On a family of results concerning direction of vorticity and regularity for the Navier–Stokes equations

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ANNALI DELL'UNIVERSITA' DI FERRARA
SEZIONE VII - SCIENZE MATEMATICHE

ISSN 0430-3202

Ann Univ Ferrara
DOI 10.1007/s11565-014-0206-3

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Received: 19 September 2013 / Accepted: 7 February 2014

Abstract These notes concern existence, and suitable formulation, of meaningful conditions on the direction of the vorticity which guarantee the regularity of the solutions to the evolution Navier–Stokes equations. A main concern here is to compare the different situations which appear in considering slip and no-slip boundary conditions. The paper reviews mainly results obtained in some of the references cited.

Keywords Navier–Stokes equations · Direction of vorticity · Regularity of solutions · Geometrical constraints

Mathematics Subject Classification 35B65 · 35Q30 · 36D03 · 36D05

1 Introduction. Some general comments.

We start by recalling the evolution Navier–Stokes equations

$$\begin{cases} u_t + (u \cdot \nabla)\, u - \nu \, \triangle u + \nabla p = 0\,, \\ \qquad\qquad\qquad\qquad \nabla \cdot u = 0 \qquad \text{in } \Omega \times (0, T], \\ \qquad\qquad\qquad u(x,0) = u_0(x) \qquad \text{in } \Omega\,, \end{cases} \tag{1}$$

where u_0 is divergence free. For simplicity, we assume that external force vanishes. We assume that the initial data u_0 belongs to the Sobolev space $H^1(\Omega)$, plus suitable boundary conditions, depending on the problem. If we want solutions which are regular

Dedicated to the Memory of Mariolina Padula.

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Published online: 08 March 2014

including the time $t = 0$, then u_0 should be assumed more regular, and satisfying suitable compatibility conditions. This kind of problem is out of real interest here.

Ω may denote the whole space R^3, the half space R^3_+, or an open, connected, bounded subset of R^3, with a smooth boundary $\Gamma = \partial\,\Omega$. In this last case, equations are supplemented, on $\Gamma \times (0, T)$, with the slip boundary condition ("stress-free" boundary condition)

$$\begin{cases} u \cdot n = 0\,, \\ \omega \times n = 0\,, \end{cases} \tag{2}$$

or with the no-slip boundary condition

$$u = 0. \tag{3}$$

Here $\omega = \nabla \times u = \mathrm{curl}\, u$ is the vorticity field, while n denotes the exterior unit normal vector to the boundary. In the case of flat boundaries, the above condition (2) coincides with the classical Navier boundary conditions without friction, see [28], and also [31].

We will not repeat well know notation as, for instance, Sobolev spaces notation, and so on. Solutions $u \in L^2(0,\, T;\; H^1(\Omega)\,) \cap L^\infty(0,\, T;\; L^2(\Omega)\,)$ are defined here in the well known Leray-Hopf weak sense, and are assumed to be weakly continuous from the right at time $t = 0$. We say that a Leray-Hopf weak solution u is strong if it belongs to $L^\infty(0,\, T;\; H^1(\Omega)\,) \cap L^2(0,\, T;\; H^2(\Omega)\,)$. It is well known that strong solutions are smooth, if data and domain are also smooth.

We look for significant, geometrical, conditions on the vorticity-direction which guarantee that a given Leray-Hopf solution is regular on the domain of existence, as long as our geometrical conditions hold. We set

$$\theta(x, y, t) \stackrel{def}{=} \angle(\omega(x, t), \omega(y, t))\,,$$

where the symbol "$\angle$" denotes the amplitude of the angle between two vectors. We are interested in sufficient conditions on $\sin\theta(x, y, t)$ to guarantee the regularity of the solutions.

The following explanation should be clear to readers acquainted with the basic theory of the Navier–Stokes equations. We claim that, in the sequel, we may argue by assuming that solutions under consideration are smooth. In fact, it is well know that each solution is strong (hence, smooth) in a "small" time interval $(0,\, \epsilon_0)$, where $\epsilon_0 > 0$ depends on the H^1 norm of u_0. On the other hand, in the following, we prove that if a *regular solution* satisfies our geometrical conditions in some interval $(0,\, t_0)$, then it necessarily belongs to $L^\infty(0,\, t_0;\; H^1(\Omega) \cap L^2(0,\, t_0;\; H^2(\Omega)\,)$. So such a solution belongs, in particular, to $C(\,[0,\, t_0\,];\; H^1(\Omega)\,)$. Hence, roughly speaking, we may start from $u(t_0)$ as a new "initial data". It follows, as for $t = 0$, that the solution is unique and strong up to time $t_0 + \epsilon$, for some positive ϵ, which depends on the H^1 norm of $u(t_0)$. We also show that the H^1 norm of our solution keeps bounded from above in $[0,\, T\,]$. The above picture shows that in proving the estimates below, it is sufficient to deal with smooth solutions.

In this section we appeal to the simplest situation, namely the whole space case, to describe and discuss some general aspects of the basic theory, which do not depend on boundary conditions. Then, in Sect. 2, we describe, and compare, the main obstacles in the proofs under slip and no-slip boundary conditions. In Sect. 3 the above considerations will be illustrate in more detail, by going inside the proofs.

We start from the very fundamental pioneering paper [16], by P. Constantin and Ch. Fefferman, where the authors prove that solutions to the evolution Navier–Stokes equations in the whole space are smooth if the direction of the vorticity is Lipschitz continuous with respect to the space variables. More precisely, they show the following result.

Theorem 1.1 *Let be $\Omega = R^3$, and let u be a weak solution of* (1) *in* $[0, T)$, *with* $u_0 \in H^1(R^3)$ *and* $\nabla \cdot u_0 = 0$. *If*

$$\sin\theta(x, y, t) \leq g(t)\,|x - y|$$

for some $g(t, x) \in L^{12}(0, T;\ L^\infty(\mathbb{R}^3))$, *then the solution u is regular. Clearly, the result holds if g is a constant.*

As in all the following results, conditions on $\sin\theta(x, y, t)$ are assumed for almost all x and y in Ω, and almost all t in $(0, T)$. Furthermore, these conditions are needed merely for points x and y such that $|x - y| < \delta$, for an arbitrary positive constant δ.

In reference [7], L.C. Berselli and the present author show that regularity still holds in the whole space by replacing Lipschitz continuity by $\frac{1}{2}$–Hőlder continuity. The following theorem was proved.

Theorem 1.2 *Let Ω, u, and u_0, be as in the previous theorem. Further, suppose that there exists* $\beta \in [1/2, 1]$ *and* $g \in L^a(0, T; L^b(\Omega))$, *where*

$$\frac{2}{a} + \frac{3}{b} = \beta - \frac{1}{2} \quad \textit{with} \quad a \in \left[\frac{4}{2\beta - 1}, \infty\right], \tag{4}$$

such that

$$\sin\theta(x, y, t) \leq g(t, x)\,|x - y|^\beta \tag{5}$$

holds in $\Omega \times (0, T)$. *Then, the solution u is strong. In particular, the regularity result holds if*

$$\sin\theta(x, y, t) \leq c\,|x - y|^{1/2}. \tag{6}$$

In the subsequent paper [2], we consider the case $\beta \leq \frac{1}{2}$. The proof is simply a quite obvious variant of the proof of theorem 1.2 in reference [7]. The result is the following.

Theorem 1.3 *Let u be a weak solution of* (1) *in* $[0, T)$ *with* $u_0 \in H^1$ *and* $\nabla \cdot u_0 = 0$. *Let* $\beta \in [0, 1/2]$ *and assume that*

$$\sin\theta(x, y, t) \leq c|x - y|^\beta \tag{7}$$

holds in $\Omega \times (0, T)$. *In addition, suppose that*

$$\omega \in L^2(0, T; L^r), \tag{8}$$

where

$$r = \frac{3}{\beta + 1}. \tag{9}$$

Then the solution u is strong in $(0, T)$ *and, consequently, is regular. In particular* $\sin\theta(x, y, t) \leq c\,|x - y|^{1/2}$ *is sufficient for regularity.*

Following the aim of these notes, we like to recall separately references [6,9–11]. In particular, in reference [9], regularity is proved for the Cauchy problem, without any continuity assumption on $\sin\theta(x, y, t)$. This kind of condition is replaced by a smallness assumption. Essentially, it is proved that there is a sufficiently small constant C_1 (an explicit estimate for C_1 is given) such that regularity holds if

$$\sin\theta(x, y, t) \leq C_1.$$

Clearly, there are many very interesting papers related to our contributions. We recall here, without any claim of completeness, the related papers [6,9,11–15,17–27,29,34], and references therein.

Finally, in the appendix, we present some reflections upon the global structure and significance of our results, taken as a whole.

2 Boundary value problems. Comparison between the slip and the no-slip cases

In reference [3] we consider the Navier–Stokes equations in $[0, T) \times \mathbb{R}^3_+$, endowed with the slip boundary condition (2), and prove the following result.

Theorem 2.1 *Assume that* $u_0 \in H^1(\mathbb{R}^3_+)$ *is divergence free, and tangential to the boundary. Let u be a weak solution of the Navier–Stokes equations* (1) *in* $[0, T) \times \mathbb{R}^3_+$, *endowed with the slip boundary condition* (2). *Let* $\beta \in [0, 1/2]$ *and assume that* (7), (8)*, and* (9) *hold. Then the solution* u *is strong in* $(0, T)$ *and, consequently, is regular.*

In particular,

$$\sin\theta(x, y, t) \leq c\,|x - y|^{1/2}$$

is sufficient for regularity.

The last claim follows from the fact that weak solutions satisfy (8) for $r = 2$. Hence, if $\beta = 1/2$, assumption (8) is superfluous.

The Theorem 2.1 was proved in reference [3] by appealing, separately, to the classical Dirichlet and Neumann *Green functions*, in the half space. This can be done for flat boundaries since the boundary conditions (2) can be written, on Γ, in the form

$$\begin{cases} u_3 = 0, \\ \frac{\partial u_j}{\partial x_3} = 0, \quad 1 \leq j \leq 2. \end{cases} \tag{10}$$

The third equation follows from $\omega_1 = \omega_2 = 0$ on Γ, plus differentiation of $u_3 = 0$ with respect to the first to independent variables. As shown in Sect. 3, we will control the non-linear convective term, present in the vorticity equation (13), by treating the boundary conditions via the related Green functions. Clearly, the independence, and the classical form, of the boundary conditions imposed to the three components of the velocity by Eq. (10), makes this task much easier.

Another basic tool in the Proof of Theorem 2.1 are the equations

$$\begin{cases} \omega_1 = \omega_2 = 0, \\ \frac{\partial \omega_3}{\partial x_3} = 0 \end{cases} \tag{11}$$

on Γ. The third equation follows from tangential differentiation of the first two equations together with $\nabla \cdot \omega = 0$.

The picture concerning the no-slip boundary condition (3) is, in a certain sense, opposite to that concerning the slip boundary condition. In both cases the very starting point is Eq. (13) below. The terms to be controlled are the integral involving nonlinear terms (22) (which arises from integration by parts of a viscous term), and the non-linear integral (which arises from the vortex stretching term). In the slip case, Eq. (11) immediately shows that the boundary integral vanishes. On the contrary, in the no-slip boundary case, the non zero viscosity prevent us from a suitable control of the boundary integral (see [4]).

On the other hand, concerning the control of the integral involving nonlinear terms, the situation is reversed. This control is easier under the no-slip boundary condition, even in the half-space, since in this case appeal to the Green's function for the Dirichlet boundary value problem is clearly sufficient. The Green's function for the Neumann boundary value problem is not needed here. An even more crucial difference is that, in the no-slip case, the half-space simplified approach followed in the Proof of Theorem 2.1 applies also for regular domains Ω. The reason is that independence of the three boundary conditions, namely

$$u_1 = u_2 = u_3 = 0\,, \tag{12}$$

holds also for non-flat boundaries. So, we simply apply to the Green function for the Dirichlet problem, exactly as for the half-space case. On the contrary, in the case of non-flat boundaries under the slip boundary condition, the separation present in Eq. (10) is no more true, and the problem becomes much harder. As shown in reference [8], appeal to Green functions theory is still possible, but particularly delicate. In fact, in the above reference, Berselli and the author succeed in extending the Theorem 2.1 to the case in which $\Omega \subset R^3$ is an open, bounded set with a smooth boundary. Since we can not appeal, separately, to the Dirichlet and the Neumann Green functions, as in the Proof of Theorem 2.1, now we have to localize the problem, a not trivial and quite technical matter. So, in the proof, the authors appeal to representation formulas for Green's matrices derived in Solonnikov's fundamental work [32,33]. With the aid of these explicit formulas, original local representation formulas for the velocity (in terms of the vorticity) were introduced. In this way useful estimates for the vortex stretching terms were proved. The proof is particularly involved.

Remark 2.1 For $\beta = 1$ (the classical Lipschitz continuity assumption considered in Constantin and Fefferman's paper), a much simpler proof of the sharp result stated in reference [8] is given in reference [5].

3 Inside the proofs

In this section we opt to consider the no-slip boundary condition (3) as being the "running assumption", and to implement the argument with suitable remarks concerning analogies and differences with the slip boundary condition. The very basic ideas are similar in booth approaches, but substantially more complex to describe in the slip case. Clearly, we took into account that our option precludes some interesting features, peculiar to the slip boundary case. On the other hand, as already remarked, we are not able to prove the final result under the no-slip boundary condition (this is still an open problem), due to the stronger boundary-effect of viscosity under this last condition. However, this fact turns into a factor of enrichment of our exposition.

In the following, when we consider the slip boundary conditions, the assumptions are clearly that described in the Theorem 2.1 where, for simplicity, we assume the main case $\beta = \frac{1}{2}$, namely

$$\sin\theta(x, y, t) \leq c\,|x - y|^{1/2}.$$

In the case of the no-slip boundary condition, the assumptions are the same, except that the initial data u_0 vanishes on the boundary. As already remarked, in the sequel we mostly assume the no-slip boundary condition, except if a different situation is explicitly indicated.

By applying the curl operator to equation (1) we get the well-known equation

$$\omega_t + (u \cdot \nabla)\,\omega - \nu\,\Delta\omega = (\omega \cdot \nabla)\,u.$$

Scalar multiplication by ω, integration in Ω, and integrations by parts show that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \nu\|\nabla\omega\|_2^2 - \nu\int_\Gamma \frac{\partial\omega}{\partial n}\cdot\omega\,d\Gamma = \int_\Omega (\omega\cdot\nabla)\,u\,\cdot\omega\,dx. \tag{13}$$

Set, for each triad (j, k, l), $j, k, l \in \{1, 2, 3\}$,

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i, j, k) \text{ is an even permutation}\,, \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation}\,, \\ 0 & \text{if two indexes are equal.} \end{cases}$$

These are the components of the totally anti-symmetric Ricci tensor. One has

$$(a \times b)_j = \epsilon_{jkl}\,a_k\,b_l\,, \qquad (\nabla \times v)_j = \epsilon_{jkl}\,\partial_k v_l. \tag{14}$$

The usual convention about summation of repeated indexes is assumed.

Since

$$-\Delta u = \nabla \times (\nabla \times u) - \nabla(\nabla \cdot u)$$

it follows that

$$\begin{cases} -\Delta u = \nabla \times \omega \quad \text{in} \quad \Omega\,, \\ u = 0 \quad \text{on} \quad \Gamma\,, \end{cases} \tag{15}$$

for each t. Let now $G(x, y)$ be the Green's function for the Dirichlet boundary value problem in Ω. Since the boundary Γ is regular, it is a classical result that

$$G(x, y) = \frac{1}{4\pi\,|x - y|} + \gamma(x, y)\,,$$

where $\gamma(x, y)$ is smooth. In particular

$$\left|\frac{\partial^2 G(x, y)}{\partial y_k \partial x_i}\right| \leq \frac{c}{|x - y|^3}. \tag{16}$$

From (15) it follows that

$$u(x) = \int_\Omega G(x,\ y)\,\nabla \times \omega(y)\,dy\,, \tag{17}$$

for $x \in \Omega$.

By considering in equation (17) a single component u_j, by appealing to (14), and by taking into account that $G(x,\ y) = 0$ if $y \in \Gamma$, an integration by parts yields

$$u_j(x) = \int_\Omega G(x, y)\,\epsilon_{jkl}\,\partial_k\,\omega_l(y)\,dy = -\int_\Omega \epsilon_{jkl}\,\frac{\partial\,G(x, y)}{\partial\,y_k}\,\omega_l(y)\,dy.$$

Hence

$$\frac{\partial\,u_j(x)}{\partial\,x_i} = -P.V. \int_\Omega \epsilon_{jkl}\,\frac{\partial^2\,G(x, y)}{\partial\,x_i \partial\,y_k}\,\omega_l(y)\,dy.$$

It readily follows that

$$\begin{aligned} \mathcal{K}(x) := ((\omega \cdot \nabla)\,u \cdot \omega)\,(x) &= \tfrac{\partial\,u_j(x)}{\partial\,x_i}\,\omega_i(x)\omega_j(x) \\ &= -\int_\Omega \epsilon_{jkl}\,\tfrac{\partial^2\,G(x,y)}{\partial\,y_k \partial\,x_i}\,\omega_i(x)\,\omega_j(x)\,\omega_l(y)\,dy.n \end{aligned}$$

Recall that (Eq. (13))

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \nu\,\|\nabla\omega\|_2^2 - \nu \int_\Gamma \frac{\partial\omega}{\partial n}\cdot\omega\, d\Gamma = \int_\Omega \mathcal{K}(x)\, dx.$$

Since $-\epsilon_{jkl}\,\omega_j(x)\,\omega_l(y) = \big(\omega_j(x)\times\omega_l(y)\big)_k$, it follows that

$$\mathcal{K}(x) = P.V. \int_\Omega \frac{\partial^2 G(x,y)}{\partial y_k\,\partial x_i}\,\omega_i(x)\,\big(\omega_j(x)\times\omega_l(y)\big)_k\, dy.$$

By appealing to (16) one shows that

$$|\,\mathcal{K}(x)\,| \le \int_\Omega \frac{c}{|x-y|^3}\,|\omega(x)|^2\,|\omega(y)|\,\sin\theta(x,y,t)\,dy.$$

Furthermore, by (6), one has

$$|\mathcal{K}(x)| \le c\,|\omega(x)|^2 \int_\Omega |\omega(y)|\,\frac{dy}{|x-y|^{3-\frac{1}{2}}} = c\,|\omega(x)|^2\,I(x)\,, \tag{18}$$

where the Riesz potential $I(x)$ satisfies

$$\|I\|_3 \le c\,\|\omega\|_2. \tag{19}$$

Recall that, in general, if

$$I(x) = \int_\Omega |\omega(y)|\,\frac{dy}{|x-y|^{n-\beta}}\,,$$

where $0<\beta<n$ and $\omega\in L^r(\Omega)$, for some $1<r<n$, then

$$\|I\|_q \le c\,\|\omega\|_r\,,$$

where $1/q = 1/r - \beta/n$. See [30].

From (18) and (19), one gets

$$\begin{aligned}\int_\Omega |\mathcal{K}(x)|\,dx &\le c\int_\Omega |\omega(x)|^2\,I(x)\,dx \le c\,\|I\|_3\,\|\omega\|_2\,\|\omega\|_6\\ &\le c\,\|\omega\|_2^2\,(\,\|\nabla\omega\|_2 + \|\omega\|_2\,) \le C_\epsilon\,\|\omega\|_2^4 + c\,\|\omega\|_2^3 + \epsilon\,\|\nabla\omega\|_2^2.\end{aligned} \tag{20}$$

So,

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \nu\,\|\nabla\omega\|_2^2 - \nu \int\limits_{\Gamma} \frac{\partial\omega}{\partial n}\cdot\omega\, d\Gamma = \int\limits_{\Omega} \mathcal{K}(x)\, dx$$
$$\leq C\,(\,\|\omega\|_2^2 + \|\omega\|_2\,)\cdot\|\omega\|_2^2 + \epsilon\,\|\nabla\,\omega\|_2^2. \tag{21}$$

Unfortunately, under the slip boundary condition, we are not able to control the boundary integral

$$\nu \int\limits_{\Gamma} \frac{\partial\omega}{\partial n}\cdot\omega\, d\Gamma. \tag{22}$$

On the contrary, under the *slip boundary condition* in the half-space case, the above boundary integral vanishes, as immediately follows from Eq. (11). Further, estimates on $\mathcal{K}$ are the same also in the slip case. So, in this last case, we drop the boundary integral in Eq. (21). Further, since weak solutions satisfy

$$\|\omega(t)\|_2^2 \in L^1(0,\,T)\,,$$

it follows from Eq. (21), together with Gronwall's lemma, that $\omega \in L^\infty(0,\,T;\,L^2(\Omega)) \cap L^2(0,\,T;\,H^1(\Omega)\,)$. Hence

$$u \in L^\infty(0,\,T;\,H^1(\Omega)\,)\cap L^2(0,\,T;\,H^2(\Omega)\,).$$

This shows the regularity of the solution under the slip boundary condition, since we can prove that Eq. (21) also holds under this boundary assumption. See [3], equation (70). However, the proof of (21) becomes quite more involved.

Appendix

As remarked at the end of the introduction, we present here some reflections upon the global structure and significance of our results, taken as a whole.

As the reader has verified, the statements presented above are split into two families of sufficient conditions for regularity, namely, $\beta \geq \frac{1}{2}$ and $\beta \leq \frac{1}{2}$. In Theorem 1.2, the advantage of assuming $\beta > \frac{1}{2}$ is counterbalanced by replacing in Eq. (5) the constant c by a function $g \in L^a(0,T;L^b(\Omega))$. On the other hand, in Theorem 1.3, we mitigate the penalizing situation $\beta < \frac{1}{2}$ by assuming (8). This situation may give the wrong idea that the two families of results are relatively independent. On the contrary, the above formal separation is not substantial. In fact, the two families glue perfectly at the intersection point $\beta = \frac{1}{2}$ since the conclusion (namely, "condition (6) implies regularity") is the same in both cases. On the other hand, a step by step analysis of the proofs given for each of the two above theorems, shows that, inside each class, the results have the same "strength", independently of the value of the parameter β.

Since the two families "glue" at point $\frac{1}{2}$, we conclude that we have just one family of strictly connected results, all having an equivalent "strength".

We may also show the "equal strength" of the above sufficient conditions for regularity by appealing to scaling techniques. Let us illustrate this possibility by showing that the sufficient conditions for regularity $\sin\theta(x,y,t) \le g(t,x)|x-y|^\beta$, as β goes from $\frac{1}{2}$ to 1, enjoy the same strength.

Assume that $((u(x,t),\ p(x,t))$ is a solution to the Navier–Stokes equations in $(0,+\infty)\times R^3$. Then

$$((u_\lambda(x,t),\ p_\lambda(x,t)) \equiv ((\lambda\, u(\lambda x,\lambda^2 t),\ \lambda^2 p(\lambda x,\lambda^2 t))$$

is a solution in the same domain. In particular

$$\omega_\lambda(x,t) \equiv \operatorname{curl} u_\lambda(x,t) = \lambda^2\,\omega(\lambda x,\lambda^2 t).$$

Set

$$\theta_\lambda(x,y,t) \stackrel{def}{=} \angle(\omega_\lambda(x,t),\omega_\lambda(y,t)).$$

Then, by appealing to

$$\sin\theta(x,y,t) = \frac{|\,\omega(x,t)\times\omega(y,t)\,|}{|\omega(x,t)|\,|\omega(y,t)|}\,,$$

it follows that

$$\sin\theta_\lambda(x,y,t) = \sin\theta(\lambda x,\lambda y,\lambda^2 t).$$

Assume now that the solution $u(x,t)$ satisfies

$$\sin\theta(x,y,t) \le g(t,x)|x-y|^\beta\,,$$

for some $\beta\in[\frac{1}{2},1]$, where $g\in L^a(0,+\infty;L^b(R^3))$, and the exponents are defined by Eq. (4). It follows that

$$\sin\theta_\lambda(x,y,t) \le g_\lambda(x,t)|x-y|^\beta\,,$$

where the function g_λ is given by

$$g_\lambda(x,t) \stackrel{def}{=} \lambda^\beta\, g(\lambda x,\lambda^2 t).$$

It follows that

$$\|\,g_\lambda\,\|_{L^a(0,+\infty;\,L^b(R^3))} = \lambda^{\frac{1}{2}}\,\|\,g\,\|_{L^a(0,+\infty;\,L^b(R^3))},$$

for all $\beta \in [\frac{1}{2}, 1]$. The equivalence of the "strength" of the different sufficient conditions for regularity follows from the independence of the exponent $\frac{1}{2}$ with respect to β. The reader may verify that weaker (resp. stronger) sufficient conditions for regularity lead to larger (resp. smaller) exponents.

Finally we show that the above common strength is at the same level as a classical "Prodi-Serrin" integrability conditions for regularity. In fact, for $\beta = 0$, condition (7) is superfluous, since it holds automatically. Furthermore, condition (8) simply reads $\omega \in L^2(0, T; L^3(\Omega))$. This means $u \in L^2(0, T; H^{1,3}(\Omega))$, which is a class of regularity, see [1]. This class is formally equivalent, in an obvious sense, to the classical "Prodi-Serrin" condition $u \in L^2(0, T; L^\infty(\Omega))$.

The above argument lead us to call all the above family of $\beta-$dependent results, *sharp results*. Note that, in Theorem 1.2, the weak regularity allowed by (4) to the coefficients $g(t, x)$ is fundamental to obtain sharp results. This is the reason why proving the "minimal regularity" for the coefficients $g(t, x)$, is taken here into considerable attention. A similar remark applies in relation to (9) and Theorem 1.3.

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