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Research Article

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On nonlinear potential theory, and regular boundary points, for the p-Laplacian in N space variables

Abstract: We turn back to some pioneering results concerning, in particular, nonlinear potential theory and non-homogeneous boundary value problems for the so-called p-Laplace operator. Unfortunately these results, obtained at the very beginning of the seventies, were kept in the shade. We believe that our proofs are still of interest, in particular due to their extreme simplicity. Moreover, some contributions seem to improve the results quoted in the current literature.

Keywords: p-Laplacian, non-homogeneous Dirichlet problem, barriers, capacitary potentials, regular boundary points

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1 Introduction

At the very beginning of the seventies we proved a set of results concerning nonlinear potential theory related to the so-called p-Laplace operator. Following [3], here we use the symbol "t" instead of the nowadays more common "p" to denote the leading integrability exponent (see (2.4)). In the 1972 paper [3] (see also [2]) we considered, in a nonlinear setting, notions such as barriers, order preservation, capacitary potentials, regular boundary points, and so on. This contribution seems almost forgotten in the subsequent literature. However we believe that its topicality and interest still remain, or have even grown. In fact, the basic ideas on which the theory is founded was emphasized by the original simplicity of the broad lines. In Part I, we turn back to the results published in reference [3]. We keep the presentation as close as possible to the original paper. However, addition of suitable remarks, together with some changes in notation, may help the reader. By the way, we warn the reader that [3] is full of small misprints, luckily very easy to single out and correct. In Part II we turn back to an unpublished proof of a result stated in reference [3] (Theorem 7.3 below), and to a related result proved in reference [5] (Theorem 7.4 below), both concerning regularity of boundary points for p-Laplacian equations. The contribution of [5] to this last problem was to prove Hölder continuity of the solutions to the obstacle problem in the lower dimension $N-1$. Below we merely prove the continuity of the above solutions, since this weaker property is sufficient here.

The main object of this work is the Dirichlet boundary value problem (2.7), whose prototype is the following problem:

$$\begin{cases} \operatorname{div}(|\nabla u|^{t-2}\nabla u) = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

For $t = 2$ we get the classical Laplace equation. It is worth noting that the theory developed in references [3] and [5] could have been extended to similar, but more general, equations. However, at that time, we were only interested in the basic picture. Regular boundary points for the above Dirichlet problem is here the core subject. In equation (2.7), arbitrary continuous boundary data ϕ are allowed. This leads us to consider two distinct notions of solutions, generalized and variational.

We recall that a boundary point y is said to be regular if to each continuous boundary data ϕ the corresponding solution is continuous in y. Theorem 2.10 below (called Theorem A in reference [3]) states that a point y is regular if and only if there is at y a system of nonlinear barriers, see Definition 2.8. By appealing to this last result, we prove Theorem 2.13 (called Theorem B in reference [3]), which establishes that a point $y \in \partial\Omega$ is regular if and only if the t-capacitary potentials of the sets E_ρ satisfy (2.24), for each positive real m, and each sufficiently small radius ρ, where $E_\rho = (\complement\Omega)(y, \rho)$ denotes the complementary set of Ω with respect to the closed ball $\overline{I(y, \rho)}$.

In Part II, by appealing to Theorem 2.13, we establish two explicit, geometrical, sufficient condition for regularity. Let us briefly illustrate these results.

Denote by

$$\sigma_0(\rho) = \frac{|E_\rho|}{|I(y, \rho)|} \tag{1.2}$$

the density (with respect to the N-dimensional Lebesgue measure) of E_ρ with respect to the sphere $I(y, \rho)$. In Theorem 7.3 it is stated that there is a positive constant Λ_0 such that if

$$[\sigma_0(\rho)]^{\frac{t}{t-1}} \geq \Lambda_0(\log\log\rho^{-1})^{-1}, \tag{1.3}$$

for small positive values of ρ, then the boundary point y is regular. Note that

$$\lim_{\rho\to 0} \sigma_0(\rho) = 0 \tag{1.4}$$

is included, so the above condition is stronger than the usual N-dimensional, external, cone property, and similar notions. This result was already stated in the introduction of reference [3] (due to a misprint, the second exponent -1 in (1.3) was overlooked). At that time we did not publish the proof, since we had used similar ideas in reference [5], where it was proved (still, appealing to Theorem 2.13) that a boundary point y is regular if an $(N-1)$-dimensional external cone property is satisfied at the point y (a Lipschitz image of such a cone being sufficient). See Theorem 7.4 below.

In fact, Theorems 7.3 and 7.4 are corollaries of the same result, Theorem 7.2, where it is proved that the necessary and sufficient condition for regularity stated in Theorem 2.13 holds under the assumption (7.3). The proofs of Theorems 7.2, 7.3 and 7.4 are shown in Part II below.

Our proofs do not require knowledge of particularly specialized results. They appeal, in particular, to a suitable extension of De Giorgi's truncation method to nonlinear variational inequalities with obstacles, following in particular reference [2] (see also [1]). De Giorgi's truncation method was also used in reference [11] to obtain the following sufficient condition for regularity:

$$\limsup_{\rho\to 0} \operatorname{cap}(E_\rho)\rho^{t-N} > 0, \tag{1.5}$$

where $\operatorname{cap} \equiv \operatorname{cap}_t$ denotes (here and in the sequel) the capacity of order t. Since $|E|^{\frac{N-t}{N}} \leq C \operatorname{cap}_t E$, condition (1.5) leads to

$$\limsup_{\rho\to 0} \frac{|E_\rho|}{|I(y, \rho)|} > 0, \tag{1.6}$$

which, basically, is equivalent to the N-dimensional external cone property, as well as the corkscrew condition, stated in [13, Theorem 6.31]. This treatise furnishes a wide-ranging excursion into the above and related results. See, in particular, Section 9.

Readers interested in a quick overlook on the main results should go directly to Definition 2.8 and Theorem 2.10, to Definitions 2.11 and 2.12, and Theorem 2.13, and, in Part II, to Theorems 7.3 and 7.4.

Part I

2 Some definitions and main results

We are concerned with the differential operator

$$\mathcal{L}u =: \operatorname{div} A(\nabla u), \tag{2.1}$$

where $A(p)$ denotes a continuous map from $\mathbb{R}^N$ into itself, u is a real function defined on an open subset of $\mathbb{R}^N$, and ∇u is its gradient. We assume the following conditions on $A(p)$:

$$A(0) = 0, \tag{2.2}$$
$$\big(A(p) - A(q)\big) \cdot (p - q) > 0 \qquad \text{if } p \neq q, \tag{2.3}$$
$$A(p) \cdot p \geq a|p|^t \qquad \text{if } |p| \geq p_0, \tag{2.4}$$
$$|A(p)| \leq a^{-1}|p|^{t-1} \qquad \text{if } |p| \geq p_0, \tag{2.5}$$

where $a > 0$, $p_0 \geq 0$, and $t > 0$ are constants. Further, $|x|$ and $x \cdot y$ denote, respectively, the norm and the scalar product in $\mathbb{R}^N$. Note that the above assumptions imply $A(p) \cdot p > 0$, for all $p \in \mathbb{R}^N$.

In the following, Ω is an open bounded subset of $\mathbb{R}^N$, with boundary denoted by $\partial\Omega$. We define $H^{1,t}(\Omega)$ as the completion of $C^1(\overline{\Omega})$ (or equivalently, $\mathrm{Lip}(\overline{\Omega})$) with respect to the norm $\|v\|_{1,t} = \|v\|_t + \|\nabla v\|_t$. Here $C^1(\overline{\Omega})$ is the set of functions which belong to $C^0(\overline{\Omega})$ and have continuous first order partial derivatives in Ω, which can be extended continuously to $\overline{\Omega}$. Furthermore, $H_0^{1,t}(\Omega)$ denotes the closure in $H^{1,t}(\Omega)$ of $C_0^1(\overline{\Omega})$, the set of the $C^1(\overline{\Omega})$ functions with compact support in Ω. See, for instance [16]. Furthermore, $H_{\mathrm{loc}}^{1,t}(\Omega)$ denotes the set consisting of functions defined in Ω whose restriction to any $\Omega' \subset\subset \Omega$ belongs to $H^{1,t}(\Omega')$.

We recall here the following property. Let $\phi(t)$ be a real, Lipschitz continuous function of the real variable t, with, at most, a finite number of points of non-differentiability. Further, let $v \in H^{1,t}(\Omega)$. Then $\phi(v(x)) \in H^{1,t}(\Omega)$, moreover $\partial_i\phi(v(x)) = \phi'(v(x))\partial_i v(x)$ a.e. in Ω. In particular

$$\partial_i \max\{v(x), k\} = \begin{cases} \partial_i v(x) & \text{if } v(x) \geq k, \\ 0 & \text{if } v(x) \leq k, \end{cases} \tag{2.6}$$

a.e. in Ω.

For convenience, we set

$$\mathbb{V} = \mathbb{V}(\Omega) = H^{1,t}(\Omega), \quad \mathbb{V}_0 = \mathbb{V}_0(\Omega) = H_0^{1,t}(\Omega),$$

and so on.

In the sequel we are interested in the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ \quad u = \phi & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

where $\mathcal{L}u$ is defined by (2.1), and $\phi \in C^0(\partial\Omega)$. In the sequel we show that to each $\phi \in C^0(\partial\Omega)$ there corresponds a unique solution $u \in H_{\mathrm{loc}}^{1,t}(\Omega) \cap C^0(\Omega)$ to problem (2.7), see Theorem 2.4. This solution will be called generalized solution.

Since $A(p)$ may be merely continuous, local solutions of the problem $\mathcal{L}u = 0$ in Ω are understood in the following, well-known, weak sense. One considers the form

$$\mathfrak{a}(v, \psi) := \int_\Omega A(\nabla v) \cdot \nabla\psi \, dx, \tag{2.8}$$

defined on $\mathbb{V} \times \mathbb{V}$, or on $H_{\mathrm{loc}}^{1,t}(\Omega) \times D(\Omega)$, and give the following definition.

Definition 2.1. We say that a function u is a weak solution in Ω of the problem

$$\mathcal{L}u \equiv \operatorname{div} A(\nabla u) = 0 \tag{2.9}$$

if u belongs to $H_{\mathrm{loc}}^{1,t}(\Omega)$ and satisfies the condition

$$\mathfrak{a}(u, \psi) = 0 \quad \text{for all } \psi \in \mathcal{D}(\Omega). \tag{2.10}$$

Note that it immediately follows that (2.10) holds for all $\psi \in H^{1,t}(\Omega)$ with compact support in Ω.

The above definition does not take into account boundary values. The definition of a generalized solution to the boundary value problem (2.7), where $\phi \in C^0(\partial\Omega)$, is given below, see Definition 2.3. Generalized solutions to the boundary value problem are defined as limits of suitable sequences of variational solutions. In reference [3] we have used in both cases the term "solution". However, for clarity, we decided to use in these notes the two notions, "variational" and "generalized", to denote related but distinct concepts.

Next, we recall the definition of a variational solution. Let $\phi \in \mathbb{V}(\Omega)$. We set

$$\mathbb{V}_\phi(\Omega) = \{v \in \mathbb{V}(\Omega) : v - \phi \in \mathbb{V}_0(\Omega)\}. \tag{2.11}$$

Properties (i) to (iv) below are easily shown.

(i) $\mathfrak{a}(v, v-u) - \mathfrak{a}(u, v-u) \geq 0$ for all pairs $u, v \in \mathbb{V}(\Omega)$ (monotonicity).

(ii) $\mathfrak{a}(u + tv, w)$ is a continuous function of the real variable t for all triads $u, v, w \in \mathbb{V}(\Omega)$ (hemicontinuity).

(iii) $\mathfrak{a}(v, v-u) - \mathfrak{a}(u, v-u) = 0$ implies $\nabla u = \nabla v$ in Ω. Moreover, if $u - v \in \mathbb{V}_0(\Omega)$, then $u = v$.

(iv) One has (coercivity)

$$\lim_{\|v\|_{1,t}\to\infty} \frac{\mathfrak{a}(v,v)}{\|v\|_{1,t}} = +\infty, \tag{2.12}$$

where $v \in \mathbb{V}_\phi(\Omega)$.

Existence and uniqueness of the solution to the following variational problem are well known:

$$u_1 \in \mathbb{V}_\phi(\Omega), \quad \mathfrak{a}(u_1, v) = 0 \quad \text{for all } v \in \mathbb{V}_0(\Omega). \tag{2.13}$$

Clearly, these solutions are weak solutions of (2.9) in Ω. All this was already classical in the sixties.

Definition 2.2. The function $u = u_1$ in (2.13) is, by definition, the *variational solution* to the Dirichlet problem (2.7) when the boundary data is defined by means of an element $\phi \in H^{1,t}(\Omega)$. In this case, $u = \phi$ on $\partial\Omega$ means that $u - \phi \in H_0^{1,t}(\Omega)$.

In the sequel, our first step is to extend the notion of solution to all continuous boundary data ϕ. This will be done as in reference [3]. Given $\phi \in C^0(\partial\Omega)$, we consider an arbitrary sequence of functions $\phi_n \in C^1(\overline{\Omega})$, which converge uniformly to ϕ on $\partial\Omega$, and we consider the sequence $u_n(x)$ consisting of the variational solutions to the Dirichlet problem (2.7), with boundary data ϕ_n. Then we prove (Theorem 4.4) that the sequence $u_n(x)$ converges uniformly in Ω to a function $u(x) \in H^{1,t}_{\mathrm{loc}}(\Omega) \cap C^0(\Omega)$. Moreover, we show that $u(x)$ is a weak solution in Ω of problem (2.9), and also that it does not depend on the particular sequence ϕ_n. So, to each continuous boundary data ϕ there corresponds a unique element $u(x) \in H^{1,t}_{\mathrm{loc}}(\Omega) \cap C^0(\Omega)$, obtained by the above procedure. The above argument leads to the following, natural, definition.

Definition 2.3. Let $\phi \in C^0(\partial\Omega)$ be given. By definition, the above, unique, element $u(x) \in H^{1,t}_{\mathrm{loc}}(\Omega) \cap C^0(\Omega)$ is the generalized solution to the Dirichlet problem (2.7) with the continuous boundary data ϕ.

We anticipate the following result.

Theorem 2.4. *To each boundary value $\phi \in C^0(\partial\Omega)$ there corresponds a unique generalized solution to the Dirichlet problem* (2.7).

It is worth noting that the auxiliary variational solutions $u_n(x)$ used above are not necessarily continuous up to the boundary, even though $\phi_n \in C^1(\overline{\Omega})$. Even more, this negative situation holds for generalized solutions. Hence, a crucial problem is to study the possible continuity up to a boundary point y of the solutions to the Dirichlet problem. In this direction we give the following definition.

Definition 2.5. We say that a point $y \in \partial\Omega$ is regular with respect to Ω and $\mathcal{L}$ if given an arbitrary data $\phi \in C^0(\partial\Omega)$, the corresponding generalized solution u of Dirichlet problem (2.7) satisfies the condition

$$\lim_{\substack{x\to y\\ x\in\Omega}} u(x) = \phi(y). \tag{2.14}$$

As proved in Theorem 6.2 below, the notion of regular point has a local character.

We remark that in the above definition, as in the following, we do not assume (in any sense) that the continuous boundary data ϕ is the trace on $\partial\Omega$ of an element of $H^{1,t}(\Omega)$.

For the Laplace operator, $A(p) = p$, regular points have been characterized by Wiener; see [23, 24] and Frostman [10]. For linear operators with discontinuous coefficients,

$$A_i(p) = \sum_j a_{i,j}(x) p_j,$$

where

$$\sum_j a_{i,j}(x)\xi_i\xi_j \geq \nu|\xi|^2,$$

$\nu > 0$, and $a_{i,j} \in L^\infty(\Omega)$, $i, j = 1, \dots, N$, such a characterization was given by Littman, Stampacchia, and Weinberger in [16].

The following definitions are crucial to the theory (see [21, Definition 1.1 and remarks]).

Definition 2.6. Let Σ be an open, bounded, set and $E \subset \overline{\Sigma}$ be a measurable set. We say that $v \in H^{1,t}(\Sigma)$ satisfies the inequality $v \geq 0$ on E in the $H^{1,t}(\Sigma)$ sense if there is a sequence $v_n \in C^1(\overline{\Sigma})$ convergent to v in $H^{1,t}(\Sigma)$ and satisfying $v_n \geq 0$ on E. Similarly, we define $v \leq 0$ on E, in the $H^{1,t}(\Sigma)$ sense. Further, $v = 0$ on E if, simultaneously, $v \geq 0$ and $v \leq 0$. Finally, $v \geq w$ on E, in the $H^{1,t}(\Sigma)$ sense if $v - w \geq 0$ on E, and so on.

Furthermore, we denote respectively by $\sup_E v$ and $\inf_E v$ the upper bound and the lower bound of v on E in the $H^{1,t}(\Sigma)$ sense. Essential upper bounds and lower bounds (i.e., up to sets of zero Lebesgue measure) are denoted by the symbols $\mathrm{Sup}_E\, v$ and $\mathrm{Inf}_E\, v$, respectively.

It is worth noting that the above definition is meaningless if the $(N-2)$-dimensional measure of set E vanishes. This claim is in general not true if we replace $N-2$ by $N-1$. Let us consider the following specific example, related to our results. Assume that E is an $(N-1)$-dimensional truncated cone (see for instance (7.6)) contained in a given sphere Σ. Since the elements $v \in H^{1,t}(\Sigma)$ do have a trace (for instance, in the usual Sobolev's spaces sense) on the surface E, it follows that if $v \in H^{1,t}(\Sigma) \cap C^0(\Sigma)$ satisfies $v \geq m > 0$ on E, in the $H^{1,t}(\Omega)$ sense, then $v \geq m$ pointwisely on E. However, if E is an $(N-2)$-dimensional cone and $t < N$, the result is not true in general. For instance the continuous, constant, function $v = 0$ in Σ satisfies $v \geq m > 0$ on E, in the sense of Definition 2.6.

To illustrate the results obtained in this work, we need additional definitions and results. Given a point $y \in R^N$ and $\rho > 0$, we denote by $I(y, \rho)$ the open sphere with center in y and radius ρ. If $B \subset \mathbb{R}^N$, we set $B(y, \rho) = B \cap I(y, \rho)$. By $\complement B$ and $\overline{B}$ we denote the complementary set and the closure of B in R^N, respectively.

As in [3], we give the following definitions.

Definition 2.7. We say that $v \in H^{1,t}_{\mathrm{loc}}(\Omega)$ is a *supersolution* (resp., a *subsolution*) in Ω with respect to the operator $\mathcal{L}$ if

$$\mathfrak{a}(v, \psi) \geq 0 \quad \text{for all } \psi \in \mathcal{D}(\Omega), \quad \psi \geq 0 \quad (\text{resp.} \psi \leq 0). \tag{2.15}$$

Obviously, if $v \in \mathbb{V} = H^{1,t}(\Omega)$, then $\mathcal{D}(\Omega)$ may be replaced by $\mathbb{V}_0$. Formally, a supersolution satisfies $\mathcal{L}v \leq 0$ in Ω.

The following definition generalizes Perron's notion of barrier (see for instance Perron [18] and Courant–Hilbert [7, pp. 306–312 and 341]).

Definition 2.8. We say that there is a system of barriers at a point $y \in \partial\Omega$ with respect to $\mathcal{L}$ if, given two positive arbitrary reals ρ and m, there exist a supersolution $V \geq 0$ and a subsolution $U \leq 0$ which belong to $\mathbb{V} \cap C^0(\Omega)$ and satisfy the following conditions:

(j) $V \geq m$ and $U \leq -m$ on $(\partial\Omega) \cap \complement I(y, \rho)$,

(jj) $\lim_{x\to y} V(x) = \lim_{x\to y} U(x) = 0$.

In Definition 2.8, and in the sequel, inequalities like $V \geq m$, $U \leq -m$, and so on, are to be intended in the sense introduced in Definition 2.6. Note that the above definition does not change by restriction of the range of the radius ρ to values smaller than some positive $\rho_0(y)$.

Under suitable symmetry conditions, Definition 2.8 may be simplified as follows.

Remark 2.9. Define

$$B(p) = -A(-p). \tag{2.16}$$

The continuous function $B(p)$ inherits the properties (2.2)–(2.5). Furthermore, consider the operator

$$\overline{\mathcal{L}}w = \operatorname{div} B(\nabla w).$$

The transformation $w \to -w$ maps solutions of (2.21), relative to one of the operators $\mathcal{L}$ or $\overline{\mathcal{L}}$, onto the solutions of (2.22) relative to the other operator, and reciprocally. Further, the same transformation maps supersolutions, solutions, and subsolutions, relative to one of the operators onto, respectively, subsolutions, solutions, and supersolutions, relative to the other operator.

In particular, if

$$A(-p) = -A(p), \tag{2.17}$$

the transformation $w \to -w$ maps supersolutions onto subsolutions, and reciprocally. In this case, it is sufficient in Definition 2.8 to consider upper-solutions V.

Finally, if the function $A(p)$ is positively homogeneous,

$$A(sp) = s^{t-1}A(p) \quad \text{for all } s > 0, \tag{2.18}$$

it is sufficient in Definition 2.8 to consider the value $m = 1$.

Main examples: $A(p) = (1 + |p|^2)^{\frac{(t-2)}{2}} p$ satisfies (2.17), and $A(p) = |p|^{t-2}p$ satisfies (2.17)–(2.18). The differential equations associate to this functions are the Euler equations to the extremals of the integrals $\int (1 + |\nabla u|^2)^{\frac{t}{2}}\, dx$ and $\int |\nabla u|^t\, dx$, respectively.

In Section 5 we prove the following result (see [3, Theorem A]):

Theorem 2.10. *A point y is regular if and only if there is at y a system of barriers.*

As in reference [3], the symbols Ω and Σ denote suitable open sets. However, in this rewriting, we make the reading easier by a better use of the above symbols.

Let Σ be an open bounded set, $E \subset \Sigma$ be a closed set, and m be a positive constant (the fact that Σ is assumed to be a sphere is not necessary here). We introduce the following convex, closed, subsets of $\mathbb{V}_0(\Sigma)$:

$$\mathbb{K}_m(\Sigma) = \{v \in \mathbb{V}_0(\Sigma) : v \geq m \text{ on } E\}, \tag{2.19}$$

and

$$\mathbb{K}_{-m}(\Sigma) = -\mathbb{K}_m(\Sigma) = \{v \in \mathbb{V}_0(\Sigma) : v \leq -m \text{ on } E\}. \tag{2.20}$$

The inequalities are in the $H^{1,t}(\Sigma)$ sense.

Obviously, properties (i)–(iii) hold with Ω replaced by Σ. Moreover, as easily shown, the coercivity property (iv) holds by replacing $v \in \mathbb{V}_\phi(\Omega)$ by $v \in \mathbb{K}_m(\Sigma)$, or by $v \in \mathbb{K}_{-m}(\Sigma)$. Hence, from properties (i)–(iv), together with well-known general theorems (see Hartman–Stampacchia [12] and J.-L. Lions [15]), existence and uniqueness of solutions to the following two problems follow:

$$u_2 \in \mathbb{K}_m(\Sigma), \quad \mathfrak{a}(u_2, v - u_2) \geq 0 \quad \text{for all } v \in \mathbb{K}_m(\Sigma), \tag{2.21}$$

$$u_3 \in \mathbb{K}_{-m}(\Sigma), \quad \mathfrak{a}(u_3, v - u_3) \geq 0 \quad \text{for all } v \in \mathbb{K}_{-m}(\Sigma). \tag{2.22}$$

Next we introduce the t-capacitary potentials. The following definition is related to the notion of capacity used by Serrin in [19].

Definition 2.11. Let Σ be an open sphere, $E \subset \Sigma$ be a closed set, and m be a positive real. The solutions to problems (2.21) and (2.22) are called t-capacitary potentials of the set E with respect to the nonlinear operator $\mathcal{L}$, the real m and the sphere Σ. Since t is fixed, we drop the label t.

In Definition 2.11, the dependence on the particular fixed sphere Σ is without significance. In particular, the numerical values of the related capacities remain equivalent provided that the distances from the sets E to the boundary $\partial\Sigma$ have a positive, fixed, lower bound. From now on we fix, once and for all, a sphere

$$\Sigma = I(y_0, 2R)$$

such that

$$\Omega \subset I(y_0, R).$$

So

$$\operatorname{dist}(\Omega, \partial\Sigma) \geq R.$$

Further, for each couple y, ρ, where $y \in \partial\Omega$ and $0 < \rho < \frac{R}{2}$, we set

$$E_\rho = (\complement\Omega) \cap \overline{I(y,\rho)}. \tag{2.23}$$

Definition 2.12. We denote by $u_{m,\rho}$ and $u_{-m,\rho}$ the capacitary potentials of the above sets E_ρ relative to the values m and $-m$ respectively.

In Section 6 we prove the following result (see [3, Theorem B]):

Theorem 2.13. *A point $y \in \partial\Omega$ is regular if and only if the capacitary potentials of the sets E_ρ are continuous in y. More precisely, a point $y \in \partial\Omega$ is regular if and only if*

$$\begin{cases} \lim\limits_{x\to y} u_{m,\rho}(x) = m, \\ \lim\limits_{x\to y} u_{-m,\rho}(x) = -m, \end{cases} \tag{2.24}$$

for each couple ρ, m as above (or, equivalently, for a sequence (ρ_n, m_n) such that $(\rho_n, m_n) \to (0, +\infty)$).

From Theorem 2.13, together with the immersion of $H^{1,t}(\Sigma)$ in $C^{0,1-\frac{N}{t}}(\overline{\Sigma})$, one gets the following result.

Corollary 2.14. *Any boundary point is regular with respect to the operator $\mathcal{L}$ if $t > N$.*

3 Maximum principles and related results

In this section we state some results concerning maximum principles, order preservation, and similar notions. Related results may be found, for instance, in [9, 17, 20].

The section is divided into two subsections. The first one concerns variational solutions in Ω to the nonlinear boundary value problem (2.7). The second one concerns solutions to the variational inequalities (2.21) and (2.22), which describe obstacle problems in Σ.

3.1 Variational solutions in Ω

We denote by $|B|$ the Lebesgue measure of a set B. By c, c_0, c_1, etc., we denote positive constants that depend, at most, on t, N, a and p_0. The same symbol may be used to denote different constants of the same type.

One has the following *maximum principle*.

Lemma 3.1. *The (variational) solution $u = u_1$ of problem* (2.13) *satisfies the estimates*

$$\inf_{\partial\Omega} \phi \leq \operatorname*{Inf}_{\Omega} u \leq \operatorname*{Sup}_{\Omega} u \leq \sup_{\partial\Omega} \phi. \tag{3.1}$$

Proof. We prove that $\operatorname{Sup}_\Omega u \leq k$, where $k = \sup_{\partial\Omega} \phi$. For convenience we set

$$A(k) = \{x \in \Omega : u(x) \geq k\}.$$

If $|A(k)| = 0$, the thesis is obvious. Assume that $|A(k)| > 0$, and set $v = \max\{u - k, 0\}$. Since $v \in H_0^{1,t}(\Omega)$, it follows from (2.13) that

$$\int_{A(k)} A(\nabla u) \cdot \nabla u \, dx = 0.$$

This equation, together with (2.2) and (2.3), shows that $\nabla u = 0$ on $A(k)$, so $u = k$ on $A(k)$. This proves our thesis. A similar argument proves the first inequality (3.1). □

Lemma 3.2 (Order preserving). *Let w be a subsolution, z a be supersolution, and assume that $w \leq z$ on $\partial\Omega$. Then $w(x) \leq z(x)$ a.e. in Ω.*

Proof. Set $\eta = \min\{0, z - w\}$. It follows that $\eta \in H_0^{1,t}(\Omega)$, moreover $\eta(x) \leq 0$. By taking into account Definition 2.7, we may write

$$\int_\Omega A(\nabla w) \cdot \nabla \eta \, dx = \int_{\{w \geq z\}} A(\nabla w) \cdot \nabla (z - w) \, dx \geq 0 \tag{3.2}$$

and

$$\int_{\{w \geq z\}} A(\nabla z) \cdot \nabla (z - w) \, dx \geq 0, \tag{3.3}$$

where $\{w \geq z\} = \{x \in \Omega : w(x) \geq z(x)\}$. From (3.2) and (3.3) it follows that

$$\int_{\{w \geq z\}} (A(\nabla z) - A(\nabla w)) \cdot (\nabla z - \nabla w) \, dx \leq 0. \tag{3.4}$$

This inequality together with (2.17) implies $\nabla(w - z) = 0$ on $\{w - z \geq 0\}$. By appealing to the hypothesis $w - z \leq 0$ on $\partial\Omega$, the thesis follows. □

Corollary 3.3. *If u and v are two (variational) solutions which belong respectively to $\mathbb{V}_\phi$ and $\mathbb{V}_\psi$, then*

$$\operatorname*{Sup}_\Omega |u - v| \leq \sup_{\partial\Omega} |\phi - \psi|. \tag{3.5}$$

Proof. Set $\eta = \sup_{\partial\Omega} |\phi - \psi|$. The function $w = v + \eta$ is a variational solution in $H^{1,t}_{\psi+\eta}(\Omega)$, moreover $u \leq w$ on $\partial\Omega$. By Lemma 3.2 it follows that $u \leq w = v + \eta$ a.e. in Ω, that is, $u - v \leq \eta$ a.e. in Ω. Similarly, one proves that $v - u \leq \eta$ a.e. in Ω. These two relations yield the thesis. □

3.2 Variational inequalities in Σ

In this subsection Σ, E and m are as in Definition 2.11.

Lemma 3.4. *Let $u = u_2$ be the solution of problem (2.21). Then $u(x) \leq m$ a.e. in Σ. In particular, $u = m$ on E.*

Analogously, the solution $u = u_3$ of (2.22) satisfies the inequality $u(x) \geq -m$ a.e. in Σ. In particular, $u = -m$ on E.

Proof. Let $u = u_2$, and set $v = \min\{u, m\}$. Since $v \in \mathbb{K}_m(\Sigma)$, from (2.21) we get

$$\int_\Sigma A(\nabla u) \cdot \nabla (v - u) \, dx \geq 0,$$

that is,

$$\int_{B_m} A(\nabla u) \cdot \nabla u \, dx \leq 0. \tag{3.6}$$

From (3.6), (2.2), and (2.3) it follows that $\nabla u = 0$ a.e. on the set $\{x \in \Sigma : u(x) \geq m\}$. From this last property, since $u \in \mathbb{K}_m(\Sigma)$, it readily follows that $u = m$ on E.

The second part of the lemma may be proved in a similar way, or as a consequence of the first part, together with Remark 2.9. □

Lemma 3.5. *The solution $u = u_2$ of problem (2.21) solves in $\Sigma - E$ the problem*

$$\int_{\Sigma - E} A(\nabla u) \cdot \nabla v \, dx = 0 \quad \textit{for all } v \in H_0^{1,t}(\Sigma - E). \tag{3.7}$$

Moreover, u_2 is a super-solution in Σ. Similarly, the solution $u = u_3$ of (2.22) solves in $\Sigma - E$ problem (3.7) and is a sub-solution in Σ.

Proof. Equation (2.21) may be written in the form

$$\int_{\Sigma} A(\nabla u) \cdot \nabla(w - u)\, dx \geq 0 \quad \text{for all } w \in \mathbb{K}_m(\Sigma). \tag{3.8}$$

Given $v \in H_0^{1,t}(\Sigma - E)$, denote by $\bar{v}$ the function equal to v in $\Sigma - E$, and vanishing on E. By the construction, the functions $u + \bar{v}$ and $u - \bar{v}$ belong to $\mathbb{K}_m(\Sigma)$. By replacing these functions in equation (3.8) we obtain (3.7).

Furthermore, u is a super-solution. In fact, let $\psi \in C_0^\infty(\Omega)$ be non-negative. Then the function $w = u + \psi$ belongs to $\mathbb{K}_m(\Sigma)$. By using it as test function in equation (3.8), one proves (2.15).

The second part of the lemma may be obtain similarly or, alternatively, by appealing to Remark 2.9. □

4 A convergence result. Proof of the existence Theorem 2.4

In this section we associate to each boundary data $\phi \in C^0(\partial\Omega)$ a weak solution u in Ω of equation (2.9). Recall that, by definition, u is a weak solution of (2.9) in Ω if $u \in H^{1,t}_{\mathrm{loc}}(\Omega)$ satisfies (2.10), namely

$$\int_{\Omega} A(\nabla u) \cdot \nabla\psi\, dx = 0 \quad \text{for all } \psi \in \mathcal{D}(\Omega).$$

As already remarked, it immediately follows that (2.10) holds for all $\psi \in H^{1,t}(\Omega)$ with compact support in Ω.

Remark 4.1. Every $L^\infty(\Omega)$ solution to equation (2.10) necessarily belongs to $C^0(\Omega)$.

In fact, the above solution is locally Hölder continuous in Ω, see Ladyzhenskaya–Ural'tseva [14]. Actually, continuity may be proved by appealing to a simplification of the argument used in Part II below.

Lemma 4.2. *A family of solutions to equation* (2.10) *equi-bounded in* $L^\infty(\Omega)$ *is necessarily equi-bounded in* $H^{1,t}(\Omega')$ *for each* $\Omega' \subset\subset \Omega$.

Proof. From the properties of $A(p)$ it immediately follows that

$$\begin{cases} A(p) \cdot p \geq a|p|^t - a p_0^t, \\ \quad |A(p)| \leq a^{-1}|p|^{t-1} + d_0, \end{cases} \tag{4.1}$$

where d_0 is a non-negative constant. Let $k > 0$ and consider the family $\mathcal{F}$ consisting of the solutions to (2.10) for which $\mathrm{Sup}_\Omega |u(x)| \leq k$. Equi-boundedness of $\|u\|_{t,\Omega}$ is obvious. Let us proof the equi-boundedness of $\|\nabla u\|_{t,\Omega}$. Let Λ be an open set such that $\Omega' \subset\subset \Lambda \subset\subset \Omega$, and let ϕ be a regular function, $0 \leq \phi(x) \leq 1$, equal to 1 in Ω' and vanishing on $\Omega - \Lambda$. One easily shows that

$$\int_{\Lambda} A(\nabla u) \cdot (\nabla u)\phi^t\, dx \leq t \int_{\Lambda} |A(\nabla u)||\nabla\phi||u|\phi^{t-1}\, dx. \tag{4.2}$$

By appealing to Hölder's inequality one gets

$$\int_{\Lambda} A(\nabla u) \cdot (\nabla u)\phi^t\, dx \leq C\Big(\int_{\Lambda} |A(\nabla u)|^{\frac{t}{t-1}} \phi^t\, dx\Big)^{\frac{t-1}{t}},$$

where $C = tk\|\nabla\phi\|_{t,\Lambda}$. The last inequality together with (4.1) leads to

$$a\int_{\Lambda} |\nabla u|^t \phi^t\, dx \leq C_0\Big(\int_{\Lambda} |\nabla u|^t \phi^t\, dx\Big)^{\frac{t-1}{t}} + C_1.$$

Since $\frac{t-1}{t} < 1$, it readily follows that the integral on the left hand side of the above inequality is bounded by a constant C_2. So

$$\int_{\Omega'} |\nabla u|^t\, dx \leq \int_{\Lambda} |\nabla u|^t \phi^t\, dx \leq C_2. \qquad \square$$

Lemma 4.3. *Let $\{u_n\}$ be a sequence of solutions to* (2.10)*, equi-bounded in $L^\infty(\Omega)$ and uniformly convergent in Ω to a function $u(x)$. Then $u(x)$ is a solution to* (2.10) .

Proof. Note that $u_n \in C^0(\Omega)$, as follows from Remark 4.1. Lemma 4.2 shows that $u \in H^{1,t}(\Omega')$ for each Ω' as above. Let $u^0 \in H^{1,t}(\Omega')$ be the variational solution in Ω' of the problem $\mathcal{L}u^0 = 0$ in Ω', $u^0 - u \in H_0^{1,t}(\Omega')$. By applying Corollary 3.3 to the functions u^0 and u_n, it follows that

$$\operatorname*{Sup}_{\Omega'} |u_n - u^0| \le \sup_{\partial\Omega'} |u_n - u|. \tag{4.3}$$

Since $u_n(x) \to u(x)$ uniformly in Ω, from (4.3) it follows that $u_n(x) \to u^0(x)$ uniformly in Ω'. So, $u^0(x) = u(x)$ in Ω'. In particular, $\mathcal{L}u = 0$ in Ω'. From the arbitrarity of Ω', the thesis follows (note that local uniform convergence in Ω would be sufficient here). □

The following statement corresponds to [3, Theorem 2.4].

Theorem 4.4. *To each $\phi \in C^0(\partial\Omega)$ there corresponds a (unique) function $u(x)$ such that the following holds:*

Let $\{\phi_n\}$ be an arbitrary sequence of functions in $C^1(\overline{\Omega})$ uniformly convergent to ϕ on $\partial\Omega$ (it is well know that these sequences exist). Further, denote by $u_n(x)$ the variational solutions to the problem $\mathcal{L}u_n = 0$, $u_n \in \mathbb{V}_{\phi_n}$. Then the sequence $\{u_n\}$ converges uniformly in Ω to a function $u(x)$. Moreover, the function $u(x)$, which belongs to $H^{1,t}_{\mathrm{loc}}(\Omega) \cap C^0(\Omega)$, is a weak solution in Ω, i.e. u solves (2.10).

Proof. Let ϕ, ϕ_n and u_n be as in the above statement. The variational solutions u_n are continuous in Ω, see Remark 4.1. Clearly, they are also equi-bounded. By Corollary 3.3 it follows that, for all couple of indexes m, n,

$$\operatorname*{Sup}_{\Omega} |u_n - u_m| \le \sup_{\partial\Omega} |\phi_n - \phi_m|. \tag{4.4}$$

So the sequence $\{u_n(x)\}$ is uniformly convergent in Ω to some $u(x) \in C^0(\Omega)$. Lemma 4.3 shows that $u(x)$ is a weak solution in Ω. Moreover, by appealing to Lemma 4.2, we get $u \in H^{1,t}_{\mathrm{loc}}(\Omega)$. Furthermore, the limit u is independent of the particular sequence $\{\phi_n\}$, as follows from (4.4) applied to two distinct, arbitrary, sequences (ϕ_n, u_n) and (ψ_n, v_n). This argument also proves the uniqueness of the solution u. □

Theorem 4.4 justifies Definition 2.3 of generalized solution given in Section 2, and also proves the existence and uniqueness Theorem 2.4.

It is worth noting that from Definition 2.3, Lemma 3.1 and Corollary 3.3, it follows that if u and v are the solutions corresponding to the continuous data ϕ and ψ, then

$$\min_{\partial\Omega} \phi \le \operatorname*{Inf}_{\Omega} u \le \operatorname*{Sup}_{\Omega} u \le \max_{\partial\Omega} \phi$$

and

$$\operatorname*{Sup}_{\Omega} |u - v| \le \max_{\partial\Omega} |\phi - \psi|.$$

Minimum and maximum are used here in the very classical sense.

5 Proof of Theorem 2.10

In this section we prove Theorem 2.10. We denote by $C^1(\partial\Omega)$ the functional space consisting on the restrictions to $\partial\Omega$ of functions in $C^1(\overline{\Omega})$.

Lemma 5.1. *A point $y \in \partial\Omega$ is regular if and only if condition* (2.14) *holds for each $\phi \in C^1(\partial\Omega)$.*

Proof. Let u be the solution corresponding to a given data $\phi \in C^0(\partial\Omega)$, and let $\{\phi_n\}$ and $\{u_n\}$ be as in Theorem 4.4 (by the way, note that the solutions u_n are variational and generalized). Define $\overline{u}_n(x)$ by $\overline{u}_n(x) = u_n(x)$ in Ω, $\overline{u}_n(y) = \phi_n(y)$, and define $\overline{u}(x)$ by $\overline{u}(x) = u(x)$ in Ω, $\overline{u}(y) = \phi(y)$. The functions $\overline{u}_n(x)$ are, by assumptions, continuous in $\Omega \cup \{y\}$, and uniformly convergent in $\Omega \cup \{y\}$ to the function $\overline{u}(x)$. So, $\overline{u}(x)$ is continuous in $\Omega \cup \{y\}$. □

Proof of Theorem 2.10. Necessary condition: Assume that $y \in \partial\Omega$ is regular. Given ρ and m, consider the restriction to $\partial\Omega$ of the function $h(x) = m\frac{|x-y|^2}{\rho^2}$. This function belongs to $C^1(\partial\Omega)$. Let $V(x)$ be the solution with $h(x)$ as boundary data. By construction, $V(x)$ satisfies condition (j) in Definition 2.8. Further, by the definition of a regular point, $V(x)$ satisfies condition (jj). Similarly, by considering the data $-h(x)$, one proves the existence of the function $U(x)$ required in Definition 2.8.

Sufficient condition: Assume that, at the point y, there exists a system of barriers. By Lemma 5.1 we may assume that $k(x) \in C^1(\partial\Omega)$. Let $u(x)$ be the corresponding solution and set

$$M = \sup_{\partial\Omega} |k(x)|. \tag{5.1}$$

Given $\epsilon > 0$, there is $\rho_\epsilon > 0$ such that

$$|k(x) - k(y)| < \frac{\epsilon}{2} \quad \text{if } |x-y| < \rho_\epsilon,\ x \in \partial\Omega. \tag{5.2}$$

Let V and U be barriers related to the values $\rho = \rho_\epsilon$ and $m = M$. Then (see also [16])

$$V(x) \ge M \quad \text{and} \quad U(x) \le -M \quad \text{on } (\partial\Omega) \cap \complement I(y,\rho). \tag{5.3}$$

By appealing to (5.1), (5.2), and (5.3), we show that

$$\begin{cases} k \le k(y) + \dfrac{\epsilon}{2} + V, \\ k \ge k(y) - \dfrac{\epsilon}{2} + U, \end{cases} \tag{5.4}$$

on $\partial\Omega$.

From (5.4) and Lemma 3.2 it follows that

$$\begin{cases} u(x) \le k(y) + \dfrac{\epsilon}{2} + V(x), \\ u(x) \ge k(y) - \dfrac{\epsilon}{2} + U(x), \end{cases} \tag{5.5}$$

a.e. in Ω, since $k(y) + \frac{\epsilon}{2} + V(x)$ is a super-solution, etc. Furthermore, property (jj) in Definition 2.8 implies the existence of $\overline{\rho}_\epsilon > 0$ such that

$$x \in \Omega \cap I(y, \overline{\rho}_\epsilon) \implies |V(x)| \le \frac{\epsilon}{2} \text{ and } |U(x)| \le \frac{\epsilon}{2}. \tag{5.6}$$

From (5.5) and (5.6) we show that

$$x \in \Omega \cap I(y, \overline{\rho}_\epsilon) \implies -\epsilon \le u(x) - h(y) \le \epsilon.$$

So, $\lim_{x \to y} u(x) = h(y)$. Hence y is regular. □

6 Proof of Theorem 2.13

We start with some preliminary results.

Lemma 6.1. *The Lipschitz continuous function*

$$u(x) = \alpha|x-y| + \beta,$$

where α and β are constants, is a super-solution if $\alpha > 0$, and u is a sub-solution if $\alpha < 0$.

Proof. Without loss of generality we assume that $u(x) = \alpha r$, where $r = |x|$. We start by assuming that $A(p)$ is indefinitely differentiable. By taking into account the monotony assumptions, we easily show (for instance, by appealing to the first order Taylor's formula with Lagrange form of the remainder) that the Jacobian matrix $DA(p)$ of the transformation $A(p)$ is positive semi-definite at each point $p \in \mathbb{R}^N$. So, for each unit vector $\xi \in \mathbb{R}^N$,

$$DA(p)\xi \cdot \xi \le \operatorname{tr} DA(p) \quad \text{for all } \xi \in \mathbb{R}^N, \tag{6.1}$$

since the trace coincides with the sum of the eigenvalues.

Let us denote the generical element of the Jacobian matrix $DA(p)$ by $A_{ij}(p)$. By setting $p = \nabla u$ one has

$$\operatorname{div} A(\nabla u(x)) = \sum_{i,j} A_{ij}(p)\partial_i p_j. \tag{6.2}$$

Since $p = \alpha x r^{-1}$, it follows that $\partial_i p_j = \alpha\, r^{-1}(\delta_{ij} - r^{-2}x_i x_j)$. So, from (6.2) we get

$$\operatorname{div} A(\nabla u(x)) = \alpha\, r^{-1}\{\operatorname{tr} DA(\alpha x r^{-1}) - DA(\alpha x r^{-1})(x r^{-1}) \cdot (x r^{-1})\} \tag{6.3}$$

for each $x \neq 0$. From (6.3) and (6.1) it follows that $\operatorname{div} A(\nabla u(x))$ has the sign of the constant α, for each $x \neq 0$.

Let ϕ be a non-negative, indefinitely differentiable function in $\mathbb{R}^N$. Fix $R > 0$ such that

$$\operatorname{supp} \phi \subset I(0, R).$$

Next, fix a function $\gamma(x) \in D(\mathbb{R}^N)$ such that $0 \leq \gamma(x) \leq 1$, and $\gamma(x) = 1$ for $|x| \leq 1$. To fix ideas, assume that $\operatorname{supp} \gamma \subset I(0, 2)$. Further define, for each $s > 0$,

$$\gamma_s(x) = \gamma(s^{-1}x) \quad \text{and} \quad \phi_s(x) = \phi(x)(1 - \gamma_s(x)).$$

Note that, for all $s \in (0, R)$,

$$\operatorname{supp} \phi_s \subset I(0, R) - I(0, s).$$

Hence, by integration by parts,

$$\alpha \int\limits_{I(0,R)} A(\nabla u(x)) \cdot \nabla \phi_s(x)\, dx = -\alpha \int\limits_{\complement I(0,s)} \operatorname{div} A(\nabla u(x))\phi_s(x)\, dx \leq 0,$$

where $u(x) = \alpha r$. Note that, on the left hand side, we may replace $I(0, R)$ by $\mathbb{R}^N$. We want to show that

$$\lim_{s \to 0} \int\limits_{I(0,R)} A(\nabla u(x)) \cdot \nabla \phi_s(x)\, dx = \int\limits_{I(0,R)} A(\nabla u(x)) \cdot \nabla \phi(x)\, dx. \tag{6.4}$$

This proves that

$$\alpha \int\limits_{\mathbb{R}^N} A(\nabla u) \cdot \nabla \phi\, dx \leq 0,$$

which is our thesis.

Straightforward calculations show that

$$\nabla \phi_s(x) = (1 - \gamma_s(x))\nabla \phi(x) - s^{-1}\phi(x)(\nabla \gamma)(s^{-1}x). \tag{6.5}$$

Since $(1 - \gamma_s(x))\nabla\phi(x)$ converges point-wisely to $\nabla\phi(x)$, $x \neq 0$, as $s \to 0$, it readily follows, by Lebesgue dominated convergence theorem, that (6.4) holds by replacing, on the left hand side, $\nabla\phi_s$ by $(1 - \gamma_s(x))\nabla\phi(x)$.

Let us see that on the left hand side of (6.4) the contribution due to the second term on the right hand side of (6.5) tends to zero. One has

$$\begin{aligned} s^{-1} \int\limits_{I(0,R)} |\phi(x)(\nabla\gamma)(s^{-1}x)|\, dx &\leq s^{-1} \int\limits_{I(0,2)} |\phi(sy)(\nabla\gamma)(y)| s^N\, dx \\ &\leq 2^N V_N s^{N-1} \|\phi\|_{L^\infty(\mathbb{R}^N)} \|\nabla\gamma\|_{L^\infty(\mathbb{R}^N)}, \end{aligned}$$

where V_N denotes the volume of the unit sphere. Since $A(\nabla u(x))$ is uniformly bounded in $I(0, R)$, the thesis follows.

Assume now that $A(p)$ is merely continuous. Let $j_\epsilon(\eta)$ be, for each $\epsilon > 0$, a real, non-negative function, indefinitely differentiable with compact support contained in the sphere $I(0, \epsilon)$ and integral equal to 1. Set

$$A_\epsilon(p) = \int A(\eta) j_\epsilon(p - \eta)\, d\eta.$$

These functions are indefinitely differentiable. Furthermore,

$$A_\epsilon(p) - A_\epsilon(q) = \int [A(p - \xi) - A(q - \xi)]\, j_\epsilon(\xi)\, d\xi. \tag{6.6}$$

In particular, this last inequality implies that $A_\epsilon(p)$ satisfies the monotony hypothesis (2.3) (note that assumptions (2.2), (2.4), and (2.5) were not used here). From the first part of the proof it follows that

$$\alpha \int A_\epsilon(\nabla u) \cdot \nabla \phi \, dx \le 0 \tag{6.7}$$

for all non-negative $\phi \in D(\Omega)$. Further, since

$$A_\epsilon(p) - A(p) = \int [A(\eta) - A(p)] j_\epsilon(p - \eta) \, d\eta$$

and since $A(p)$ is uniformly continuous on compact sets, it follows that $A_\epsilon(p) \to A(p)$ uniformly on compact sets. So, letting $\epsilon \to 0$ in equation (6.7), one gets the thesis. □

The next result concerns the local character of the notion of a regular point.

Theorem 6.2. *Let Ω and Λ be two open bounded sets, and let $y \in \partial\Omega \cap \partial\Lambda$. Assume, moreover, that there exists a sphere $I(y, r)$ such that*

$$I(y, r) \cap \Omega = I(y, r) \cap \Lambda. \tag{6.8}$$

Then y is regular with respect to Ω if and only if it is regular with respect to Λ.

Proof. Due to Theorem 2.10, it is sufficient to show that there is a system of barriers with respect to Λ if and only if there is a system of barriers with respect to Ω.

Let y be regular with respect to Λ. Assume, for the time being, that $\Lambda \subset \Omega$. Given ρ and m, $0 < \rho < r$ and $0 < m$, let $V(x)$ be the variational solution in Ω to problem (2.7) with boundary data given by

$$h(x) = m|x - y|^2 \rho^{-2}.$$

By construction, $V(x)$ satisfies condition (j) in Definition 2.8. Let us show that it also satisfies condition (jj). Let

$$M \ge \max\Big\{1, m^{-1} \operatorname*{Sup}_{\Omega} |V(x)|\Big\},$$

and let $V'(x)$ be the solution in Λ with boundary data $h'(x) = Mm|x - y|^2\rho^{-2}$. Clearly, $V(x)$ is a solution in Λ. Furthermore, from the definition of M it follows that $V' \ge V$ on $\partial\Omega$. From this last inequality, together with Lemma 3.2, we show that $V'(x) \ge V(x)$ a.e. in Λ. From this last assertion, together with the regularity of y with respect to Λ, it follows that

$$0 \le \lim_{\substack{x \to y \\ x \in \Omega}} V(x) \le \lim_{\substack{x \to y \\ x \in \Lambda}} V'(x) = 0.$$

This proves condition (jj). By appealing to Theorem 2.10 we conclude that y is regular with respect to Ω. The existence of the function $U(x)$ referred in Definition 2.8 may be shown by a similar argument, or by appealing to Remark 2.9.

Reciprocally, assume that y is regular with respect to Ω. Given ρ and m, $0 < \rho < r$ and $0 < m$, we construct below the corresponding barrier $V(x)$ in Λ, according to Definition 2.8. Let $V(x)$ be the solution in Ω with boundary data $h(x) = m|x - y|\rho^{-1}$ on $\partial\Omega$. The function $V(x)$ is a solution in Λ and satisfies condition (ii) since y is regular with respect to Ω and $h(x) = 0$. Further, since $V = h$ on $\partial\Omega$ and $h(x)$ is a sub-solution in Ω (Lemma 6.1), it must be $V(x) \ge h(x)$ a.e. in Ω. In particular, $V \ge h$, so $V \ge m$ on $\partial\Lambda \cap \complement I(y, \rho)$, as desired.

Finally, if Λ is not contained in Ω, consider the open set $D = I(y, r) \cap \Lambda = I(y, r) \cap \Omega$, and take into account that $D \subset \Lambda$ and $D \subset \Omega$. □

We end this section by proving Theorem 2.13.

Proof of Theorem 2.13. Necessary condition: Let y be regular. By Theorem 6.2 it follows that y is regular with respect to $\Sigma - E_\rho$. Since the capacitary potential $u_{\rho,m}$ is the solution in $\Sigma - E_\rho$ (Lemma 3.5) with data m on ∂E_ρ and 0 on $\partial\Sigma$ (Lemma 3.4), the first equation (2.24) follows. A similar argument applies to $u_{\rho,-m}$.

Sufficient condition: We assume that the hypothesis (2.24) holds, and we prove the existence of a system of barriers at y. Given $\rho > 0$ and $m > 0$, we construct the function $V(x)$ referred in Definition 2.8. Let R_0 be such that $\Sigma \subset I(y, R_0)$, and define $k > 0$ by

$$\frac{(k + 2m)\rho}{2R_0} = m. \tag{6.9}$$

For convenience, we denote by u the capacitary potential $u = u_{\frac{\rho}{2},-(m+k)}$. Furthermore, we define in Σ the function $V = u + (m + k)$. Then V is a solution in $\Sigma - E_{\frac{\rho}{2}}$ (Lemma 3.5) and, in particular, it is a solution in Ω. Since $\lim_{x\to y} u(x) = -(m + k)$, the function V satisfies condition (jj) in Definition 2.8. Obviously $V(x) \geq 0$ a.e. in Σ, as follows from Lemma 3.4.

Next we prove condition (j). Consider in $I(y, R_0)$ the function $f(x) = (k + 2m)R_0^{-1}|x - y| - (k + 2m)$. This function is a sub-solution in $I(y, R_0)$ (Lemma 6.1) and, in particular, is a sub-solution in $I(y, R_0) - I(y, \frac{\rho}{2})$. Since $\Sigma \subset I(y, R_0)$, it follows that $f \leq 0$ on $\partial\Sigma$. Further, from (6.9) it follows that $f = -(k + m)$ on $\partial I(y, \frac{\rho}{2})$. So

$$f \leq u \quad \text{on } \partial I\Big(y, \frac{\rho}{2}\Big). \tag{6.10}$$

By appealing to (6.10), to the inequality $f \leq 0$ on $\partial\Sigma$, and to Lemma 3.2 applied in $\Sigma - I(y, \frac{\rho}{2})$, it follows that $f(x) \leq u(x)$ a.e. in this last set. So

$$V(x) \geq f(x) + (m + k) \geq m \quad \text{a.e. on } \Sigma - I(y, \rho). \tag{6.11}$$

In particular, (6.11) implies that $V \geq m$ on $\partial\Omega - I(y, \rho)$, hence condition (j) holds. □

Part II

7 Main results

We start by remarking that the proofs presented in Part II strongly rely on ideas and techniques used in reference [21], to which the reader is referred.

The aim of the second part of this work is to state sufficient conditions for regularity of a given boundary point y. This task is done by appealing to Theorem 2.13. The sufficient conditions obtained here consist in assumptions on the sets

$$E_\rho = (\complement\Omega)(y, \rho), \tag{7.1}$$

the complementary sets of Ω with respect to the closed balls $\overline{I(y, \rho)}$. They always concern sufficiently small values of the radius ρ.

The cornerstone result of Part II, Theorem 7.2, has an “abstract” feature due to Assumption 7.1. However we show that this assumption holds if simple geometrical conditions are fulfilled. This leads to the statements in Theorems 7.3 and 7.4 below.

Assumption 7.1. Let $y \in \partial\Omega$ be a given boundary point. There is a strictly positive function $\sigma(\rho)$ such that, for each positive ρ in an arbitrarily small neighborhood of zero, the estimate

$$|v(x)| \leq \sigma(\rho)^{-1} \int\limits_{I(y,\rho)} \frac{|\nabla v(z)|}{|x - z|^{N-1}}\, dz \tag{7.2}$$

holds a.e. in $I(y, \rho)$, for all $v \in H^{1,t}(I(y, \rho))$ vanishing identically on E_ρ.

The next theorem and the related Theorems 7.3 and 7.4 below are the main results in Part II.

Theorem 7.2. *There is a constant Λ which depends only on t, N, a and p_0 such that if Assumption 7.1 holds for some $\sigma(\rho)$ satisfying*

$$[\sigma(\rho)]^{\frac{t}{t-1}} \geq \Lambda(\log\log\rho^{-1})^{-1} \tag{7.3}$$

for small positive values of ρ, then the point $y \in \partial\Omega$ is regular with respect to the operator $\mathcal{L}$.

Actually, the next two theorems are corollaries of Theorem 7.2.

Theorem 7.3. *There is a constant Λ_0 which depends only on t, N, a and p_0 such that if*

$$\left(\frac{|E_\rho|}{|I(y,\rho)|}\right)^{\frac{t}{t-1}} \geq \Lambda_0(\log\log\rho^{-1})^{-1}$$

for small positive values of ρ, then the point $y \in \partial\Omega$ is regular with respect to the operator $\mathcal{L}$.

Note that this condition is stronger than the usual cone condition since the right hand side goes to zero with ρ.

The next statement is [5, Theorem 5.5] (see also [3, p. 5]).

Theorem 7.4. *A point $y \in \partial\Omega$ is regular with respect to the operator $\mathcal{L}$ if y satisfies an $(N-1)$-dimensional external cone property. The $(N-1)$-dimensional external cone property may be replaced by a generalized $(N-1)$-dimensional external cone property (see Definition 7.7 below).*

We end this section by proving that Assumption 7.1 follows from the geometrical assumptions required both in Theorem 7.3 and in Theorem 7.4. So, as soon as this purpose is fulfilled, our task will be merely to present the proof of Theorem 7.2. This proof is postponed to the next sections.

The proofs of the next lemma and corollary strictly follow [21, proof of Theorem 6.2]. We denote by S the surface of the N-dimensional unit sphere. Further, if $\Theta \subset S$, we denote by $|\sphericalangle\Theta|$ the $(N-1)$-dimensional spherical measure of Θ,

$$|\sphericalangle\Theta| = \int_\Theta dS.$$

Lemma 7.5. *Set $I = I(y,\rho)$, and let $E = E_\rho$ be given by* (7.1)*. Furthermore, let a point $x \in I$, $x \notin E$, be given, and denote by S the surface of the unit sphere centered in x. Finally, consider the set*

$$\Theta = \Theta(x) = \{\xi \in S : \text{there exists } t = t(\xi) \in \mathbb{R} \text{ such that } x + t\xi \in E\}.$$

Then the estimate

$$|v(x)| \leq \frac{1}{|\sphericalangle\Theta(x)|}\int_I \frac{|\nabla v(z)|}{|x-z|^{N-1}}\,dz$$

holds for any function $v \in C^1(I)$ vanishing on E.

Proof. Let $\xi \in \Theta$ and $t(\xi) \in \mathbb{R}$ be such that $x + t(\xi)\xi \in E$. Since

$$|v(x+t(\xi)\xi) - v(x)| \leq \int_0^{t(\xi)} |\nabla v(x+r\xi)|\,dr$$

and $|x-z|^{N-1}\,dSdr = dz$, it follows that

$$|\sphericalangle\Theta||v(x)| = \int_\Theta |v(x)|\,dS \leq \int_\Theta\int_0^{t(\xi)} |\nabla v(x+r\xi)|\,drdS \leq \int_I \frac{|\nabla v(z)|}{|x-z|^{N-1}}\,dz. \qquad \square$$

Corollary 7.6. *Let $v \in C^1(I)$ vanish on E. Assume that*

$$|\sphericalangle\Theta| \equiv \inf_x |\sphericalangle\Theta(x)| > 0. \tag{7.4}$$

Then

$$|v(x)| \leq \frac{1}{|\sphericalangle\Theta|}\int_I \frac{|\nabla v(z)|}{|x-z|^{N-1}}\,dz \tag{7.5}$$

for all $x \in I$. Furthermore, if $v \in H^{1,t}(I)$ vanishes on E in the $H^{1,t}(I)$ sense, then (7.5) *holds a.e. in I.*

Note that the estimate (7.5) is obvious if $x \in E$. The last assertion in the corollary follows from well-known results on the continuity of the linear map defined by convolution with the kernel $|z|^{-(N-1)}$. Actually, this map is continuous from L^r to L^{r^*} where $1/r^{r^*} = 1/r - 1/n$. See, for instance, [22, Chapter V].

Further, the "volumetric" estimate $(1/N)|\sphericalangle\Theta(x)|(\operatorname{diam} I)^N \geq |E|$ shows that in equation (7.5) one has

$$|\sphericalangle\Theta| \geq \frac{NV_N}{2^N}\frac{|E_\rho|}{|I(y,\rho)|}.$$

Hence Assumption 7.1 holds for $\sigma(\rho)$ given by the right hand side of the above inequality. So, Theorem 7.3 follows from Theorem 7.2.

Next, we prove Theorem 7.4. Some details are left to the reader. By "cone" (in any dimension) we mean a right circular cone, truncated by a sphere with center the vertex of the cone. For instance, the $(N-1)$-dimensional "truncated cones" with vertex $y = 0$ have the form

$$C_{\rho,\omega} = \{x \in \mathbb{R}^N : x_1 \geq 0,\ x_N = 0,\ |x| \leq \rho,\ |x|^2 \leq (1+\omega){x_1}^2\}, \tag{7.6}$$

where ρ and ω are positive constants. Note that, by setting $x = (x_1, x', x_N)$, the above condition means that $|x'|^2 \leq \omega x_1^2$.

Definition 7.7. We say that a point $y \in \partial\Omega$ satisfies an $(N-1)$-dimensional external cone property if there exists an $(N-1)$-dimensional cone C with vertex at y and contained in $\complement\Omega$. Similarly, we define the generalized $(N-1)$-dimensional cone property at the point y, by replacing the cone C by a Lipschitz image of itself.

The proof of Theorem 7.4 follows immediately from Theorem 7.2 and Corollary 7.6, by a small modification of the argument used to prove Theorem 7.3. As above, we appeal to Corollary 7.6. Roughly speaking, as for Theorem 7.3, we would like to show that there is a positive lower bound $|\sphericalangle\Theta|$ for the values of the solid angles $|\sphericalangle\Theta(x)|$ from which the set E_ρ can be "watched" from points $x \in I(y,\rho)$. Clearly, this is false in general, since (for instance) x and E_ρ may belong to an $(N-1)$-dimensional hyperplane. However the same argument applies here. Let us prove that equation (7.2) holds for a positive $\sigma(\rho)$, independent of ρ. To show this claim, note that geometry and estimates for a generical value ρ can immediately be brought back to the case $\rho = 1$, by a suitable homothety. Next, note that the estimates in play are invariant under Lipschitz maps, up to multiplication by positive constants. So, we may fold up the original $(N-1)$-dimensional cone into a "non flat" $(N-1)$-dimensional "twisted cone", which contains N distinct pieces of surface, each one orthogonal to a single x_i-direction, $i = 1, \dots, N$. Now, from each point $x \in I(y,1)$, one "watches", at least, one of the above pieces of surface, from a positive solid angle $|\sphericalangle\Theta(x)|$. Moreover, the lower bound $|\sphericalangle\Theta|$ of the values of solid angles is positive. This proves Theorem 7.4.

Note that it would be sufficient to prove that the lower bounds behaves like $\sigma(\rho)$ in equation (7.3), as ρ goes to zero.

8 A recursive estimate for the local oscillation

In the sequel, to avoid unessential devices, we assume in equations (2.4) and (2.5) that $p_0 = 0$. One easily extends the proof to the general situation by appealing to (4.1). This leads to the appearance of "lower order" terms, easy to control.

We prove Theorem 7.3 by showing that (2.24) holds. More precisely, we fix a couple of positive constants ρ_0 and m, and prove that

$$\lim_{x\to y} u_{m,\rho_0}(x) = m.$$

The proof of the second equation (2.24) is absolutely identical. Alternatively, we may appeal to Remark 2.9, to refer the proof to that of the first equation.

In the sequel the "large" ball Σ, the point $y \in \partial\Omega$, and the positive constants m and ρ_0 are assumed to be fixed, once and for all. The capacitary potential $u_{m,\rho_0}(x)$ of E_{ρ_0} will be simply denoted by $u(x)$. Furthermore, without loss of generality, we place the origin at y, so

$$y = 0.$$

We set $I(r) = I(0,r)$. The following result is well known.

Lemma 8.1. *One has*

$$\|v\|_{t^*,r} \le c\|\nabla v\|_{t,r} \quad \text{for all } v \in H_0^{1,t}(r), \tag{8.1}$$

where $\frac{1}{t^*} = \frac{1}{t} - \frac{1}{N}$.

We define the sets

$$B(k,r) = \{x \in \Omega(y,r) : u(x) \le k\} \tag{8.2}$$

and introduce the cut-off function

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \le \rho, \\ \frac{R-|x|}{R-\rho} & \text{if } \rho \le |x| \le R, \\ 0 & \text{if } R \le |x|. \end{cases} \tag{8.3}$$

In the sequel, $0 < \rho < R < \rho_0$. For brevity, we set

$$B(k) = B(k,R).$$

The following kind of estimates is well known.

Lemma 8.2. *Assume that* $0 < \rho < R < \overline{R}$ *and* $0 < h < k$. *Let* $v \in H^{1,t}(\overline{R})$. *Then, the following estimates hold:*

$$\begin{cases} \displaystyle\int_{B(h,\rho)} (h-u)^t \, dx \le c\Big((R-\rho)^{-t} \int_{B(k)} (k-u)^t \, dx + \int_{B(k)} |\nabla u|^t \phi^t \, dx\Big)|B(k)|^{\frac{t}{N}}, \\ \displaystyle |B(h,\rho)|(k-h)^t \le \int_{B(h,\rho)} (k-u)^t \, dx \le \int_{B(k)} (k-u)^t \, dx. \end{cases} \tag{8.4}$$

For the proof of the first estimate see, for instance, [4, proof of the first inequality (6.12)]. The second estimate (8.4) is obvious.

Theorem 8.3. *Let* ϕ *be given by* (8.3). *Then, for each real* k,

$$\int_{B(k,R)} |\nabla u|^t \phi^t \, dx \le c(R-\rho)^{-t} \int_{B(k,R)} |u-k|^t \, dx. \tag{8.5}$$

Proof. By the definition of $u_{m,\rho_0}(x)$ one has

$$\int_{\Sigma} (A(\nabla u), \nabla(v-u)) \, dx \ge 0 \quad \text{for all } v \in \mathbb{K}_m(\Sigma), \tag{8.6}$$

where (recall (2.19))

$$\mathbb{K}_m(\Sigma) = \{v \in H_0^{1,t}(\Sigma) : v \ge m \text{ on } E_{\rho_0}\}.$$

By setting $v = u - \phi^t \min(u-k, 0)$ in equation (8.6), it follows that

$$\int_{B(k)} (A(\nabla u), \nabla u)\phi^t \, dx \le -t \int_{B(k)} (A(\nabla u), \nabla \phi)(u-k)\phi^{t-1} \, dx. \tag{8.7}$$

From (8.7), by appealing to Hölder's inequality and to properties enjoyed by ϕ and $A(p)$, we show that

$$a \int_{B(k)} |\nabla u|^t \phi^t \, dx \le t a^{t-1} \Big(\int_{B(k)} |\nabla u|^t \phi^t \, dx\Big)^{\frac{t-1}{t}} \Big(\int_{B(k)} |u-k|^t |\nabla \phi|^t \, dx\Big)^{\frac{1}{t}}. \tag{8.8}$$

Equation (8.8) leads to

$$\int_{B(k)} |\nabla u|^t \phi^t \, dx \le c \int_{B(k)} |u-k|^t |\nabla \phi|^t \, dx.$$

Since $|\nabla \phi| \le (R-\rho)^{-1}$, the thesis follows. □

The next result follows by appealing to Theorem 8.3 and Lemma 8.2.

Lemma 8.4. *Assume that* $0 < \rho < R$, *and* $0 < h < k$. *The following estimates hold:*

$$\begin{cases} \int\limits_{B(h,\rho)} (h-u)^t \, dx \le c_1 |B(k)|^{\frac{t}{N}} (R-\rho)^{-t} \int\limits_{B(k)} (k-u)^t \, dx, \\ |B(h,\rho)|(k-h)^t \le \int\limits_{B(k)} (k-u)^t \, dx. \end{cases} \tag{8.9}$$

For brevity we set

$$\begin{cases} u(h,\rho) = \int\limits_{B(h,\rho)} (h-u)^t \, dx, \\ b(h,\rho) = |B(h,\rho)|. \end{cases} \tag{8.10}$$

So, equation (8.4) takes the form

$$\begin{cases} u(h,\rho) \le c_1 b(k,R)^{\frac{t}{N}} (R-\rho)^{-t} u(k,R), \\ b(h,\rho)(k-h)^t \le u(k,R). \end{cases} \tag{8.11}$$

Next, we define

$$\psi(h,\rho) = u(h,\rho)^{\theta \frac{N}{t}} b(h,\rho), \tag{8.12}$$

where

$$\theta = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{t}{N}} > 1.$$

Straightforward calculations show that

$$\psi(h,\rho) \le c_1^{\frac{N}{t}\theta} \frac{1}{(R-\rho)^{N\theta}} \frac{1}{(k-h)^t} \psi(k,R)^\theta. \tag{8.13}$$

Note that $\frac{t}{N} + \theta = \theta^2$. We point out that the above choice of θ is the only choice possible to get an estimate of the form (8.13).

Lemma 8.5. *Let* $0 < r_0 \le \frac{\rho_0}{2}$, $k_0 \in \mathbb{R}$ *and* $d > 0$. *Define, in correspondence to each non-negative integer* m, *the following quantities:*

$$\begin{cases} r_m = \frac{r_0}{2} + \frac{r_0}{2^{m+1}}, \\ k_m = k_0 - d + \frac{d}{2^m}, \end{cases} \tag{8.14}$$

$$\begin{cases} a_m = |B(k_m, r_m)|, \\ u_m = \int\limits_{B(k_m,r_m)} (k_m - u)^t \, dx, \end{cases} \tag{8.15}$$

and

$$\psi_m = u_m^{\theta \frac{N}{t}} b_m. \tag{8.16}$$

Then

$$\left| B\left(k_0 - d, \frac{r_0}{2}\right) \right| = 0 \tag{8.17}$$

if

$$d \ge c_1^{\frac{N\theta}{t^2}} \frac{2^{\frac{\beta\theta}{t}}}{(2r_0)^{\frac{N\theta}{t}}} \psi_0^{\frac{\theta-1}{t}} \equiv C \frac{\psi_0^{\frac{\theta-1}{t}}}{r_0^{\frac{N\theta}{t}}}. \tag{8.18}$$

Proof. Note that a_m, u_m and ψ_m are non-increasing sequences. By setting, in equation (8.13), $(k,R) = (k_m, r_m)$ and $(h,\rho) = (k_{m+1}, r_{m+1})$, one shows that

$$\psi_{m+1} \le c_1^{\frac{N}{t}\theta} \frac{1}{d^t} \frac{1}{(2r_0)^{N\theta}} 2^{(m+1)(t+N\theta)} \psi_m^\theta. \tag{8.19}$$

We want to prove, by induction, that

$$\psi_m \leq \frac{\psi_0}{2^{\beta m}} \quad \text{for all } m \geq 0, \tag{8.20}$$

where

$$\beta = \frac{t + N\theta}{\theta - 1}.$$

For $m = 0$, (8.20) is obvious. Assume it for some $m \geq 0$. By appealing to (8.19) and (8.20) straightforward calculations show that

$$\psi_{m+1} \leq c_1^{\frac{N}{t}\theta} \frac{1}{d^t} \frac{2^{\beta\theta}}{(2r_0)^{N\theta}} \psi_0^{\theta-1} \frac{\psi_0}{2^{\beta(m+1)}}. \tag{8.21}$$

This proves (8.20) under the assumption (8.18). In particular, $\psi_m \to 0$ as $m \to \infty$. Since

$$\left|B\left(k_0 - d, \frac{r_0}{2}\right)\right| \Bigg\{ \int_{B(k_0-d,\frac{r_0}{2})} ((k_0 - d) - u)^t \, dx \Bigg\}^{\theta\frac{N}{t}} \leq \psi_m,$$

the thesis of the theorem follows. □

Corollary 8.6. *There is a constant C, independent of r_0 and k_0, such that*

$$\operatorname*{Inf}_{I(\frac{r_0}{2})} u \geq k_0 - C\Bigg\{\frac{1}{r_0^N} \int_{B(k_0,r_0)} (k_0 - u)^t \, dx\Bigg\}^{\frac{1}{t}} \Bigg\{\frac{1}{r_0^N} |B(k_0, r_0)|\Bigg\}^{\frac{\theta-1}{t}}. \tag{8.22}$$

In particular,

$$\operatorname*{Inf}_{I(\frac{r_0}{2})} u \geq k_0 - C\Bigg\{\frac{1}{r_0^N} \int_{B(k_0,r_0)} (k_0 - u)^t \, dx\Bigg\}^{\frac{1}{t}}. \tag{8.23}$$

The proof of the first estimate follows immediately from (8.17), by taking into account that the C term on the right hand sice of (8.22) is equal to the C term on the right hand side of (8.18). The second estimate follows from the first one (here, we change the value of the constant C). Since C does not depend on r_0 and k_0, we drop the index 0. Further, we define

$$i(r) = \operatorname*{Inf}_{I(r)} u, \quad s(r) = \operatorname*{Sup}_{I(r)} u, \quad \omega(r) = s(r) - i(r).$$

By setting, in (8.23), $k = i(2r) + \eta\omega(2r)$, where $\eta > 0$, and by taking into account that for $x \in B(k, r)$ one has

$$0 \leq k - u(x) \leq \eta\omega(2r),$$

it follows that

$$i(\frac{r}{2}) \geq i(2r) + \eta\omega(2r) - C\Big\{\frac{1}{r^N} |B(k, r)|\Big\}^{\frac{1}{t}} \eta\omega(2r).$$

Hence,

$$\omega(\frac{r}{2}) \leq \Bigg\{1 - \eta\Bigg[1 - C\Big(\frac{1}{r^N} |B(k, r)|\Big)^{\frac{1}{t}}\Bigg]\Bigg\}\omega(2r).$$

For convenience we replace r by $2r$ in the next result.

Proposition 8.7. *Let $k = i(4r) + \eta\omega(4r)$. Then*

$$\omega(r) \leq \Bigg\{1 - \eta\Bigg[1 - C\Big(\frac{1}{r^N} |B(k, 2r)|\Big)^{\frac{1}{t}}\Bigg]\Bigg\}\omega(4r). \tag{8.24}$$

Remark 8.8. In [4, equation (6.21)] it was proved that

$$|B(h, \rho)|(k - h)^t \leq c\Big((R - \rho)^{-t} \int_{B(k)} (k - u)^t \, dx + \int_{B(k)} |\nabla u|^t \phi^t \, dx\Big)|B(h, \rho)|^{\frac{t}{N}}. \tag{8.25}$$

This estimate, together with (8.5), shows that

$$|B(h,\rho)|^{1-\frac{t}{N}}(k-h)^t \le c_1(R-\rho)^{-t}\int\limits_{B(k)}(k-u)^t\,dx. \tag{8.26}$$

If we appeal to this estimate (instead of appealing to the second estimate (8.9)), we get (8.22) with the exponent $\frac{\theta-1}{t}$ replaced by $\frac{1}{N\theta_1}$, where $\frac{t}{N-t}+\theta_1=\theta_1^2$.

9 Proof of Theorem 7.2

We start this section by stating a well-known potential theory result.

Lemma 9.1. *Let μ be a compact supported, bounded variation measure in $\mathbb{R}^N$, and let*

$$U_1^\mu(x)=\int\frac{d\mu(z)}{|x-z|^{N-1}} \tag{9.1}$$

be the potential of order 1 generated by μ. Then there is a positive constant c such that

$$|\{x\in\mathbb{R}^N:|U_1^\mu(x)|\ge\tau\}|\le\left(\frac{c\int|d\mu|}{\tau}\right)^{\frac{N}{N-1}} \tag{9.2}$$

for each $\tau>0$.

For potentials of order 2, the above result is essentially due to E. Cartan, see [6, Lemma 4]. The result is easily extended to potentials of arbitrary order α. For $\alpha=1$, it asserts that

$$\mathrm{cap}_1^*\{x\in\mathbb{R}^N:|U_1^\mu(x)|\ge\tau\}\le\frac{2^{N-1}\int|d\mu|}{\tau}$$

for each $\tau>0$, where $\mathrm{cap}_1^*(E)$ denotes the internal capacity of order 1 of the set E. Equation (9.2) follows by appealing to the classical estimate

$$|E|\le c(N)(\mathrm{cap}_1^*(E))^{\frac{N}{N-1}}.$$

Next we prove the following result.

Lemma 9.2. *Let $0\le h<k\le m$, and $0<r<\frac{\rho_0}{2}$. Then*

$$|B(h,2r)|^{\frac{t(N-1)}{N(t-1)}}\le c[(k-h)\sigma(2r)]^{-\frac{t}{t-1}}(|B(k,2r)|-|B(h,2r)|)\Big((2r)^{-t}\int\limits_{B(k,4r)}|u-k|^t\,dx\Big)^{\frac{1}{t-1}}. \tag{9.3}$$

Proof. Set

$$v=\begin{cases}k-h & \text{if } u\le h,\\ k-u & \text{if } h\le u\le k,\\ 0 & \text{if } k\le u,\end{cases}$$

and

$$\mu(z)=\begin{cases}|\nabla v(z)| & \text{on } I(0,2r),\\ 0 & \text{on } (\complement I)(0,2r).\end{cases} \tag{9.4}$$

Since v vanishes on E_{2r}, from Assumption 7.1 it follows $|v(x)|\le c\sigma(2r)^{-1}U_1^\mu(x)$ on $I(2r)$. Hence, by Lemma 9.1, we show that

$$|\{x\in I(2r):|v(x)|\ge\tau\}|\le c\Big((\sigma(2r)\tau)^{-1}\int\limits_{I(2r)}|\nabla v(z)|\,dz\Big)^{\frac{N}{N-1}} \tag{9.5}$$

for each $\tau>0$. Let $\tau=k-h-\epsilon$, where $\epsilon>0$.

By appealing to the definition of v, we prove that

$$|B(h,2r)| \le |\{x \in I(2r) : v(x) \ge \tau\}| \le c\Big([\sigma(2r)(k-h-\epsilon)]^{-1} \int_{B(k,2r)-B(h,2r)} |\nabla v(z)|\, dz \Big)^{\frac{N}{N-1}}.$$

Further, by letting $\epsilon \to 0$ in the last equation, and by appealing to Hölder's inequality, we obtain the estimate

$$|B(h,2r)|^{\frac{N-1}{N}} \le c\Big([\sigma(2r)(k-h)]^{-1} \Big(\int_{B(k,2r)} |\nabla u|^t\, dx \Big)^{\frac{1}{t}} \cdot (|B(k,2r)| - |B(h,2r)|)^{\frac{t-1}{t}} \Big). \tag{9.6}$$

Finally, by raising both terms of the last equation to the power $\frac{t}{t-1}$ and by appealing to Theorem 8.3 (with $\rho = 2r$ and $R = 4r$), the thesis follows. □

Theorem 9.3. *Let $0 < r < 4^{-1}\rho_0$. There is a constant C_1 which depends at most on a, p_0, d, t and N such that if $n_0 = n_0(r)$ satisfies* (9.13) *below, then*

$$\omega(r) \le (1 - 2^{-1}\eta_{n_0})\omega(4r), \tag{9.7}$$

where

$$\eta_{n_0} = 2^{-(n_0+1)}.$$

Proof. Let $l = i(4r)$, $\omega = \omega(4r)$, and set, for each non-negative integer j,

$$\begin{cases} \eta_j = 2^{-(j+1)}, \\ k_j = i(4r) + \eta_j \omega(4r), \end{cases} \tag{9.8}$$

and $b_j = |B(k_j, 2r)|$. By Lemma 9.2 with $k = k_j$ and $h = k_{j+1}$, we obtain

$$b_{j+1}^{\frac{t(N-1)}{N(t-1)}} \le c[2^{-(j+2)}\omega\sigma(2r)]^{-\frac{t}{t-1}}(b_j - b_{j+1}) \cdot [(2r)^{-t} V_N (4r)^N (2^{-(j+1)}\omega)^t]^{\frac{1}{t-1}}.$$

Straightforward calculations show that

$$b_{j+1}^{\frac{t(N-1)}{N(t-1)}} \le c r^{\frac{N-t}{t-1}} \sigma(2r)^{-\frac{t}{t-1}} (b_j - b_{j+1}), \tag{9.9}$$

where, for convenience, the value of the constant c may change from equation to equation (clearly, it depends only on fixed quantities like N, t, etc.).

Denote by $n_0 = n_0(r)$ an arbitrary positive integer to be fixed later on. From (9.9) it follows that

$$b_{n_0}^{\frac{t(N-1)}{N(t-1)}} \le b_{j+1}^{\frac{t(N-1)}{N(t-1)}} \le c r^{\frac{N-t}{t-1}} \sigma(2r)^{-\frac{t}{t-1}} (b_j - b_{j+1})$$

for each j, $0 \le j \le n_0 - 1$. Consequently,

$$n_0 b_{n_0}^{\frac{t(N-1)}{N(t-1)}} \le c r^{\frac{N-t}{t-1}} \sigma(2r)^{-\frac{t}{t-1}} \sum_{j=0}^{n_0-1} (b_j - b_{j+1}) \le c_0 \sigma(2r)^{-\frac{t}{t-1}} (2r)^{\frac{t(N-1)}{t-1}}.$$

Hence,

$$\Big(\frac{b_{n_0}}{(2r)^N} \Big)^{\frac{1}{t}} \le C n_0^{-\frac{N(t-1)}{t^2(N-1)}} \sigma(2r)^{-\frac{N}{t(N-1)}}. \tag{9.10}$$

On the other hand, from (8.24), one has

$$\omega(r) \le \Big\{ 1 - 2^{-(n_0+1)} \Big[1 - C\Big(\frac{b_{n_0}}{r^N} \Big)^{\frac{1}{t}} \Big] \Big\} \omega(4r). \tag{9.11}$$

Finally, from (9.10) and (9.11),

$$\omega(r) \le \Big\{ 1 - 2^{-(n_0+1)} \Big[1 - C_0 n_0^{-\frac{N(t-1)}{t^2(N-1)}} \sigma(2r)^{-\frac{N}{t(N-1)}} \Big] \Big\} \omega(4r). \tag{9.12}$$

Next, we want to single out an index $n_0 = n_0(r)$ such that the expression under square brackets is less than or equal to $\frac{1}{2}$ for each positive (small) radius r. This leads to

$$n_0(r) \ge C_1 \sigma(2r)^{-\frac{t}{t-1}}, \tag{9.13}$$

where C_1 is a constant which depends at most on a, p_0, d, t and N. In the sequel we denote by $n_0(r)$ the smallest integer for which (9.13) holds. Hence

$$C_1\sigma(2r)^{-\frac{t}{t-1}} \le n_0(r) < 1 + C_1\sigma(2r)^{-\frac{t}{t-1}}. \tag{9.14}$$

The proof is complete. □

Lemma 9.4. *Let C_1 be the constant in equation (9.13). If*

$$[\sigma(r)]^{\frac{t}{t-1}} \ge C_1(\log 2)(\log\log(r^{-1}))^{-1} \tag{9.15}$$

for each positive r in an arbitrarily small neighborhood of zero (clearly, $r < 1$ is assumed), then

$$\lim_{r\to 0}\omega(r) = 0. \tag{9.16}$$

In particular, the boundary point y is regular.

Proof. Fix a positive r_0 such that

$$\omega(r) \le (1 - 4^{-1}\eta_{n_0})\omega(4r) \quad \text{for all } r < r_0. \tag{9.17}$$

This choice is possible by (9.7). Further, define, for each non-negative index i,

$$r_i = 4^{-i}r_0. \tag{9.18}$$

Furthermore, set $n_0(i) = n_0(r_i)$. From (9.17) it follows that $\omega(r_i) \le (1 - 4^{-1}\eta_{n_0(i)})\omega(r_{i-1})$ for each $i \ge 1$, so

$$\omega(r_i) \le \prod_{k=1}^{i}(1 - 4^{-1}\eta_{n_0(k)})\omega(r_0). \tag{9.19}$$

From (9.14) and (9.15) it follows that

$$n_0(r) < 1 + (\log 2)^{-1}\log(\log(2r)^{-1}).$$

Hence

$$2^{n_0(k)+1} \le 4e^{\log(\log(2r)^{-1})} = 4\log(2r)^{-1},$$

where $r = r_k$. It follows that

$$\eta_{n_0(k)} \ge 4^{-1}(\log(2r_k)^{-1})^{-1} \quad \text{for all } k \ge 1. \tag{9.20}$$

Further, by appealing to (9.18), one gets

$$\eta_{n_0(k)} \ge \frac{1}{4(k\log 4 - \log(2r_0))}. \tag{9.21}$$

Since $\log(1-x) \le -x$, we get

$$\log(1 - 4^{-1}\eta_{n_0(k)}) \le \frac{-1}{4^2(k\log 4 - \log(2r_0))}.$$

So

$$\sum_{k=1}^{+\infty}\log(1 - 4^{-1}\eta_{n_0(k)}) = -\infty.$$

Hence

$$\prod_{k=1}^{+\infty}(1 - 4^{-1}\eta_{n_0(k)}) = 0. \tag{9.22}$$

Equation (9.16) follows from (9.19) and (9.22) . □

Remark 9.5. In the more general situation (2.3)–(2.5), one has to appeal to (4.1). In this case (9.7) is replaced by

$$\omega(r) \le (1 - 2^{-1})\omega(4r) + (c + \eta_{n_0}^{-1})r. \tag{9.23}$$

So, in the proof of Lemma 9.4, one has to consider also the event of the non-existence of a positive r_0 for which (9.17) holds.

Proof of Theorem 7.2. From Lemma 9.4 we conclude that the capacitary potentials $u_{\rho,m}(x)$ are continuous at the point y. Since $u_{\rho,m} = m$ on E_{ρ_0}, and $|E_\rho| > 0$ for each positive ρ, it must be $u_{\rho,m}(y) = m$. The continuity of the potentials $u_{\rho,-m}(x)$ at y, and $u_{\rho,-m}(y) = -m$, are proved in a totally similar way or, alternatively, by appealing to Remark 2.9.

Finally, the regularity of the boundary point y follows from Theorem 2.13. □

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