Research Article

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Moduli of continuity, functional spaces, and elliptic boundary value problems. The full regularity spaces $C^{0,\lambda}_{\alpha}(\overline{\Omega})$

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Abstract: Let $\boldsymbol{L}$ be a second order uniformly elliptic operator, and consider the equation $\boldsymbol{L}u = f$ under the boundary condition $u = 0$. We assume data f in generical subspaces of continuous functions $D_{\overline{\omega}}$ characterized by a given *modulus of continuity* $\overline{\omega}(r)$, and show that the second order derivatives of the solution u belong to functional spaces $D_{\widehat{\omega}}$, characterized by a *modulus of continuity* $\widehat{\omega}(r)$ expressed in terms of $\overline{\omega}(r)$. Results are optimal. In some cases, as for Hölder spaces, $D_{\widehat{\omega}} = D_{\overline{\omega}}$. In this case we say that full regularity occurs. In particular, full regularity occurs for the new class of functional spaces $C^{0,\lambda}_{\alpha}(\overline{\Omega})$ which includes, as a particular case, the classical Hölder spaces $C^{0,\lambda}(\overline{\Omega}) = C^{0,\lambda}_{0}(\overline{\Omega})$. Few words, concerning the possibility of generalizations and applications to non-linear problems, are expended at the end of the introduction and also in the last section.

Keywords: Linear elliptic boundary value problems, data spaces of uniformly continuous functions, uniform continuity properties of higher order derivatives, intermediate and full regularity, new functional spaces

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1 Introduction

The proofs and results shown below are essentially contained in ArXiv reference [7] (see also [6]). We start with some notation. By Ω we denote an open, bounded, connected set in $\mathbb{R}^n$, locally situated on one side of its boundary Γ. To simplify, we assume that the boundary Γ is of class C^3. The notation $\Omega_0 \subset\subset \Omega$ means that the open set Ω_0 satisfies the property $\overline{\Omega}_0 \subset \Omega$.

By $C(\overline{\Omega})$ we denote the Banach space of all real continuous functions f defined in $\overline{\Omega}$. The "sup" norm is denoted by $\|f\|$. We also appeal to the classical spaces $C^k(\overline{\Omega})$ endowed with their usual norms $\|u\|_k$, and to the Hölder spaces $C^{0,\lambda}(\overline{\Omega})$, endowed with the standard semi-norms and norms. The space $C^{0,1}(\overline{\Omega})$, sometimes denoted by $\mathrm{Lip}(\overline{\Omega})$, is the space of Lipschitz continuous functions in $\overline{\Omega}$. We set

$$I(x;r) = \{y : |y-x| \leq r\}, \quad \Omega(x;r) = \Omega \cap I(x;r).$$

Symbols c and C denote generical positive constants. We may use the same symbol to denote different constants.

We start by recalling an old, but related, result. In [3] (dedicated to the two-dimensional Euler equations, see also [8]) we were led to the study of the auxiliary problem

$$\begin{cases} \boldsymbol{L}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

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where

$$\boldsymbol{L} = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j, \tag{1.2}$$

is a second order, uniformly elliptic operator. Without loss of generality, we assume that the matrix of coefficients $a_{ij}(x)$ is symmetric. To avoid conditions depending on the single case, we assume once and for all that the operator's coefficients are Lipschitz continuous in $\overline{\Omega}$. Lower order terms can be considered without difficulty.

In [3] we looked for Banach spaces $C_*(\overline{\Omega}) \subset C(\overline{\Omega})$, *as large as possible*, for which the following result holds ([3, Theorem 4.5]).

Theorem 1.1. *Let $f \in C_*(\overline{\Omega})$ and let u be the solution of problem* (1.1). *Then $u \in C^2(\overline{\Omega})$, moreover,*

$$\|\nabla^2 u\| \le c\|f\|_*.$$

This result was stated for constant coefficients operators, however the proof applies without any modification to variable coefficients case, since it depends only on the behavior of the related Green's function (by following the same ideas we have shown, see [4], that the solution $(\boldsymbol{u}, p)$ to the Stokes system belongs to $\boldsymbol{C}^2(\overline{\Omega}) \times C^1(\overline{\Omega})$ if $\boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega})$).

For convenience we recall the definition and main properties of $C_*(\overline{\Omega})$ (see [3] and, for complete proofs, [4]). Define, for $f \in C(\overline{\Omega})$, and for each $r > 0$,

$$\omega_f(r) \equiv \sup_{x,y\in\Omega,\, 0<|x-y|\le r} |f(x) - f(y)|, \tag{1.3}$$

and consider the semi-norm

$$[f]_* = [f]_{*,R} \equiv \int_0^R \omega_f(r)\,\frac{dr}{r}, \tag{1.4}$$

where $R > 0$ is fixed. The finiteness of the above integral is known as *Dini's continuity condition*. We define the functional space

$$C_*(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_* < \infty\}$$

normalized by $\|f\|_* = [f]_* + \|f\|$. Norms defined for two distinct values of R are equivalent. We have shown that $C_*(\overline{\Omega})$ is a Banach space, that the embedding $C_*(\overline{\Omega}) \subset C(\overline{\Omega})$ is compact, and that the set $C^\infty(\overline{\Omega})$ is dense in $C_*(\overline{\Omega})$.

The regularity Theorem 1.1 for data in $C_*(\overline{\Omega})$ raise a number of new questions. Contrary to the case of Hölder continuity, where full regularity is restored ($\nabla^2 u$ and f has the same regularity), no significant additional regularity is obtained for data in $C_*(\overline{\Omega})$, besides mere continuity of $\nabla^2 u$. So, we are here in the presence of two totally opposite behaviors. This picture leads us to study regularity in the framework of general Banach spaces $D_\omega(\overline{\Omega})$, characterized by a given *modulus of continuity* function $\omega(r)$. For clearness, when the space $D_\omega(\overline{\Omega})$ plays the particular role of f data space, we will use the symbol $D_{\overline{\omega}}(\overline{\Omega})$. In this last case $D_{\widehat{\omega}}(\overline{\Omega})$ denotes the corresponding regularity space (i.e., the space to which the second order derivatives of solutions belong). To each suitable $\overline{\omega}(r)$ there corresponds a $\widehat{\omega}(r)$ such that $\nabla^2 u \in D_{\widehat{\omega}}$ for $f \in D_{\overline{\omega}}$, see Theorem 3.2. This general regularity result is always optimal, in the sharp sense introduced in Definition 3.3. Clearly, $\overline{\omega}(r) \le c\widehat{\omega}(r)$, for some $c > 0$. If a reverse inequality $\widehat{\omega}(r) \le c\overline{\omega}(r)$ holds, then full regularity occurs, see Theorem 3.4. This is the situation for data in Hölder spaces. However *intermediate regularity* (between *mere continuity* and *full regularity*) may also occur. This holds, for instance, for data in Log spaces $D^{0,\alpha}(\overline{\Omega})$, simply defined by replacing in the expression of the classical modulus of continuity of α-Hölder spaces the quantity $1/|x-y|$ by $\log(1/|x-y|)$. Log spaces are significant also for arbitrarily large values of α. The related regularity result is the following. If $f \in D^{0,\alpha}$, for some $\alpha > 1$, then $\nabla^2 u \in D^{0,\alpha-1}$.

In other cases, as for Hölder spaces, full regularity occurs. This is the more interesting situation. A very significant case is that of the new family of functional spaces $C^{0,\lambda}_\alpha(\overline{\Omega})$, $0 \le \lambda < 1$, $\alpha \in \mathbb{R}$, called here Hölog spaces. For $\lambda > 0$ and $\alpha = 0$, $C^{0,\lambda}_0(\overline{\Omega}) = C^{0,\lambda}(\overline{\Omega})$, is a Hölder classical space. For $\lambda = 0$ and $\alpha > 0$, $C^{0,0}_\alpha(\overline{\Omega}) = D^{0,\alpha}(\overline{\Omega})$ is a Log space. Main point is that, for $\lambda > 0$, $\nabla^2 u$ and f enjoy the same $C^{0,\lambda}_\alpha(\overline{\Omega})$ regularity (full regularity). See Theorem 9.2.

The assumptions on the data spaces $D_{\overline{\omega}}(\overline{\Omega})$ required in Theorems 3.2 and 3.4 can be substantially weakened. However, explicit statements in this direction would not add particularly significant features, at the cost of more involved manipulations.

Concerning generalizations, it looks clear that the same type of results can be proved for derivatives of order higher than two, and extended to more general elliptic boundary value problems. Clearly, specific and significant variations are expected, as already happens in the sequel. Such a program should start by imitating the classical main lines followed, for long time, in the framework of Hölder spaces. Concerning applications to non-linear problems, see a couple of remarks in the last section.

Looking for references, we realized that other authors, see [1, 12, 17, 19], have previously stated related results, in general obtained by quite different methods (like, for instance, harmonic analysis). Below we simply appeal to very classical potential theory. We hope that results, particularly complete presentation, and detailed proofs, are of real interest to many readers.

2 The spaces $D_\omega(\overline{\Omega})$. General properties

In this section we define the spaces $D_\omega(\overline{\Omega})$ and state some general properties. We consider real, *continuous*, *non-decreasing* functions $\omega(r)$, defined for $0 \leq r < R$, for some $R > 0$. Furthermore, $\omega(0) = 0$, and $\omega(0) > 0$ for $r > 0$. These three conditions are assumed everywhere in the sequel. The functions $\omega(r)$ will be used here to measure the uniform continuity of functions. To abbreviate, we mostly use the term *oscillations* instead of *modulus of continuity*.

Recalling (1.3), we set

$$[f]_\omega = \sup_{0< r<R} \frac{\omega_f(r)}{\omega(r)}.$$

Hence,

$$\omega_f(r) \leq [f]_\omega \omega(r) \quad \text{for all } r \in (0, R). \tag{2.1}$$

Further, we define the linear space

$$D_\omega(\overline{\Omega}) = \{f \in C(\overline{\Omega}) : [f]_\omega < \infty\}.$$

One easily shows that $[f]_\omega$ is a semi-norm in $D_\omega(\overline{\Omega})$. We introduce a norm by setting

$$\|f\|_\omega = [f]_\omega + \|f\|.$$

Two norms with distinct values of the parameter R are equivalent, due to the addition of $\|f\|$ to the semi-norms.

It is worth noting that, beyond the three conditions on $\omega(r)$ introduced above, any other property assumed in the sequel is merely needed in an arbitrarily small neighborhood of the origin. This fact may be used without a continual reference. In the sequel, to avoid continual specification, we introduce the following definitions.

Definition 2.1. We say that $\omega(r)$ is *concave* if it is concave in a neighborhood of the origin, and say that $\omega(r)$ is *differentiable* if it is point-wisely differentiable (not necessarily continuously differentiable), for each $r > 0$, in a neighborhood of the origin.

Next we establish some useful properties of the above functional spaces.

Proposition 2.2. *If*

$$0 < k_0 \leq \frac{\omega(r)}{\omega_0(r)} \leq k_1 < +\infty, \tag{2.2}$$

for r in some neighborhood of the origin, then $D_\omega(\overline{\Omega}) = D_{\omega_0}(\overline{\Omega})$, *with* equivalent norms.

The proof is immediate.

Lemma 2.3. *If* $\|f_n\|_\omega \leq C_0$, *and* $f_n \to f$ *in* $C(\overline{\Omega})$, *then* $\|f\|_\omega \leq C_0$.

The proof is immediate.

Theorem 2.4. *The space $D_\omega(\overline{\Omega})$ is a Banach space.*

Proof. Let f_n be a Cauchy sequence in $D_\omega(\overline{\Omega})$. It follows, in particular, that $f_n \to f$ in $C(\overline{\Omega})$, where $f \in D_\omega(\overline{\Omega})$. On the other hand, for $|x-y| = r$,

$$\frac{|(f(x)-f_n(x))-(f(y)-f_n(y))|}{\omega(r)} = \lim_{m\to\infty} \frac{|(f_m(x)-f_n(x))-(f_m(y)-f_n(y))|}{\omega(r)} \leq \limsup_{m\to\infty}[f_m - f_n]_\omega .$$

Hence

$$[f-f_n]_\omega \leq \limsup_{m\to\infty}[f_m - f_n]_\omega .$$

From the Cauchy sequence hypothesis it readily follows that

$$\lim_{n\to\infty}[f-f_n]_\omega = 0.$$

□

Next we consider compact embedding properties.

Theorem 2.5. *If*

$$\lim_{r\to 0}\frac{\omega(r)}{\omega_1(r)} = 0 \tag{2.3}$$

holds, then the embedding

$$D_\omega(\overline{\Omega}) \subset D_{\omega_1}(\overline{\Omega})$$

is compact.

Proof. By assumption,

$$\|f_n\|_\omega = [f_n]_\omega + \|f_n\| \leq C_0 \quad \text{for all } n.$$

From (2.3) it follows that $\omega(r) \leq \omega_1(r)$ for $r \in (0, R_0)$, for some $R_0 > 0$. For $r \in (R_0, R)$ one has

$$\omega(r) \leq \frac{\omega(R)}{\omega_1(R_0)}\omega_1(r).$$

So there is a positive constant C such that

$$\omega(r) \leq C\omega_1(r) \quad \text{for all } r \in (0,R).$$

By the Ascoli–Arzela Theorem, the embedding

$$D_\omega(\overline{\Omega}) \subset C(\overline{\Omega})$$

is compact. Hence, by appealing to Lemma 2.3, one shows that there is a subsequence, still denoted f_n, which converges uniformly to some $f \in D_\omega(\overline{\Omega})$. Without loss of generality, we assume that $f = 0$.

Let $|x-y| = r$. One has

$$\frac{|f_n(x)-f_n(y)|}{\omega_1(r)} = \frac{|f_n(x)-f_n(y)|}{\omega(r)}\,\frac{\omega(r)}{\omega_1(r)} \quad \text{for all } n.$$

Given $\epsilon > 0$, it follows from (2.3) that there is $R_0(\epsilon) > 0$ such that

$$0 < r \leq R_0(\epsilon) \implies \frac{\omega(r)}{\omega_1(r)} < \epsilon.$$

Hence, for $0 < |x-y| \leq R_0(\epsilon)$,

$$\frac{|f_n(x)-f_n(y)|}{\omega_1(r)} \leq C_0\epsilon \quad \text{for all } n. \tag{2.4}$$

On the other hand, if $r \in (R_0(\epsilon), R)$, one has

$$\frac{|f_n(x)-f_n(y)|}{\omega_1(r)} \leq \frac{2}{\omega_1(R_0(\epsilon))}\|f_n\|.$$

Since the sequence $\|f_n\|$ converges to zero, there is an index $N(\epsilon)$ such that, for each $n > N(\epsilon)$, the right-hand side of the last inequality is smaller than ϵ. This fact, together with (2.4), shows that (2.4) holds for $0 < |x-y| \leq R$ and $n > N(\epsilon)$ (increase the constant C_0, if necessary). So,

$$\lim_{n\to+\infty}[f_n]_\omega = 0.$$

□

Lemma 2.6. *Assume that ω is concave. Then*

$$\omega(kr) \le k\omega(r) \quad \text{for all } k \ge 1.$$

The proof is immediate. Recall that $\omega(0) = 0$.

Theorem 2.7. *Assume that $\omega(r)$ is concave and that (2.3) holds. Then $D_\omega(\overline{\Omega})$ is not dense in $D_{\omega_1}(\overline{\Omega})$.*

Proof. We assume that the origin belongs to Ω, and argue in a neighborhood $I = I(0, \delta) \subset \Omega$. Define f by setting $f(x) = \omega_1(|x|)$. We show that $[f - g]_{\omega_1} \ge 1$, for each $g \in D_\omega(\overline{\Omega})$. It is sufficient to consider the one-dimensional case. One has

$$\frac{|(f(x) - g(x)) - (f(0) - g(0))|}{\omega_1(|x|)} = \left|1 - \frac{g(x) - g(0)}{\omega_1(|x|)}\right|.$$

Hence $[f - g]_{\omega_1} \ge 1$ follows if we show that

$$\lim_{x \to 0} \frac{g(x) - g(0)}{\omega_1(|x|)} = 0.$$

Let us prove this last inequality. One has, as $x \to 0$,

$$\lim \frac{g(x) - g(0)}{\omega_1(|x|)} = \lim \frac{g(x) - g(0)}{\omega(|x|)} \cdot \lim \frac{\omega(|x|)}{\omega_1(|x|)} = 0.$$

Note that in the above proof we did not explicitly appeal to the concavity assumption. This assumption is introduced here merely to guarantee that $f(x) = \omega_1(|x|)$ belongs to D_{ω_1} in a neighborhood of the origin. This holds if

$$\omega_1(s) \le \omega_1(r) + c\omega_1(s - r) \quad \text{for } 0 < r < s < \rho, \tag{2.5}$$

for some constant $c \ge 1$, and some $\rho > 0$. Concave oscillations satisfy (2.5) with $c = 1$. □

The above result shows, in particular, that $C^{0,\mu}(\overline{\Omega})$ is not dense in $C^{0,\lambda}(\overline{\Omega})$ for $1 \ge \mu > \lambda > 0$. In particular, $\mathrm{Lip}(\overline{\Omega})$, hence $C^1(\overline{\Omega})$, is not dense in $C^{0,\lambda}(\overline{\Omega})$ (a result sometimes appealed in the literature).

We end this section by stating an extension theorem, where $\Omega_\delta \equiv \{x : \mathrm{dist}(x, \Omega) < \delta\}$.

Theorem 2.8. *Assume that Ω is convex or, alternatively, that $\omega(r)$ is concave (concavity may be replaced by condition (3.9)). Then there is a $\delta > 0$ such that the following holds. There is a linear continuous map T from $C(\overline{\Omega})$ to $C(\overline{\Omega}_\delta)$, and from $D_\omega(\overline{\Omega})$ to $D_\omega(\overline{\Omega}_\delta)$, such that $Tf(x) = f(x)$, for each $x \in \overline{\Omega}$.*

The proof follows by appealing to the argument used to prove [4, Theorem 2.3]. Note that the classical proof of approximation of functions on compact subsets of Ω by appealing to mollification does not work here. Otherwise, the density property refused by Theorem 2.7 would hold.

3 Spaces $D_{\overline{\omega}}(\overline{\Omega})$ and $D_{\widehat{\omega}}(\overline{\Omega})$, and regularity. The main theorems

In this section we state Theorems 3.2 and 3.4. Recall that we use the symbol $D_{\overline{\omega}}(\overline{\Omega})$ when the space $D_\omega(\overline{\Omega})$ plays the role of f data space. In this case, we use the symbol $D_{\widehat{\omega}}(\overline{\Omega})$ to denote the corresponding regularity space, to which belong the second order derivatives of solutions.

From now on we assume that the modulus of continuity $\overline{\omega}(r)$ satisfy the condition

$$\int_0^R \overline{\omega}(r) \frac{dr}{r} \le C_R, \tag{3.1}$$

for some constant C_R. Assumption (3.1) is equivalent to the inclusion $D_{\overline{\omega}}(\overline{\Omega}) \subset C_*(\overline{\Omega})$. This assumption is almost necessary to obtain $\nabla^2 u \in C(\overline{\Omega})$.

We put each suitable oscillation $\overline{\omega}(r)$ in correspondence with a unique, related oscillation $\widehat{\omega}(r)$ defined by setting $\widehat{\omega}(0) = 0$, and

$$\widehat{\omega}(r) = \int_0^r \overline{\omega}(s)\frac{ds}{s}$$

for $0 < r \leq R$. Hence, to a functional space $D_{\overline{\omega}}(\overline{\Omega})$ there corresponds a well-defined functional space $D_{\widehat{\omega}}(\overline{\Omega})$. Obviously, $\widehat{\omega}$ satisfies all the properties described in section 2 for generical oscillations. In particular, Banach spaces

$$D_{\widehat{\omega}}(\overline{\Omega}) = \{f \in C(\overline{\Omega}) : [f]_{\widehat{\omega}} < \infty\}$$

turn out to be well defined.

Next we discuss some additional restrictions on the data spaces $D_{\overline{\omega}}(\overline{\Omega})$. We start by excluding $\mathrm{Lip}(\overline{\Omega})$ as data space since this *singular* case, largely considered in literature, is borderline. So, we impose the *strict* limitation

$$\mathrm{Lip}(\overline{\Omega}) \subset D_{\overline{\omega}}(\overline{\Omega}) \subset C_*(\overline{\Omega}).$$

Exclusion of $\mathrm{Lip}(\overline{\Omega})$ means that $\overline{\omega}(r)$ does not verify $\overline{\omega}(r) \leq cr$, for any positive constant c. Hence we obtain $\limsup(\overline{\omega}(r)/r) = +\infty$, as $r \to 0$. We simplify, by assuming that

$$\lim_{r\to 0} \frac{\overline{\omega}(r)}{r} = +\infty. \tag{3.2}$$

In particular, the graph of $\overline{\omega}(r)$ is tangent to the vertical axis at the origin (as for Hölder and Log spaces). It follows that *concavity* of the graph is here a quite natural assumption. Concavity implies that left and right derivatives are well defined, for $r > 0$. By also taking into account that $\overline{\omega}(r)$ is non-decreasing, we realize that pointwise differentiability of $\overline{\omega}(r)$, for $r > 0$, is not a particularly restrictive assumption. This last claim is reenforced by the equivalence result for norms, under condition (2.2). This equivalence allows regularization of oscillations $\overline{\omega}(r)$, staying inside the same original functional space $D_{\overline{\omega}}(\overline{\Omega})$. Summarizing, *differentiability* and *concavity* (recall Definition 2.1) are natural assumptions here.

If $\overline{\omega}(r)$ is concave, not flat, and differentiable, it follows that

$$\frac{\overline{\omega}(r)}{r\overline{\omega}'(r)} > 1,$$

for $r > 0$. This justifies the assumption

$$\lim_{r\to 0} \frac{\overline{\omega}(r)}{r\overline{\omega}'(r)} = C_1 > 1, \tag{3.3}$$

where $C_1 = +\infty$ is admissible. Assumption (3.3) is reenforced by the particular situation in Lipschitz, Hölder, and Log cases. The limit exists and is given by, respectively, 1, $\frac{1}{\lambda}$, and $+\infty$. As expected, the Lipschitz case stays outside the admissible range. Note that, basically, the larger is the space, the larger is the limit.

The above consideration allow us to assume in Theorems 3.2 and 3.4 that oscillations $\overline{\omega}(r)$, are concave, differentiable, and satisfy conditions (3.1), (3.2), and (3.3).

Note that, due to a possible loss of regularity, it could happen that a $D_{\widehat{\omega}}(\overline{\Omega})$-space is not contained in $C_*(\overline{\Omega})$, as happens in Theorem 8.2 if $1 < \alpha < 2$. In other words, $\widehat{\omega}(r)$ does not necessarily satisfy (3.1).

Next, we define the quantity

$$B(r) =: \frac{r\int_r^R \frac{\overline{\omega}(s)}{s^2}\,ds}{\int_0^r \frac{\overline{\omega}(s)}{s}\,ds}.$$

The following result holds.

Lemma 3.1. *Assume that $\overline{\omega}(r)$ is concave and satisfies assumptions* (3.1), (3.2) *and* (3.3). *Then*

$$\lim_{r\to 0} B(r) = \frac{1}{C_1 - 1}. \tag{3.4}$$

In particular, there is a positive constant C_2 such that

$$B(r) \leq C_2 \tag{3.5}$$

in some neighborhood of the origin.

Proof. By appealing to (3.1), (3.2) and to a de L'Hôpital's rule one shows that

$$\lim_{r\to 0} \frac{1}{r} \int_0^r \frac{\overline{\omega}(s)}{s}\, ds = +\infty. \tag{3.6}$$

On the other hand

$$\lim_{r\to 0} B(r) = \lim_{r\to 0} \frac{\int_r^R \frac{\overline{\omega}(s)}{s^2}\, ds}{\frac{1}{r}\int_0^r \frac{\overline{\omega}(s)}{s}\, ds}. \tag{3.7}$$

Equation (3.6) shows that the denominator $g(r)$ of the fraction on the right-hand side of (3.7) goes to $+\infty$ as r goes to zero. Furthermore, its derivative

$$g'(r) = \frac{1}{r^2}\Big(\overline{\omega}(r) - \int_0^r \frac{\overline{\omega}(s)}{s} ds\Big)$$

is strictly negative for positive r in a neighborhood of the origin, as follows from the inequality

$$\overline{\omega}(r) - \int_0^r \frac{\overline{\omega}(s)}{s}\, ds < 0,$$

for $r > 0$, which we are going to show. Since the left-hand side of the inequality goes to zero with r, it is sufficient to show that its derivative is strictly negative for $r > 0$. This follows easily by appealing to (3.3). The above results allow us to apply to the limit (3.7) one of the well-known forms of de L'Hôpital's rule. Straightforward calculations, together with (3.3), show (3.4). □

Next we state our main results, Theorems 3.2 and 3.4. In the first theorem constant coefficients are assumed.

Theorem 3.2. *Assume that the coefficients of the operator* $\boldsymbol{L}$ *are constant. Further, let the concave and differentiable oscillation* $\overline{\omega}(r)$ *satisfy conditions* (3.1), (3.2), *and* (3.3). *Assume that* $\Omega_0 \subset\subset \Omega$, $f \in D_{\overline{\omega}}(\overline{\Omega})$, *and let* u *be the solution of problem* (1.1). *Then* $\nabla^2 u \in D_{\widehat{\omega}}(\Omega_0)$ *and*

$$\|\nabla^2 u\|_{\widehat{\omega},\Omega_0} \le C\|f\|_{\overline{\omega}},$$

for some positive constant $C = C(\Omega_0, \Omega)$. *The result is optimal in the sharp sense, see Definition 3.3 below. Regularity holds up to flat boundary points.*

A point $x \in \partial\Omega$ is said to be a *flat boundary point* if the boundary is flat in a neighborhood of the point. The meaning of *sharp optimality* is the following (our abbreviate notation seems clear).

Definition 3.3. We say that a given regularity statement of type $\overline{\omega} \to \widehat{\omega}$ is sharp if any regularity statement $\overline{\omega} \to \widehat{\omega}_0$, obtained by replacing $\widehat{\omega}$ by any other $\widehat{\omega}_0$, implies the existence of a constant c for which $\widehat{\omega}(r) \le c\widehat{\omega}_0(r)$.

The sharp regularity claimed in Theorem 3.2 will be proved in Section 10.

Much stronger results hold if the constant C_1 in equation (3.3) is positive and finite. In this case one has

$$D_{\widehat{\omega}}(\overline{\Omega}) = D_{\overline{\omega}}(\overline{\Omega}). \tag{3.8}$$

In fact, by the de l'Hôpital rule, one shows that

$$\lim_{r\to 0} \frac{\widehat{\omega}(r)}{\overline{\omega}(r)} = \lim_{r\to 0} \frac{\overline{\omega}(r)}{r\overline{\omega}'(r)}$$

if the second limit exists. Hence, under this last hypothesis, identity (3.8) holds if (actually, and only if) the limit is positive and finite. Clearly, (3.8) holds by merely assuming the inequality required in Proposition 2.2. In this case the operator $\boldsymbol{L}$ can have variable coefficients, and full regularity occurs up to the whole (regular) boundary. More precisely, one has the following result.

Theorem 3.4. *Assume that the oscillation $\overline{\omega}(r)$, concave and differentiable, satisfies conditions* (3.1), (3.2), *and* (3.3) *for some $C_1 < +\infty$. Let $f \in D_{\overline{\omega}}(\overline{\Omega})$, and let u be the solution of problem* (1.1). *Then $\nabla^2 u \in D_{\overline{\omega}}(\overline{\Omega})$ and*

$$\|\nabla^2 u\|_{\overline{\omega}} \leq C\|f\|_{\overline{\omega}},$$

for some positive constant C.

Regularity in the sharp sense follows trivially from full regularity. But it is quite significant in dealing with intermediate regularity results, like in Theorem 3.2. See the example shown in Section 8, in the framework of Log spaces $D^{0,\alpha}(\overline{\Omega})$.

The set of conditions imposed in the above statements can be weakened as follows. We start by replacing the concavity assumption by the existence of a constant $k_1 > 1$ such that

$$\overline{\omega}(k_1 r) \leq c_1 \overline{\omega}(r) \tag{3.9}$$

for some positive constant c_1, and for r in a neighborhood of the origin. We take into account that, if (3.9) holds, then given $k_2 > 1$, there is a positive constant c_2 such that

$$\overline{\omega}(k_2 r) \leq c_2 \overline{\omega}(r), \tag{3.10}$$

for r in some δ_0-neighborhood of the origin. The proof is obvious, by a bootstrap argument. Take into account that, if $k_2 > k_1$, there is an integer m such that $k_2 \leq k_1^m$. If $\overline{\omega}(r)$ is concave Lemma 2.6 shows (3.9) for $k_2 = c_2 = 1$. It would be interesting to show that assumption (3.9) does not necessarily imply the existence of some convex oscillation $\overline{\omega}_0(r)$ equivalent, in the (2.2) sense, to the given, non-convex, $\overline{\omega}(r)$.

Actually, in the proof of Theorem 3.2 shown bellow, concavity, differentiability, and assumptions (3.1), (3.2), and (3.3), are replaced by the more general set of assumptions (3.1), (3.2), (3.9), and (3.5). The same holds for Theorem 3.4, by adding assumption (6.1).

4 A Hölder–Korn–Lichtenstein–Giraud inequality

In this section we prove Theorem 4.1 below. The proof is an adaptation of that developed in [11] to prove the so-called Hölder–Korn–Lichtenstein–Giraud inequality (see [11, Part II, Section 5, Appendix 1]) in the framework of Hölder spaces. Following [11], we considered *singular kernels* $\mathcal{K}(x)$ of the form

$$\mathcal{K}(x) = \frac{\sigma(x)}{|x|^n}, \tag{4.1}$$

where $\sigma(x)$ is infinitely differentiable for $x \neq 0$, and satisfies the properties $\sigma(tx) = \sigma(x)$, for $t > 0$, and

$$\int_S \sigma(x)\, dS = 0,$$

where $S = \{x : |x| = 1\}$. We denote by $|||\sigma|||$ the sum of the L^∞-norms of σ and of its first order derivatives on S.

It follows easily that, for $0 < \rho_1 < \rho_2$,

$$\int_{\rho_1<|x|<\rho_2} \mathcal{K}(x)\, dx = \int_{\rho_1<|x|} \mathcal{K}(x)\, dx = \int \mathcal{K}(x)\, dx = 0, \tag{4.2}$$

where the last integral is in the Cauchy principal value sense.

For continuous functions ϕ with compact support, the convolution integral

$$(\mathcal{K} * \phi)(x) = \int \mathcal{K}(x-y)\phi(y)\, dy, 5$$

extended to the whole space $\mathbb{R}^n$, exists as a Cauchy principal value and is finite.

We set $I(\rho) = \{x : |x| \leq \rho\}$, $D_{\overline{\omega}}(\rho) = D_{\overline{\omega}}(I(\rho))$, and do the same for other functional spaces, norms, and semi-norms labeled by ρ.

Theorem 4.1. *Let $\mathcal{K}(x)$ be a singular kernel enjoying the properties described above. Further, assume that the oscillation $\overline{\omega}$ satisfies* (3.1), (3.2), (3.9), *and* (3.5). *Let $\phi \in D_{\overline{\omega}}(\rho)$, vanish for $|x| \geq \rho$. Then $\mathcal{K} * \phi \in D_{\widehat{\omega}}(\rho)$. Furthermore, in the sphere $I(\rho)$, one has*

$$[(\mathcal{K} * \phi)]_{\widehat{\omega}} \leq C\|\phi\|_{\overline{\omega}}, \tag{4.3}$$

where $C = C(n, \overline{\omega}, |||\sigma|||)$.

Proof. Below we use the simplified notation $\overline{\omega}(r) = \overline{\omega}_\phi(r)$, the modulus of continuity of ϕ in $I(\rho)$, recall (1.3).

Let $x_0, x_1 \in I(\rho)$, $0 < |x_0 - x_1| = \delta < \delta_0 \leq \rho$. The positive constant δ_0 is fixed here in correspondence to the choice $k_2 = 3$ in (3.10). In the concave case (assumed, for clearness, in the statements of Theorems 3.2 and 3.4), we may set $k_2 = 1$.

From (4.2) it follows that

$$(\mathcal{K} * \phi)(x) = \int (\phi(y) - \phi(x))\mathcal{K}(x - y)\, dy.$$

Hence, with abbreviated notation,

$$\begin{aligned}(\mathcal{K} * \phi)(x_0) - (\mathcal{K} * \phi)(x_1) &= \int \{(\phi(y) - \phi(x_0))\mathcal{K}(x_0 - y) - (\phi(y) - \phi(x_1))\mathcal{K}(x_1 - y)\}\, dy \\ &= \int\limits_{|y-x_0|<2\delta} \{\dots\}\, dy + \int\limits_{2\delta<|y-x_0|<\delta_0} \{\dots\}\, dy + \int\limits_{\delta_0<|y-x_0|} \{\dots\}\, dy \equiv I_1 + I_2 + I_3. \end{aligned} \tag{4.4}$$

Since

$$\{y : |y - x_0| < 2\delta\} \subset \{y : |y - x_1| < 3\delta\},$$

it follows that

$$\begin{aligned}\int\limits_{|y-x_0|<2\delta} |\phi(y) - \phi(x_1)||\mathcal{K}(x_1 - y)|\, dy &\leq \int\limits_{|y-x_1|<3\delta} |\phi(y) - \phi(x_1)||\mathcal{K}(x_1 - y)|\, dy \\ &\leq \|\sigma\| \int_0^{3\delta} \frac{\overline{\omega}(r)}{r}\, dr \leq \|\sigma\|[\phi]_{\overline{\omega}} \int_0^{3\delta} \frac{\overline{\omega}(r)}{r}\, dr, \end{aligned} \tag{4.5}$$

where we appealed to polar-spherical coordinates with $r = |x_1 - y|$, to the fact that σ is positive homogeneous of order zero, to (4.1), and to definition (2.1).

A similar, simplified, argument shows that equation (4.5) holds by replacing x_1 by x_0 and 3δ by 2δ. So,

$$|I_1| \leq 2\|\sigma\|[\phi]_{\overline{\omega}} \int_0^{3\delta} \frac{\overline{\omega}(r)}{r}\, dr \leq c\|\sigma\|[\phi]_{\overline{\omega}} \int_0^{\delta} \frac{\overline{\omega}(r)}{r}\, dr,$$

where we have appealed to (3.10) for $k_2 = 3$. Hence,

$$|I_1| \leq c\|\sigma\|[\phi]_{\overline{\omega}}\widehat{\omega}(\delta). \tag{4.6}$$

On the other hand

$$I_2 = \int\limits_{2\delta<|y-x_0|<\delta_0} (\phi(x_1) - \phi(x_0))\mathcal{K}(x_0 - y)\, dy + \int\limits_{2\delta<|y-x_0|<\delta_0} (\phi(y) - \phi(x_1))(\mathcal{K}(x_0 - y) - \mathcal{K}(x_1 - y))\, dy.$$

The first integral vanishes, due to (4.2). Hence,

$$|I_2| \leq \int\limits_{2\delta<|y-x_0|<\delta_0} |\phi(y) - \phi(x_1)||\mathcal{K}(x_0 - y) - \mathcal{K}(x_1 - y)|\, dy.$$

Further, by the Mean-Value Theorem, there is a point x_2, between x_0 and x_1, such that

$$|\mathcal{K}(x_0 - y) - \mathcal{K}(x_1 - y)| \leq |\nabla\mathcal{K}(x_2 - y)|\delta.$$

Since

$$\partial_i \mathcal{K}(x) = \frac{1}{|x|^{n+1}}\Big[(\partial_i \sigma)\Big(\frac{x}{|x|}\Big) - n\frac{x_i}{|x|}\sigma(x)\Big],$$

it readily follows that

$$\big|\mathcal{K}(x_0 - y) - \mathcal{K}(x_1 - y)\big| \le c|||\sigma|||\frac{\delta}{|y - x_2|^{n+1}} \le c|||\sigma|||\frac{\delta}{|y - x_0|^{n+1}}. \tag{4.7}$$

Note that, for $|x_0 - y| > 2\delta$, one has

$$|x_0 - y| \le 2|x_2 - y| \le 4|x_0 - y|.$$

On the other hand, for $2\delta < |x_0 - y|$,

$$|x_1 - y| \le 3|x_0 - y|.$$

So,

$$|\phi(y) - \phi(x_1)| \le [\phi]_{\overline{\omega}}\overline{\omega}(3|x_0 - y|).$$

The above estimates show that

$$|I_2| \le c|||\sigma|||[\phi]_{\overline{\omega}}\delta\int_{2\delta}^{\delta_0}\overline{\omega}(3r)r^{-2}\,dr \le c|||\sigma|||[\phi]_{\overline{\omega}}\delta\int_{2\delta}^{\delta_0}\overline{\omega}(r)r^{-2}\,dr,$$

where we appealed to (3.10) for $k_2 = 3$. Finally, by (3.5), it readily follows that

$$|I_2| \le c|||\sigma|||[\phi]_{\overline{\omega}}\widehat{\omega}(\delta) \tag{4.8}$$

for $\delta \in (0, \delta_0)$.

Finally we consider I_3. By arguing as for I_2, in particular by appealing to (4.2) and (4.7), one shows that

$$|I_3| \le C\delta|||\sigma|||\int_{|y-x_0|>\delta_0}\frac{|\phi(y) - \phi(x_1)|}{|y - x_0|^{n+1}}\,dy \le C\delta|||\sigma|||\|\phi\| \le C|||\sigma|||\|\phi\|\widehat{\omega}(\delta). \tag{4.9}$$

Note that, by a de l'Hôpital rule, one shows that (3.2) holds with $\overline{\omega}(r)$ replaced by $\widehat{\omega}(r)$. From equation (4.4), by appealing to (4.6), (4.8), and (4.9), one shows that

$$|(\mathcal{K} * \phi)(x_0) - (\mathcal{K} * \phi)(x_1)| \le C|||\sigma|||\|\phi\|_{\overline{\omega}}\widehat{\omega}(\delta),$$

for each couple of points $x_0, x_1 \in I(\rho)$ such that $0 < |x_0 - x_1| \le \delta_0$. Hence (4.3) holds.

We may easily estimate $|(\mathcal{K} * \phi)(x_0) - (\mathcal{K} * \phi)(x_1)|$ for pairs of points x_0, x_1 for which $\delta_0 < |x_0 - x_1| < \rho$. However, this is superfluous, since δ_0 is fixed "once and for all". □

5 The interior regularity estimate in the constant coefficients case

In this section we apply Theorem 4.1 to prove the basic interior regularity result for solutions of the elliptic equation (1.1) in the framework of $D_{\overline{\omega}}$ data spaces. In this section $\boldsymbol{L}$ ia a constant coefficients operator. The proof is inspired by that developed in Hölder spaces in [11, Part II, Section 5]. For convenience, assume that $n \ge 3$.

By a fundamental solution of the differential operator $\boldsymbol{L}$ one means a distribution $J(x)$ in $\mathbb{R}^n$ such that

$$\boldsymbol{L}J(x) = \delta(x). \tag{5.1}$$

The celebrated Malgrange–Ehrenpreis Theorem states that every non-zero linear differential operator with constant coefficients has a fundamental solution (see, for instance, [20, Chapter VI, Section 10]). We recall that the analogue for differential operators whose coefficients are polynomials (rather than constants) is false, as shown by a famous Hans Lewy's counterexample.

In particular, for a second order elliptic operator with constant coefficients and only higher order terms, one can construct explicitly a fundamental solution $J(x)$ which satisfies properties (i), (ii), and (iii), claimed in [11], namely,

(i) $J(x)$ is a real analytic function for $|x| \neq 0$.

(ii) For $n \geq 3$

$$J(x) = \frac{\sigma(x)}{|x|^{n-2}}, \tag{5.2}$$

where $\sigma(x)$ is positive homogeneous of degree 0.

(iii) Equation (5.1) holds. In particular, for every sufficiently regular, compact supported, function v, one has

$$v(x) = \int J(x-y)(\boldsymbol{L}v)(y)dy.$$

For a second order elliptic operator as above, one has

$$J(x) = c\Big(\sum A_{ij}x_i x_j\Big)^{\frac{2-n}{2}},$$

where A_{ij} denotes the cofactor of a_{ij} in the determinant $|a_{ij}|$.

Following [11], we denote by $\boldsymbol{S}$ the operator

$$(\boldsymbol{S}\phi)(x) = \int J(x-y)\phi(y)\,dy = (J * \phi)(x).$$

Note that, in the constant coefficients case, the operator $\boldsymbol{T}$ introduced in reference [11] vanishes.

Point (iii) above (see also [11, "Lemma" A]) shows that if v is compact supported and sufficiently regular (for instance of class C^2), then

$$v = \boldsymbol{SL}v. \tag{5.3}$$

Due to the structure of the function $\sigma(x)$ appearing in equation (5.2), it readily follows that second order derivatives of $(\boldsymbol{S}\phi)(x)$ have the form $\partial_i\partial_j\boldsymbol{S}\phi = \mathcal{K}_{ij} * \phi$, where the $\mathcal{K}_{ij}$ enjoy the properties described for singular kernels $\mathcal{K}$ in Section 4.

We write, in abbreviated form,

$$\nabla^2\boldsymbol{S}\phi(x) = \int \mathcal{K}(x-y)\phi(y)\,dy, \tag{5.4}$$

where $\mathcal{K}(x)$ enjoys the properties described at the beginning of section 4. From (5.4) it follows that

$$\nabla^2\boldsymbol{SL}v = \int \mathcal{K}(x-y)\boldsymbol{L}v(y)\,dy.$$

Hence, by Theorem 4.1, one gets

$$[\nabla^2\boldsymbol{SL}v]_{\widehat{\omega};2\rho} \leq C[\boldsymbol{L}v]_{\overline{\omega};2\rho}. \tag{5.5}$$

By appealing to (5.3) we get the following result.

Proposition 5.1. *Assume that the differential operator $\boldsymbol{L}$ has constant coefficients and that the oscillation $\overline{\omega}$ satisfies assumptions* (3.1), (3.2), (3.9), *and* (3.5). *Let $v \in C^2(2\rho)$ be a support compact function such that $\boldsymbol{L}v \in D_{\overline{\omega}}(2\rho)$. Then*

$$[\nabla^2 v]_{\widehat{\omega};2\rho} \leq C[\boldsymbol{L}v]_{\overline{\omega};2\rho}. \tag{5.6}$$

One has the following interior regularity result. For brevity we have consider two spheres of radius ρ and R, $R > \rho$, in the particular case $R = 2\rho$.

Theorem 5.2. *Assume that the hypothesis of Proposition 5.1 hold. Let $u \in C^2(2\rho)$ be such that $\boldsymbol{L}u \in D_{\overline{\omega}}(2\rho)$. Then $\nabla^2 u \in D_{\widehat{\omega}}(\rho)$, moreover*

$$[\nabla^2 u]_{\widehat{\omega};\rho} \leq C[\boldsymbol{L}u]_{\overline{\omega},2\rho} + c(\theta)\Big(\frac{\|u\|}{\rho^3} + \frac{\|\nabla u\|}{\rho^2} + \frac{\|\nabla^2 u\|}{\rho}\Big)\frac{|x-y|}{\overline{\omega}(|x-y|)}, \tag{5.7}$$

for some positive constant C, independent of ρ. In particular,

$$[\nabla^2 u]_{\widehat{\omega};\rho} \leq C[\boldsymbol{L}u]_{\overline{\omega},2\rho} + \frac{c(\theta)}{\rho^3}\|u\|_{C^2(2\rho)}. \tag{5.8}$$

Proof. Fix a non-negative C^∞ function θ, defined for $0 \le t \le 1$ such that $\theta(t) = 1$ for $0 \le t \le \frac{1}{3}$, and $\theta(t) = 0$ for $\frac{2}{3} \le t \le 1$. Further fix a positive real ρ, for convenience $0 < \rho < \frac{1}{2}$, and define

$$\zeta(x) = \begin{cases} 1 & \text{for } |x| \le \rho, \\ \theta\Big(\frac{|x|-\rho}{\rho}\Big) & \text{for } \rho \le |x| \le 2\rho. \end{cases}$$

Next we consider $\zeta(x)$ for points x such that $\rho \le |x| \le 2\rho$, and leave to the reader different situations. Due to symmetry, it is sufficient to consider the one-dimensional case

$$\zeta(t) = \theta\Big(\frac{t-\rho}{\rho}\Big) \quad \text{for } \rho \le t \le 2\rho.$$

Hence

$$\zeta'(t) = \theta'\Big(\frac{t-\rho}{\rho}\Big)\frac{1}{\rho},$$

and

$$\zeta''(t) = \theta''\Big(\frac{t-\rho}{\rho}\Big)\frac{1}{\rho^2}.$$

Further,

$$\rho^2|\zeta''(t_2) - \zeta''(t_1)| \le \Big|\theta''\Big(\frac{t_2-\rho}{\rho}\Big) - \theta''\Big(\frac{t_1-\rho}{\rho}\Big)\Big|,$$

where

$$\Big|\frac{t_2-\rho}{\rho} - \frac{t_1-\rho}{\rho}\Big| = \Big|\frac{t_2-t_1}{\rho}\Big| \le \frac{1}{3} < 1.$$

So

$$|\zeta''(t_2) - \zeta''(t_1)| \le \frac{1}{\rho^3}[\theta'']_{\mathrm{Lip}}|t_2 - t_1|, \tag{5.9}$$

where $[\,\cdot\,]_{\mathrm{Lip}}$ denotes the usual Lipschitz semi-norm.

Set

$$v = \zeta u. \tag{5.10}$$

Note that $\boldsymbol{L}v \in D_{\overline{\omega}}(2\rho)$, moreover the support of v is contained in $|x| < 2\rho$.

On the other hand,

$$\boldsymbol{L}v = \zeta\boldsymbol{L}u + N. \tag{5.11}$$

One has

$$|(\zeta\boldsymbol{L}u)(x) - (\zeta\boldsymbol{L}u)(y)| \le \|\zeta\|[\boldsymbol{L}u]_{\overline{\omega}}\overline{\omega}(|x-y|) + \|\nabla\zeta\|\|\boldsymbol{L}u\||x-y| \le [\boldsymbol{L}u]_{\overline{\omega}}\overline{\omega}(|x-y|) + c\|\theta'\|\frac{1}{\rho}\|\nabla^2 u\||x-y|.$$

Hence,

$$[\zeta\boldsymbol{L}u]_{\overline{\omega}} \le [\boldsymbol{L}u]_{\overline{\omega}} + c\|\theta'\|\frac{1}{\rho}\|\nabla^2 u\|\frac{|x-y|}{\overline{\omega}(|x-y|)}. \tag{5.12}$$

Next we prove that

$$[N]_{\overline{\omega}} \le c(\theta)\Big(\frac{\|u\|}{\rho^3} + \frac{\|\nabla u\|}{\rho^2} + \frac{\|\nabla^2 u\|}{\rho}\Big)\frac{|x-y|}{\overline{\omega}(|x-y|}. \tag{5.13}$$

One has

$$N \cong (\nabla^2\zeta)u + (\nabla\zeta)(\nabla u) \equiv A + B.$$

By appealing in particular to (5.9), straightforward calculations show that

$$|A(x) - A(y)| \le \|\nabla u\|\|\nabla^2\zeta\||x-y| + \|u\|\frac{1}{\rho^3}[\theta'']_{\mathrm{Lip}}|x-y| \le \Big(\frac{1}{\rho^2}\|\theta''\|\|\nabla u\| + \frac{1}{\rho^3}[\theta'']_{\mathrm{Lip}}\|u\|\Big)|x-y|.$$

Hence

$$[A]_{\overline{\omega}} \le c(\theta)\Big(\frac{\|u\|}{\rho^3} + \frac{\|\nabla u\|}{\rho^2}\Big)\frac{|x-y|}{\overline{\omega}(|x-y|}. \tag{5.14}$$

Similar manipulations show that

$$[B]_{\overline{\omega}} \le c(\theta)\Big(\frac{\|\nabla u\|}{\rho^2} + \frac{\|\nabla^2 u\|}{\rho}\Big)\frac{|x-y|}{\overline{\omega}(|x-y|}. \tag{5.15}$$

Equation (5.13) follows from (5.14) and (5.15).

Lastly, from (5.11), (5.12), and (5.13) one shows that

$$[\boldsymbol{L}v]_{\overline{\omega}} \le [\boldsymbol{L}u]_{\overline{\omega}} + c(\theta)\Big(\frac{\|u\|}{\rho^3} + \frac{\|\nabla u\|}{\rho^2} + \frac{\|\nabla^2 u\|}{\rho}\Big)\frac{|x-y|}{\overline{\omega}(|x-y|}. \tag{5.16}$$

In the following not labeled norms concern the domain $I(2\rho)$.

From (5.10), (5.3), (5.5), and (5.16) one gets

$$[\nabla^2 u]_{\widehat{\omega};\rho} \le [\nabla^2 v]_{\widehat{\omega}} \le C[\boldsymbol{L}v]_{\overline{\omega}} \le C[\boldsymbol{L}u]_{\overline{\omega}} + c(\theta)\Big(\frac{\|u\|}{\rho^3} + \frac{\|\nabla u\|}{\rho^2} + \frac{\|\nabla^2 u\|}{\rho}\Big)\frac{|x-y|}{\overline{\omega}(|x-y|},$$

where $0 < 2\rho < 1$. □

6 The interior regularity estimate in the variable coefficients case

In this section we extend estimate (5.7) to uniformly elliptic operators with variable coefficients

$$\boldsymbol{L} = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j.$$

To avoid non-significant manipulations we assume that the coefficients $a_{ij}(x)$ are Lipschitz continuous in $I(2\rho)$, with Lipschitz constants bounded by a constant A. Following the same belief, we left to the reader the introduction of lower order terms.

We assume that

$$\overline{\omega}(r) \le k_1 \widehat{\omega}(r), \tag{6.1}$$

for some positive constant k_1, and r in some neighborhood of the origin. This yields $D_{\overline{\omega}}(\overline{\Omega}) = D_{\widehat{\omega}}(\overline{\Omega})$, recall Proposition 2.2. Assumption (6.1) holds if in equation (3.3) the constant C_1 is finite. In fact,

$$\lim_{r\to 0}\frac{\overline{\omega}(r)}{\widehat{\omega}(r)} = \lim_{r\to 0}\frac{r\overline{\omega}'(r)}{\overline{\omega}(r)} = \frac{1}{C_1}$$

if the second limit exists.

In the following we appeal to the constant coefficients operator

$$\boldsymbol{L}_0 = \sum_{i,j=1}^{n} b_{ij}\partial_i\partial_j,$$

where $b_{ij} = a_{ij}(0)$. Clearly,

$$\boldsymbol{L}_0 v(x) = \boldsymbol{L}v(x) + (\boldsymbol{L}_0 - \boldsymbol{L})v(x). \tag{6.2}$$

One has

$$(\boldsymbol{L}_0 - \boldsymbol{L})v(x) - (\boldsymbol{L}_0 - \boldsymbol{L})v(y) = (b_{ij} - a_{ij}(x))(\partial^2_{ij}v(x) - \partial^2_{ij}v(y)) + (a_{ij}(y) - a_{ij}(x))(\partial^2_{ij}v(y)), \tag{6.3}$$

where, for convenience, summation on repeated indexes is assumed. Straightforward calculations easily lead to the following pointwise estimate:

$$|(\boldsymbol{L}_0 - \boldsymbol{L})v(x) - (\boldsymbol{L}_0 - \boldsymbol{L})v(y)| \le cA\big(2\rho[\nabla^2 v]_{\overline{\omega}} + \|\nabla^2 v\|\frac{|x-y|}{\overline{\omega}(|x-y|)}\big)\overline{\omega}(|x-y|),$$

where norms and semi-norms concern the sphere $I(0, 2\rho)$.

Next assume that $v \in C^2(2\rho)$ has compact support in $I(0, 2\rho)$, and $\boldsymbol{L}v \in D_{\overline{\omega}}(2\rho)$. Then, by (6.2), (6.3), and (5.6) it follows that

$$[\nabla^2 v]_{\widehat{\omega};2\rho} \le C[\boldsymbol{L}v]_{\overline{\omega};2\rho} + C\rho[\nabla^2 v]_{\overline{\omega};2\rho} + C\|\nabla^2 v\|\frac{|x-y|}{\overline{\omega}(|x-y|)}.$$

In particular,

$$[\nabla^2 v]_{\widehat{\omega};2\rho} \leq C[\boldsymbol{L}v]_{\overline{\omega};2\rho} + C\rho[\nabla^2 v]_{\overline{\omega};2\rho} + C\|\nabla^2 v\|.$$

Now, from (6.1), one gets

$$(1 - Ck_1\rho)[\nabla^2 v]_{\overline{\omega};2\rho} \leq C([\boldsymbol{L}v]_{\overline{\omega};2\rho} + \|\nabla^2 v\|).$$

Next we set

$$v = \zeta u$$

and argue as done to prove (5.7). This proves the following result, in the case of variable coefficients operators.

Theorem 6.1. *Assume that the oscillation $\overline{\omega}$ satisfies conditions* (3.1), (3.2), (3.9), (3.5), *and* (6.1). *Further, assume that*

$$0 < \rho \leq \frac{1}{2Ck_1},$$

and let $\boldsymbol{L}u \in D_{\overline{\omega}}(2\rho)$, for some $u \in C^2(2\rho)$. Then $\nabla^2 u \in D_{\widehat{\omega}}(\rho)$, and

$$[\nabla^2 u]_{\widehat{\omega};\rho} \leq C[\boldsymbol{L}u]_{\overline{\omega},2\rho} + \frac{C}{\rho^3}\|u\|_{C^2(2\rho)}, \tag{6.4}$$

for suitable positive constants C, independent of ρ.

7 Proof of Theorems 3.2 and 3.4

The local estimates (estimates in Ω_0, $\Omega_0 \subset\subset \Omega$) claimed in Theorems 3.2 and 3.4 follow immediately from the interior estimates, by appealing to the classical method consisting in covering $\overline{\Omega}_0$ by a finite number of sufficiently small spheres. For brevity, we may estimate the quantities originated by the terms $\|u\|_{C^2(2\rho)}$, see the right-hand sides of equations (5.8) and (6.4), simply by appealing to Theorem 1.1, which shows that solutions u satisfy the estimate

$$\|u\|_{C^2(\overline{\Omega})} \leq c\|f\|_*.$$

Concerning regularity up to the boundary one proceeds as follows. The main point, the extension of the interior regularity estimate (5.8) from spheres to half-spheres, is obtained by following the argument described in [11, Part II, Section 5.6]. One starts by showing that the interior estimate in spheres also hold for half-spheres, under the zero boundary condition on the flat part of the boundary. One appeals here to “reflection” of u in the orthogonal direction through the flat boundary, from the half to the whole sphere, as an odd function. In this way the half-sphere problem goes back to an whole-sphere problem, absolutely similar to that considered in Section 5, see [11]. Note that it is sufficient, and simpler, to appeal to the above extension to half-spheres merely for constant coefficient operators. The regularity result “up to flat boundary points”, claimed for constant coefficients operators in Theorem 3.2, follows.

Concerning the variable coefficients case considered in Theorem 3.4, we argue as follows. Extension of the half-sphere’s estimate, from constant coefficients to variable coefficients operators, is obtained exactly as done in Section 6 for whole spheres, by appealing to the fundamental assumption (6.1). Then, sufficiently small neighborhoods of boundary points are regularly mapped, one to one, onto half-spheres, by appealing to suitable local changes of coordinates. This procedure allows extension of the local estimate to solutions u defined on sufficiently small neighborhoods of boundary points, vanishing on the boundary. A well-known finite covering argument leads to the thesis of Theorem 3.4.

Remark. Since the above extension to non-flat boundary points requires local changes of coordinates, even constant coefficients operators are transformed in variable coefficients operators. Hence our proof of local regularity up to non-flat boundary points requires, even for constant coefficients operators, assumption (6.1). This is the reason why regularity up to non-flat boundary points is not claimed in Theorem 3.2. The correspondent extension remains a challenging open problem, even in the framework of Log spaces (where counterexamples may also be tried).

8 The Log spaces $D^{0,\alpha}(\overline{\Omega})$. An intermediate regularity result

The following is a significant example of functional space $D_{\overline{\omega}}(\overline{\Omega})$ which yields intermediate (not full) regularity, based on the well-known formulae

$$\int \frac{(-\log r)^{-\alpha}}{r}\,dr = \frac{1}{\alpha-1}(-\log r)^{1-\alpha}, \tag{8.1}$$

where $0 < \alpha < +\infty$ (for $\alpha = 1$ the right-hand side should be replaced by $-\log(-\log r)$). Equation (8.1) shows that the $C_*(\overline{\Omega})$ semi-norm (1.4) is finite if, for some $\alpha > 1$ and some constant $C > 0$,

$$\omega_f(r) \le C(-\log r)^{-\alpha} \quad \text{for all } 0 < r < 1.$$

This led to define the semi-norm

$$[f]_\alpha \equiv \sup_{r\in(0,1)} \frac{\omega_f(r)}{\omega_\alpha(r)}, \tag{8.2}$$

where the oscillation $\omega_\alpha(r)$ is defined by setting

$$\omega_\alpha(r) = (-\log r)^{-\alpha}.$$

Hence $[f]_\alpha$ is the smallest constant for which the estimate

$$|f(x) - f(y)| \le [f]_\alpha \cdot \left(\log \frac{1}{|x-y|}\right)^{-\alpha} \tag{8.3}$$

holds for all couple $x, y \in \overline{\Omega}$ such that $|x - y| < 1$. Note that we have merely replaced, in the definition of Hölder spaces $C^{0,\alpha}(\overline{\Omega})$, the quantity $\frac{1}{|x-y|}$ by $\log \frac{1}{|x-y|}$, and allow α to be arbitrarily large.

Definition 8.1. For each real positive α, set

$$D^{0,\alpha}(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_\alpha < \infty\}.$$

A norm is introduced in $D^{0,\alpha}(\overline{\Omega})$ by setting $\|f\|_\alpha \equiv [f]_\alpha + \|f\|$.

We call these spaces Log spaces. We remark that in reference [7] we have called these spaces H-log spaces.

The restriction $|x - y| < 1$ in equation (8.3) is due to the behavior of the function $\log r$, for $r \ge 1$. Note that, by replacing $0 < |x - y| < 1$ by $0 < |x - y| < \rho$ in equation (8.2), for some $0 < \rho < 1$, it follows that

$$[f]_{\alpha;\rho} \le [f]_\alpha \le [f]_{\alpha;\rho} + \frac{2}{(-\log\rho)^{-\alpha}}\|f\|,$$

where the meaning of $[f]_{\alpha;\rho}$ seems clear. Hence, the norms $\|f\|_\alpha$ and $\|f\|_{\alpha;\rho}$ are equivalent. We may also avoid the restriction $|x - y| < 1$ by replacing in the denominator of the right hand side of (8.2) the quantity r by $\frac{r}{R}$, where $R = \operatorname{diam}\Omega$, and by letting $r \in (0, R)$. We rather prefer the first definition, since the second one implies more ponderous notation.

For $0 < \beta < \alpha$, and $0 < \lambda \le 1$, the (compact) embedding

$$D^{0,\alpha}(\overline{\Omega}) \subset D^{0,\beta}(\overline{\Omega}) \subset C(\overline{\Omega})$$

hold. Furthermore, for $1 < \alpha$, one has the (compact) embedding $D^{0,\alpha}(\overline{\Omega}) \subset C_*(\overline{\Omega})$. Note that $D^{0,1}(\overline{\Omega}) \not\subset C_*(\overline{\Omega})$.

It is worth noting that in reference [7] we claimed, and left the proof to the reader, that $C^\infty(\overline{\Omega})$ is dense in $D^{0,\alpha}(\overline{\Omega})$. Actually, as shown in Theorem 2.7, this result is false.

Theorem 8.2. *Let $\Omega_0 \subset\subset \Omega$, $f \in D^{0,\alpha}(\overline{\Omega})$ for some $\alpha > 1$, and u be the solution of problem* (1.1)*, where $\mathbf{L}$ has constant coefficients. Then $\nabla^2 u \in D^{0,\alpha-1}(\Omega_0)$, moreover*

$$\|\nabla^2 u\|_{\alpha-1,\Omega_0} \le C\|f\|_\alpha, \tag{8.4}$$

for some positive constant $C = C(\alpha, \Omega_0, \Omega)$. The regularity result holds up to flat boundary points. Moreover, it is optimal in the sharp sense. In particular, for $\beta > \alpha - 1$, $\nabla^2 u \in D^{0,\beta}(\Omega_0)$ is false in general.

Theorem 8.2 is a particular case of Theorem 3.2. In fact, the oscillation $\omega_\alpha(r)$ is concave and differentiable for $r > 0$, satisfies (3.1) for $\alpha > 1$, and (3.2) holds. Further, condition (3.3) follows from

$$\lim_{r\to 0} \frac{\omega_\alpha(r)}{r\omega'_\alpha(r)} = +\infty.$$

In [7] the above regularity result was claimed up to the boundary. However the proof is not complete, since extension to non-flat boundary points would require here estimates for variable coefficients operators. The reason for this requirement was explained in Section 7.

Next we illustrate, by means of a simple example, the practical meaning of *sharp* optimality, recall Definition 3.3. Sharp optimality is not confined to the particular family of spaces under consideration, but is something stronger. Let us illustrate the distinction. Set $\omega(r) = \omega_{\nabla^2 u}(r)$. Theorem 8.2 claims that

$$\omega(r) \le C_f(-\log r)^{-(\alpha-1)}, \tag{8.5}$$

for each $f \in D^{0,\alpha}(\overline{\Omega})$. Optimality of this result, *restricted* to the Log spaces' family, means that

$$\omega(r) \le C_f(-\log r)^{-\beta} \tag{8.6}$$

is false in general, for any $\beta > \alpha - 1$. This does not exclude that (for instance) for all $f \in D^{0,\alpha}(\overline{\Omega})$ the oscillation $\omega(r)$ of $\nabla^2 u$ satisfies the estimate

$$\omega(r) \le C_f\left[\log\left(\log \frac{1}{r}\right)\right]^{-1} \cdot (-\log r)^{-(\alpha-1)},$$

which is weaker than (8.6), but stronger than (8.5).

Sharp optimality avoids the above, and similar, possibilities. This fact is significant in all cases in which full regularity is not reached, as in Theorem 8.2. This is the meaning giving here to the sharpness of a regularity result.

Concerning references related to Log spaces (mostly for $n = 1$, or $\alpha = 1$), we refer the reader to the treatise [13] (see, in particular, Definition 2.2 in this reference), and to [14, 16, 18, 21–23].

9 Hölog spaces $C^{0,\lambda}_\alpha(\overline{\Omega})$ and full regularity

If, for some $\lambda > 0$, one has $\overline{\omega}(r) = \lambda\widehat{\omega}(r)$ in a neighborhood of the origin, then there is a constant $k > 0$ such that $\overline{\omega}(r) = kr^\lambda$. This fact could suggest that Hölder spaces could be the unique full regularity class inside our framework. However, *full regularity* is also enjoyed by other spaces. The following is a particularly interesting case. Consider oscillations

$$\omega_{\lambda,\alpha}(r) = r^\lambda(-\log r)^{-\alpha},$$

where $0 \le \lambda < 1$ and $\alpha \in \mathbb{R}$, and define the semi-norm

$$[f]_{\lambda,\alpha} \equiv \sup_{r\in(0,R)} \frac{\omega_f(r)}{\omega_{\lambda,\alpha}(r)},$$

for some $R > 0$ (for instance, $R = \operatorname{diam}\Omega$). Hence $[f]_{\lambda,\alpha}$ is the smallest constant for which the estimate

$$|f(x) - f(y)| \le [f]_{\lambda,\alpha}\left(\log \frac{1}{|x-y|}\right)^{-\alpha} \cdot |x-y|^\lambda$$

holds for all couple $x, y \in \overline{\Omega}$, $|x - y| < R$.

Definition 9.1. For each $0 \le \lambda < 1$ and each $\alpha \in \mathbb{R}$, set

$$C^{0,\lambda}_\alpha(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_\alpha < \infty\}.$$

A norm is introduced in $C^{0,\lambda}_\alpha(\overline{\Omega})$ by setting $\|f\|_{\lambda,\alpha} \equiv [f]_{\lambda,\alpha} + \|f\|$.

We call these spaces Hölog spaces.

For $\lambda = 0$ and $\alpha > 0$ we re-obtain the Log space $D^{0,\alpha}(\overline{\Omega})$, for $\lambda > 0$ and $\alpha = 0$ we re-obtain $C^{0,\lambda}(\overline{\Omega})$. Furthermore,

$$C_\alpha^{0,\lambda}(\overline{\Omega}) \subset C_\beta^{0,\lambda}(\overline{\Omega}) \quad \text{for } \alpha > \beta > 0,$$

and

$$C^{0,\lambda_2}(\overline{\Omega}) \subset C_\alpha^{0,\lambda}(\overline{\Omega}) \subset C^{0,\lambda}(\overline{\Omega}) \subset C_{-\alpha}^{0,\lambda}(\overline{\Omega}) \subset C^{0,\lambda_1}(\overline{\Omega}) \quad \text{for } 0 < \lambda_1 < \lambda < \lambda_2 < 1 \text{ and } \alpha > 0.$$

Theorem 2.5 shows that all the above inclusions are compact.

Note that the set

$$\bigcup_{\lambda,\alpha} C_\alpha^{0,\lambda}(\overline{\Omega})$$

is a *totally ordered set*, in the set's inclusion sense. Roughly speaking, in the chain merely consisting of classical Hőlder spaces, each $C^{0,\lambda}$ space can be replaced by the infinite chain $C_\alpha^{0,\lambda}$, $\alpha \in \mathbb{R}$. The resulting chain is still totally ordered.

To abbreviate notation, we set in this section

$$\overline{\omega}(r) \equiv \omega_{\lambda,\alpha}(r), \quad [f]_{\overline{\omega}} \equiv [f]_{\lambda,\alpha}, \quad \text{and} \quad \|f\|_{\overline{\omega}} \equiv \|f\|_{\lambda,\alpha}.$$

The following full regularity result holds.

Theorem 9.2. *Let $f \in C_\alpha^{0,\lambda}(\overline{\Omega})$ for some $\lambda \in (0,1)$ and some $\alpha \in \mathbb{R}$. Let u be the solution of problem* (1.1), *where the differential operator $\mathbf{L}$ may have variable coefficients. Then $\nabla^2 u \in C_\alpha^{0,\lambda}(\overline{\Omega})$. Moreover,*

$$\|\nabla^2 u\|_{\lambda,\alpha} \leq C\|f\|_{\lambda,\alpha},$$

for some positive constant C. The result is optimal, in the sharp sense.

Note that full regularity $\omega_{\lambda,\alpha} \to \omega_{\lambda,\alpha}$ could be a little surprising here. In fact, at the light of Theorem 8.2, we could merely expected the intermediate regularity result $\omega_{\lambda,\alpha} \to \omega_{\lambda,\alpha-1}$.

Proof. We appeal to Theorem 3.2. Assumptions (3.1) and (3.2) are trivially verified. Let us prove (3.3). Set

$$L(r) = \log\frac{1}{r}.$$

Straightforward calculations show that

$$\overline{\omega}'(r) = r^{\lambda-1}L(r)^{-\alpha}(\lambda + \alpha L(r)^{-1})$$

and that

$$\overline{\omega}''(r) = -r^{\lambda-2}L(r)^{-\alpha}\big(\lambda(1-\lambda) - (2\lambda-1)\alpha L(r)^{-1} - \alpha(\alpha+1)L(r)^{-2}\big). \tag{9.1}$$

Equation (9.1) shows that $\overline{\omega}''(r) < 0$ in a neighborhood of the origin, since $\lim_{r\to 0} L(r) = +\infty$. Hence $\overline{\omega}$ is concave. Furthermore, (3.3) holds since

$$\lim_{r\to 0}\frac{\overline{\omega}(r)}{r\overline{\omega}'(r)} = \frac{1}{\lambda} > 1. \tag{9.2}$$

To prove full regularity we appeal to the de l'Hôpital rule and to (9.2) to show that

$$\lim_{r\to 0}\frac{\widehat{\omega}(r)}{\overline{\omega}(r)} = \lim_{r\to 0}\frac{\overline{\omega}(r)}{r\overline{\omega}'(r)} = \frac{1}{\lambda}.$$

In particular. (2.2) holds for r in some neighborhood of the origin. Hence Proposition 2.2 applies. □

It would be interesting to study higher order regularity results in the framework of Hölog spaces.

10 Sharpness of the regularity results

In this section we prove the *sharpness* of our regularity results (a simple example was shown at the end of Section 8). The proof is quite adaptable to different situations, local and global results, etc. We merely show the main argument. We construct a counterexample, which concerns constant coefficients operators (we could easily deny case by case), which shows that any stronger regularity result can not occur. We start by considering the Laplace operator Δ. We remark that the argument applies to the regularity results stated in Theorems 3.2 and 3.4. However, in the second theorem, the conclusion is obvious, due to full regularity.

For convenience, we assume that $\overline{\omega}(r)$ is differentiable, and that there is a positive constant C such that

$$\frac{\overline{\omega}(r)}{r\overline{\omega}'(r)} \geq C > 0, \tag{10.1}$$

for $r > 0$, in a neighborhood of the origin. Note that (10.1) holds, with $C = 1$, if $\overline{\omega}(r)$ is concave.

Proposition 10.1. *Assume that $\overline{\omega}(r)$ satisfies the above hypothesis, and let $\widehat{\omega}_0(r)$ be a given oscillation. Assume that the results stated in Theorem 3.2 hold by replacing $\widehat{\omega}$ by $\widehat{\omega}_0$. Then there is a constant c for which $\widehat{\omega}(r) \leq c\widehat{\omega}_0(r)$.*

We may say that any regularity result better than (8.4) is false.

Proof. For simplicity, we start by assuming that $\boldsymbol{L} = \Delta$. Consider the function

$$u(x) = \widehat{\omega}(|x|)x_1x_2, \tag{10.2}$$

defined in $\mathbb{R}^n$, $n \geq 2$. Actually, we are merely interested in the behavior near the origin (see (10.4) below).

Straightforward calculations show that

$$\Delta u(x) = (n+2)\frac{x_1x_2}{|x|^2}\overline{\omega}(x) + \frac{x_1x_2}{|x|^2}|x|\overline{\omega}'(|x|).$$

In particular, $\Delta u(0) = 0$. By appealing to (10.1) one shows that

$$|\Delta u(x) - \Delta u(0)| = |\Delta u(x)| \leq C\overline{\omega}(|x|).$$

Hence, in a neighborhood of the origin, $f(x) = \Delta u(x)$ belongs to $D_{\overline{\omega}}$.

On the other hand, straightforward calculations show that

$$\partial_1\partial_2 u(x) = \widehat{\omega}(|x|) + \frac{1}{|x|^2}\left(x_1^2 + x_2^2 - 2\frac{x_1^2x_2^2}{|x|^2}\right)\cdot\overline{\omega}(|x|) + \frac{x_1^2x_2^2}{|x|^4}\cdot(|x|\overline{\omega}'(|x|)). \tag{10.3}$$

In particular, $\partial_1\partial_2 u(0) = 0$, and

$$|\partial_1\partial_2 u(x) - \partial_1\partial_2 u(0)| \geq \widehat{\omega}(|x|)$$

for $0 < |x| \ll 1$, since in equation (10.3) the coefficients of $\overline{\omega}(|x|)$ and of $|x|\overline{\omega}'(|x|)$ are nonnegative. On the other hand, if $\widehat{\omega}_0(r)$ regularity holds, one has

$$|\partial_1\partial_2 u(x) - \partial_1\partial_2 u(0)| \leq (c\|f\|_{\overline{\omega}})\widehat{\omega}_0(|x|)$$

for some $c > 0$. Hence $\widehat{\omega}(r) \leq c_0\widehat{\omega}_0(r)$, for $r > 0$, in a neighborhood of the origin.

If $\boldsymbol{L}$ is given by (1.2), we replace (10.2) by

$$u(x) = \widehat{\omega}(|x|)\sum_{i,j=1} 1^n b_{ij}x_ix_j,$$

where $B \neq 0$ is symmetric and

$$\sum_{i,j=1}^{n} a_{ij}b_{ij} = 0.$$

In particular, if a specific coefficient a_{kl} vanishes, we may simply choose $u(x) = \widehat{\omega}(|x|)x_kx_l$, as done in (10.2).

We localize the above result as follows. Assume that $0 \in \Omega$, and consider the function

$$u(x) = \psi(|x|)\widehat{\omega}(|x|)x_1x_2, \tag{10.4}$$

where $\psi(r)$ is non-negative, indefinitely differentiable, vanishes for $r \geq \rho > 0$, and is equal to 1 for $|x| < \frac{\rho}{2}$. The radius ρ is such that $I(0,\rho)$ is contained in Ω. The above truncation allows us to assume homogeneous boundary conditions in Ω (we may consider combinations of functions as above, centered in different points in Ω, with distinct radius, and distinct cut-off functions). □

It is worth noting that in the above argument the specific expressions of the coefficients of $\overline{\omega}(|x|)$ and $|x|\overline{\omega}'(|x|)$ are secondary (even if the non-negativity of these coefficients was exploited). They are homogeneous functions of degree zero, without particular influence on the minimal regularity. The crucial point is that the second order derivative $\partial_1\partial_2 u(x)$, due to the term $x_1 x_2$ in (10.2), leaves unchanged the "bad term" $\widehat{\omega}(|x|)$. This does not occur for derivatives $\partial_i^2 u(x)$, hence does not occur for $\Delta u(x)$.

It looks interesting to note that the "bad term" $\widehat{\omega}(|x|)$ can not be eliminated by the other two terms which are present in the right hand side of (10.3). Even when full regularity occurs (like in Hölder and Hölog spaces), the "bad term" $\widehat{\omega}(|x|)$ is still not eliminated. It simply is as regular as the other two terms, $\overline{\omega}(|x|)$ and $|x|\overline{\omega}'(|x|)$.

11 Further properties. Non-linear problems

Applications to non-linear problems requires, besides the linear theory, some main ingredients like product and composition properties. Concerning these two points we merely recall here some main properties. Set

$$(\omega_1 \vee \omega_2)(r) = \sup\{\omega_1(r), \omega_2(r)\},$$

and assume for simplicity that f and g are scalar fields in Ω. It readily follows

$$f \in D_{\omega_1}, g \in D_{\omega_2} \implies fg \in D_{\omega_1 \vee \omega_2},$$

and also $\|fg\|_{\omega_1 \vee \omega_2} \leq \|f\|_{\omega_1}\|g\|_{\omega_2}$. In particular, D_ω spaces are Banach algebras.

Concerning composition of functions, if $F \in \mathrm{Lip}(\mathbb{R}; \mathbb{R})$ and $u \in D_\omega$, then

$$[F(u)]_\omega \leq [F]_{\mathrm{Lip}}[u]_\omega.$$

In the particular case of Hölog spaces, the following extension to Hölder functions F may be useful. Assume that $F \in C^{0,\theta}(\mathbb{R}; \mathbb{R})$, $0 < \theta \leq 1$, and that $u \in C^{0,\lambda}_\alpha(\overline{\Omega})$. Then

$$[F(u)]_{\theta\lambda,\theta\alpha} \leq [F]_\theta [u]_{\lambda,\alpha}.$$

Let us end this paper by proposing the study of a non-linear problem which lies outside the above main lines. Let us recall the following well-known old problem (see the pioneering papers [15] and [2], and also the revision paper [5]). One looks for local geometrical conditions on the boundary which guarantee the continuity at a point $x_0 \in \Gamma$ of the solutions to the boundary value problem

$$\begin{cases} \nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases} \tag{11.1}$$

for each given $\phi \in C(\Gamma)$. For $p = 2$, the above p-Laplace operator is simply the classical Laplace operator. Clearly, in this linear case, the problem is even much older. It would be interesting to study, systematically, the following kind of related problem. Assume, for simplicity, that $\Gamma - \{x_0\}$ is smooth. We want to establish local geometrical conditions on the boundary, in the neighborhood of a point $x_0 \in \Gamma$, which guarantee that solution u to (11.1) belong to some fixed $D_{\omega_2}(\overline{\Omega})$, for each ϕ in a given boundary space $D_{\omega_1}(\Gamma)$.

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