in Q$, $|u(y)| \leqq [u]_{0,\lambda}\,\rho^\lambda$; therefore using (2.3) and $\lambda p + 2 - 2p = 0$ we obtain

$$(2.6)\qquad \int_Q |u|^p |D_{ij}^2 \varphi_\varepsilon^\pm|^p dy \leqq c[u]_{0,\lambda}^p \varepsilon^{-2p} \times \int dy' \iint_{\rho \leqq 2\varepsilon} \rho^{\lambda p} dy_{n-1}\, dy_n \leqq c[u]_{0,\lambda}^p \varepsilon^{\lambda p + 2 - 2p} = c[u]_{0,\lambda}^p.$$

Analogously using Hölder's inequality and $2(1-p/q)-p = 0$ we have

$$(2.7)\qquad \int_Q |D_j u|^p |D_i \varphi_\varepsilon^\pm|^p dy \leqq c\| D u \|_q^p \varepsilon^{-p} \times \left(\int dy' \iint_{\rho \leqq 2\varepsilon} dy_{n-1}\, dy_n \right)^{1-(p/q)} \leqq c \| D u \|_q^p.$$

By using (2.6) and (2.7) we prove (2.5′).

(2.8) LEMMA. — *If* $\psi \in W^{1-(1/p),p}(\Lambda_0^+)$ *and* $u \in C^{0,\lambda} \cap W^{1,q}$ *then* $\psi_\varepsilon^\pm \in W^{1-(1/p),p}(\Lambda)$ *and*

$$(2.8')\qquad \| \psi_\varepsilon^\pm \|_{1-(1/p),p,\Lambda} \leqq c([u]_{0,\lambda} + \| D u \|_q + \| \psi \|_{1-(1/p),p,\Lambda_0^+}),$$

with c *independent of* ε. *The function* $\tilde{\psi}$, *of* (2.4″), *will be chosen in the course of the proof.*

Proof. — Put $3\eta = \operatorname{dist}(\operatorname{supp} u, \partial Q - \Lambda)$ where ∂Q is the boundary of Q. Then (2.2) implies that $\operatorname{dist}(\operatorname{supp} \psi, \partial\Lambda) \geqq 3\eta$ where $\partial\Lambda$ is the boundary of Λ in $\mathbf{R}^{n-1}$.

On the other hand there exists a bounded linear map $\psi \to \tilde{\psi}$ defined on

$$\{ \psi \in W^{1-(1/p),p}(\Lambda_0^+) : \text{dist}(\text{supp}\,\psi, \partial\Lambda) \geqq 3\eta \}$$

with values in $\{ \tilde{\psi} \in W^{1-(1/p),p}(\Lambda) : \text{dist}(\text{supp}\,\tilde{\psi}, \partial\Lambda) \geqq 2\eta \}$ such that $\tilde{\psi}|_{\Lambda_0^+} = \psi$ (use V. § 0 and use multiplication by a suitable function).

Consider now the map $\tilde{\psi} \to \psi^* \equiv R\,\tilde{\psi}$, defined in III, paragraph 0, restricted to $\{ \tilde{\psi} \in W^{1-(1/p),p}(\Lambda) : \text{dist}(\text{supp}\,\tilde{\psi}, \partial\Lambda) \geqq 2\eta \}$. We suppose without loss of generality that the functions ψ^* vanish on a neighbourhood of $\{ y \in Q : y_n = 1 \}$. Now if we prove that

$$\| \psi^* \varphi_\varepsilon^\pm \|_{1,p} \leqq c \| \psi^* \|_{1,p} \tag{2.9}$$

it follows that

$$\| \tilde{\psi} \varphi_\varepsilon^\pm \|_{1-(1/p),p,\Lambda} \leqq c \| \psi \|_{1-(1/p),p,\Lambda_0^+}, \tag{2.10}$$

with c independent of ε. To prove (2.9) we have only to verify that

$$\| \psi^* D_i \varphi_\varepsilon^\pm \|_p \leqq c \| D\psi^* \|_p \tag{2.11}$$

the rest being trivial. Let β be the Sobolev's imbedding exponent relative to $n = 2$ i. e. $\beta = 2p/(2-p)$ and put $\omega_{y'} = \{ y \in Q : y' = \text{constant} \}$.

Then for almost all y' we have (3)

$$\left(\int_{\omega_{y'}} |\psi^*|^\beta \, dy_{n-1} \, dy_n \right)^{1/\beta} \leqq c \left[\int_{\omega_{y'}} (|D_{n-1}\psi^*|^p + |D_n \psi^*|^p) \, dy_{n-1} \, dy_n \right]^{1/p}. \tag{2.12}$$

On the other hand using (2.3), Hölder's inequality, $2(1-p/\beta)-p = 0$ and (2.12) we get (2.11) :

$$\begin{aligned}
&\int_Q |\psi^*|^p |D_i \varphi_\varepsilon^\pm|^p \, dy \\
&\leqq c\varepsilon^{-p} \int \left(\iint_{\rho \leqq 2\varepsilon} |\psi^*|^p \, dy_{n-1} \, dy_n \right) dy' \\
&\leqq c\varepsilon^{-p} \int \left\{ \left(\iint_{\rho \leqq 2\varepsilon} |\psi^*|^\beta \, dy_{n-1} \, dy_n \right)^{p/\beta} \left(\iint_{\rho \leqq 2\varepsilon} dy_{n-1} \, dy_n \right)^{1-(p/\beta)} \right\} dy' \\
&= c \int \left(\int_{\omega_{y'}} |\psi^*|^\beta \, dy_{n-1} \, dy_n \right)^{p/\beta} dy' \\
&\leqq c \int_Q (|D_{n-1}\psi^*|^p + |D_n \psi^*|^p) \, dy.
\end{aligned}$$

Finaly the term $a_{n-1,n} D_{n-1} \varphi_\varepsilon^\pm u$ is treated as in (2.6), (2.7).

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(3) If $v \in W^{1,p}(Q)$ then for almost all y' we have $v|_{\omega_{y'}} \in W^{1,p}(\omega_{y'})$ (take the B. Levi's definition of $W^{1,p}$; *cf.* Deny-Lions [4] or [9] § 2, th. 2.3). We can also prove (2.11) for smooth functions and then use density.

(2.13) THEOREM. — *Let* $u \in C^{0,\lambda} \cap W^{1,q}$ *be a solution of* (2.2) *and suppose that* $f \in L^p$ *and* $\psi \in W^{1-(1/p),p}(\Lambda_0^+)$. *Then* $u_\varepsilon^\pm \in W^{2,p}$ *and*

$$(2.13')\qquad \begin{cases} \| u_\varepsilon^- \|_{2,p} \leqq c(\| f \|_p + \| D u \|_q + [u]_{0,\lambda}), \\ \| u_\varepsilon^+ \|_{2,p} \leqq c(\| f \|_p + \| D u \|_q + [u]_{0,\lambda} + \| \psi \|_{1-(1/p),p,\Lambda_0^+}). \end{cases}$$

Proof. — From (2.4) it follows that $u_\varepsilon^- \in \hat{W}_1^{1,q}$ (4) and

$$(2.14)\qquad \int_Q a_{ij} D_i u_\varepsilon^- D_j v\, dy = \int_Q f_\varepsilon^- v\, dy, \qquad \forall v \in \hat{C}_1^1$$

which implies (*cf.* [1]) that $\| u_\varepsilon^- \|_{2,p} \leqq c \| f_\varepsilon^- \|_p$ and so (2.13') is proved for u_ε^-.

Analogously $u_\varepsilon^+ \in \hat{W}^{1,q}$ and

$$\int_Q a_{ij} D_i u_\varepsilon^+ D_j v\, dy = \int_Q f_\varepsilon^+ v\, dy + \int_\Lambda \psi_\varepsilon^+ v\, d\dot{y}, \qquad \forall v \in \hat{C}^1$$

since u_ε^+, f_ε^+ and ψ_ε^+ vanish if $y_{n-1} < -\varepsilon/2$.

Since $a_{nn}^{-1} \in C^1$ we have

$$\| \psi_\varepsilon^+ a_{nn}^{-1} \|_{1-(1/p),p,\Lambda} \leqq c \| \psi_\varepsilon^+ \|_{1-(1/p),p,\Lambda}.$$

Therefore, by IV § 0, there exists $v_\varepsilon \in W^{2,p}$ such that $v_\varepsilon = 0$ on Λ, $D_n v_\varepsilon = \psi_\varepsilon a_{nn}^{-1}$ on Λ,

$$\| v_\varepsilon \|_{2,p} \leqq c \| \psi_\varepsilon \|_{1-(1/p),p,\Lambda} \qquad \text{and} \qquad \text{dist}(\text{supp } v_\varepsilon, \partial Q - \Lambda) \geqq \eta.$$

Writing $w_\varepsilon = u_\varepsilon^+ + v_\varepsilon$ it follows from (2.15) that

$$\int_Q a_{ij} D_i w_\varepsilon D_j v\, dy = \int_Q [f_\varepsilon^+ - D_j(a_{ij} D_i v_\varepsilon)]\, v\, dy, \qquad \forall v \in \hat{C}^1,$$

where

$$\| f_\varepsilon^+ - D_j(a_{ij} D_i v_\varepsilon) \|_p \leqq c(\| f_\varepsilon^+ \|_p + \| \psi_\varepsilon \|_{1-(1/p),p,\Lambda}).$$

Using known results of [1], (2.5) and (2.8) we prove the second relation (2.13') for w_ε and this finishes the proof.

(2.16) THEOREM. — *If the conditions of theorem* (2.13) *hold then* $u \in W^{2,p}$ *and*

$$\| u \|_{2,p} \leqq c(\| f \|_p + \| D u \|_q + [u]_{0,\lambda} + \| \psi \|_{1-(1/p),p,\Lambda_0^+}).$$

Proof. — The result follows immediately from (2.13), the reflexivity of $W^{2,p}$ and $u_\varepsilon^+ + u_\varepsilon^- \to u$ in the L^p norm. Remark that $0 \leqq \varphi_\varepsilon(\theta_\varepsilon^+ + \theta_\varepsilon^-) \leqq 2$ on Q and $\varphi_\varepsilon(\theta_\varepsilon^+ + \theta_\varepsilon^-) \to 1$ pointwise on $Q - \{ y : y_{n-1} = 0 \}$.

(4) Recall that the lower index δ means that the functions vanish on Λ_δ.

3. GLOBAL REGULARITY. — *Proof of theorem* A *in case* $\varphi \equiv 0$. — Let us assume that the conditions of theorem A hold and also that $\varphi \equiv 0$. We shall prove that

$$\|u\|_{2,p,\Omega} \leqq c(\|f\|_{p,\Omega}+\|\psi\|_{1-(1/p),p,\Gamma^+}+[u]_{0,\lambda}+\|Du\|_{q,\Omega}). \tag{3.1}$$

Let U′ be an open neighbourhood of $x_0 \in \gamma$ with closure contained in U and let $\beta \in C^\infty(\overline{U})$ satisfy supp $\beta \subset U$ and $\beta(x) \equiv 1$ on U′. Puting $w = u\beta$, we see easily that

$$\int_{\Omega\cap U} a_{ij} D_i w D_j v\,dx = \int_{\Omega\cap U} gv\,dx + \int_{\Gamma^+\cap U} \psi' v\,d\Gamma, \qquad \forall v \in C^1(\overline{\Omega}), \qquad v = 0 \quad \text{on } \Gamma^-, \tag{3.2}$$

where

$$\begin{cases} g = f\beta - D_j(a_{ij}D_i\beta)u \\ \qquad -a_{ij}(D_j\beta D_i u + D_i\beta D_j u) - b_i\beta D_i u - c_0\beta u, \\ \psi' = \psi\beta - \sigma' u, \\ \sigma' = \sigma\beta - D_\nu\beta. \end{cases} \tag{3.2'}$$

Put

$$y = Tx, \qquad A = \text{absolute value } \det\left[\frac{\partial(x_1,\ldots,x_n)}{\partial(y_1,\ldots,y_n)}\right],$$

$$B_i = \det\left[\frac{\partial(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)}{\partial(y_1,\ldots,y_{n-1})}\right], \qquad B^2 = \sum_{i=1}^n B_i^2.$$

Moreover if v is a function defined in $\overline{\Omega}\cap U$ we write $\bar{v} = v\circ T^{-1}$.

Let $\bar{v} \in \hat{C}_0^1$ and put $v = \bar{v}\circ T$. With the change of coordinates $x \to y$ we get from (3.2)

$$\int_Q \bar{a}_{ij} D_k \bar{w}\frac{\partial y_k}{\partial x_i} D_l \bar{v}\frac{\partial y_l}{\partial x_j} A\,dy = \int_Q \bar{g}\bar{v}A\,dy + \int_{\Lambda_0^+} \bar{\psi}'\bar{v}B\,d\dot{y}, \qquad \forall \bar{v}\in\hat{C}_0^1. \tag{3.3}$$

Observe that the manifold $\Gamma\cap U$ is defined by the parametric representation $x_i = x_i(\dot{y})$, $\dot{y}\in\Lambda$, and so we have $d\Gamma = B\,d\dot{y}$ on $\Gamma\cap U$.

Putting $\alpha_{kl} = A\,\bar{a}_{ij}(\partial y_k/\partial x_i)(\partial y_l/\partial x_j)$ the relation (3.2) becomes

$$\int_Q \alpha_{kl} D_k\bar{w} D_l\bar{v}\,dy = \int_Q A\bar{g}v\,dy + \int_{\Lambda_0^+} B\bar{\psi}'\bar{v}\,d\dot{y}, \qquad \forall\bar{v}\in\hat{C}_0^1. \tag{3.4}$$

Remark that $A \in C^1(\overline{Q})$, $B\in C^1(\overline{\Lambda})$, A, B $\geqq$ positive constant,

$$\alpha_{kl}\xi_k\xi_l = A\bar{a}_{ij}\eta_i\eta_j \geqq A\mu|\eta|^2 \geqq c|\xi|^2$$

where

$$\eta_i = \frac{\partial y_k}{\partial x_i}\xi_k, \qquad \xi_k = \frac{\partial x_l}{\partial y_k}\eta_l, \qquad c = \text{positive constant.}$$

Using (3.4) we can apply theorem (2.16) to $\bar{w}$ and get

$$(3.5)\qquad \|\bar{w}\|_{2,p} \leqq c(\|f\|_{p,\Omega\cap U}+\|\psi\|_{1-(1/p),p,\Gamma^+\cap U}+[u]_{0,\lambda,\Omega\cap U}+\|Du\|_{q,\Omega\cap U})$$

since

$$\|A\bar{g}\|_p \leqq c(\|f\|_{p,\Omega\cap U}+\|u\|_{p,\Omega\cap U}+\|Du\|_{p,\Omega\cap U})$$
$$\leqq c(\|f\|_{p,\Omega\cap U}+[u]_{0,\lambda,\Omega\cap U}+\|Du\|_{q,\Omega\cap U}),$$

(remember that $u = 0$ on $\Gamma^- \cap U$) and

$$\|B\bar{\psi}'\|_{1-(1/p),p,\Lambda_0^+} \leqq c(\|\psi\|_{1-(1/p),p,\Gamma^+\cap U}+[u]_{0,\lambda,\Omega\cap U}+\|Du\|_{q,\Omega\cap U}).$$

Finally from (3.5) and $\|u\|_{2,p,\Omega\cap U'} \leqq \|w\|_{2,p,\Omega\cap U} \leqq c\|\bar{w}\|_{2,p}$ it follows that $\|u\|_{2,p,\Omega\cap U'}$ does not exceed the second member of (3.5).

If $x_0 \notin \gamma$ we use the results of [1] instead of th. (2.16) to prove that there exist neighbourhoods U and U′ of x_0 such that

$$\|u\|_{2,p,\Omega\cap U'} \leqq c(\|f\|_{p,\Omega\cap U}+\|\psi\|_{1-(1/p),p,\Gamma^+\cap U}+\|u\|_{1,p,\Omega\cap U}),$$

where we don't consider the term $\|\psi\|$ if $x_0 \notin \Gamma^+$. This completes the proof.

Proof of theorem A. — To prove theorem A we use the following know result

(3.6) LEMMA. — *There exists a linear map* L *continuous from* $W^{1-(1/p),p}(\Gamma^-)$ *into* $W^{2,p}(\Omega)$, *continous from* $W^{1-(1/q),q}(\Gamma^-)$ *into* $W^{1,q}(\Omega)$ *and continous from* $C^{0,\lambda}(\Gamma^-)$ *into* $C^{0,\lambda}(\bar{\Omega})$. *Moreover the trace of* $L\varphi$ *on* Γ^- *coincides with* φ.

For the sake of completeness we give the proof of lemma (3.6) in the appendix.

Put $u_0 = u-\varphi^*$ with $\varphi^* = L\varphi$. The function u_0 vanishes on Γ^-, belongs to $W^{1,q}(\Omega) \cap C^{0,\lambda}(\bar{\Omega})$ and solves the integral equation

$$\int_\Omega \{a_{ij}D_i u_0 D_j v + b_i D_i u_0 v + c_0 u_0 v\}\,dx$$
$$= \int_\Omega \{f + D_j(a_{ij}D_i\varphi^*) - b_i D_i\varphi^* - c_0\varphi^*\}\,v\,dx$$
$$+ \int_{\Gamma^+} [(\psi - \sigma\varphi^* - D_\nu\varphi^*) - \sigma u_0]\,v\,d\Gamma, \qquad \forall v \in C^1(\Omega), \qquad v = 0 \text{ on } \Gamma^-;$$

Applying the inequality (3.1) to u_0 we get easily (0.4), as desired.

APPENDIX

Proof of lemma (3.6). — Let $\{U_i\}_{i=1}^m$ be a finite covering of $\Gamma^- \cup \gamma$ where the U_i are open sets satisfying the conditions of section 0. Let $\{\xi_i\}_{i=1}^m$, $\xi_i \in C^\infty(\mathbf{R}^n)$, be a partition of unity subordinate to $\{U_i\}$, i. e. $\sum_{i=1}^m \xi_i = 1$ on $\Gamma^- \cup \gamma$ and $B_i \equiv \operatorname{supp}\xi_i \subset U_i$.

Assume that for every $i \in [1, m]$ there exists a map l_i (defined only on the functions φ such that supp $\varphi \subset B_i$) satisfying the conditions of lemma (3.6). Then the map

$$L\varphi = \sum_{i=1}^{m} l_i(\varphi\xi_i)$$

obviously satisfies the required conditions.

We shall now prove the existence of l_i. We write for simplicity U, ξ, B, l instead of U_i, ξ_i, B_i, l_i and we assume that $U \cap \gamma \neq \varnothing$.

Let $T : \overline{U} \to \overline{C}$ be the map defined in section 0 and assume that there exists a map λ (defined only on functions with support contained in T(B)) satisfying the conditions of lemma (3.6) with Λ_0^- and Q instead of Γ^- ans Ω respectively; choose a function $\zeta \in C^\infty(\overline{\Omega})$ such that supp $\zeta \subset U$ and $\zeta = 1$ on B. Then the map l defined by

$$(l\varphi)(x) = \zeta(x)[\lambda(\varphi \circ T^{-1})](Tx)$$

satisfies the required conditions.

To complete the proof we construct the map λ with standard methods :

If φ is a function defined on Λ_0^- we put

$$\varphi_1(y) = \int_{|\dot{z}|<1} R(\dot{z})\,\varphi(y_n\dot{z}+\dot{y})\,d\dot{z}, \qquad \forall\, y \in P,$$

where $P = \{ y : 0 < y_n < 1/4,\ y_n < -y_{n-1} < 1-y_n,\ |y_i| < 1-2y_n,\ i = 1, \ldots, n-2 \}$ and $R(\dot{z}) \in C^\infty(\mathbf{R}^{n-1})$, supp $R \subset \{ \dot{z} : |\dot{z}| \leq 1 \}$, $\int_{\mathbf{R}^{n-1}} R(\dot{z})\,d\dot{z} = 1$. P is a pyramid with height 1/2 truncated by the hyperplane $y_n = 1/4$. The trace of φ_1 on Λ_0^- is φ and the map $\varphi \to \varphi_1$ is linear and continuous from $W^{2-(1/p),p}(\Lambda_0^-)$ into $W^{2,p}(Q)$, from $W^{1-(1/q),q}(\Lambda_0^-)$ into $W^{1,q}(\Lambda_0^-)$ (*cf.* [5] or [9] lemma 5.6, § 2) and from $C^{0,\lambda}(\Lambda_0^-)$ into $C^{0,\lambda}(P)$.

By using a suitable regular homeomorphism of P onto $P' = Q \cap \{ y : y_{n-1} < 0 \}$ we can suppose without loss of generality that φ_1 is defined on P'. To conclude the proof we remark that the map $\varphi_1 \to \varphi_2$ defined by

$$\varphi_2(y', y_{n-1}, y_n) = \begin{cases} \varphi_1(y) & \text{if } y \in P', \\ 3\varphi_1(y', -y_{n-1}, y_n) - 2\varphi_1(y', -2y_{n-1}, y_n) & \text{if } y \in Q \cap \left\{ y : 0 < y_{n-1} < \dfrac{1}{2} \right\} \end{cases}$$

is a bounded linear map in the norms $W^{2,p}$, $W^{1,q}$ (*cf.* for instance [9] § 2, theorem 3.9) and $C^{0,\lambda}$.

REFERENCES

[1] S. AGMON, A. DOUGLIS and L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I* (*Comm. Pure Appl. Math.*, vol. 12, 1959, p. 623-727).

[2] H. BEIRÃO DA VEIGA, *Sur la régularité des solutions de l'équation* $\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u)$ *avec des conditions aux limites unilatérales et mêlées* (*Ann. Mat. Pura Appl.*, vol. 93, 1972, p. 173-230).

[3] H. BEIRÃO DA VEIGA, *Sul problema misto per le equazioni lineari ellittiche del secondo ordine* (*Am. J. Math.*, to appear).

[4] J. DENY and J.-L. LIONS, *Les espaces du type de Beppo Levi* (*Ann. Inst. Fourier*, t. 5, 1953-54, p. 305-370).

[5] E. GAGLIARDO, *Caratterizzazioni delle trace sulla frontiera relative ad alcune classi di funzioni in n variabili* (*Rend. Sem. Mat. Univ. Padova*, vol. 27, 1957, p. 284-305).

[6] J.-L. LIONS and E. MAGENES, *Problemi ai limiti non omogenei, III* (*Ann. sc. Norm. Sup.*, Pisa, vol. 15, 1961, p. 39-101).

[7] E. MAGENES, *Spazi di interpolazione ed equazioni a derivate parziali* (*Proc. of the VII Congress of the « Unione Matematica Italiana »*, Genova 1963).

[8] C. MIRANDA, *Sul problema misto per le equazioni lineari ellitiche* (*Ann. Mat. Pura Appl.*, vol. 39, 1955, p. 279-303).

[9] J. NECAS, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Paris, 1967.

[10] E. SHAMIR, *Regularization of mixed second-order elliptic problems* (*Israel J. Math.*, vol. 6, 1968, p. 150-168).

[11] G. STAMPACCHIA, *Problemi al contorno ellittici, con dati discontinui, dotati di soluzioni hölderiane* (*Ann. Mat. Pura Appl.*, vol. 51, 1960, p. 1-37).

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Hugo Beirão DA VEIGA,
Instituto de Física e Matemática,
avenida Gama Pinto, 2,
Lisboa, 4 (Portugal).