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A missed persistence property for the Euler equations and its effect on inviscid limits

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Abstract
We consider the problem of *strong* convergence, as the viscosity goes to zero, of the solutions to the three-dimensional evolutionary Navier–Stokes equations, under the Navier slip-type boundary condition (1.4), to the solution of the Euler equations under the no-penetration condition. In two dimensions, the above strong convergence holds in any smooth domain. Furthermore, in three dimensions, arbitrarily strong convergence results hold in the half-space case. In spite of the above results, recently we presented an explicit family of smooth initial data in the 3D sphere, for which the result fails. The result was proved as a by-product of the lack of time persistency for the above boundary condition under the Euler flow. Our aim here is to show a more general, and simpler proof, displayed in arbitrary, smooth domains.

Mathematics Subject Classification: 35Q30, 76D05, 76D09

1. Introduction

The general question is whether solutions of the Navier–Stokes equations in a domain Ω

$$\begin{cases} \partial_t \underline{u}^\nu + (\underline{u}^\nu \cdot \nabla)\underline{u}^\nu - \nu \Delta \underline{u}^\nu + \nabla \pi^\nu = 0, \\ \operatorname{div} \underline{u}^\nu = 0, \\ \underline{u}^\nu(0) = \underline{u}_0, \end{cases} \tag{1.1}$$

under suitable boundary conditions, tend to a solution of the Euler equations

$$\begin{cases} \partial_t \underline{u} + (\underline{u} \cdot \nabla)\underline{u} + \nabla \pi = 0, \\ \operatorname{div} \underline{u} = 0, \\ \underline{u}(0) = \underline{u}_0, \end{cases} \tag{1.2}$$

under the classical no-penetration condition

$$\underline{u} \cdot \underline{n} = 0, \tag{1.3}$$

as the viscosity ν goes to zero. Here, and in the following, $\underline{n}$ denotes the outer unit normal to $\partial\Omega$.

For the Cauchy problem, inviscid limit results in $C([0, T]; X)$ spaces, where X is the initial data's space, were proved in [23]. See also the more recent papers [4, 27]. We also refer the reader to the classical paper [15]. Concerning boundary value problems, the condition mostly used for the Navier–Stokes equations is the Dirichlet boundary condition $\underline{u} = 0$, which prescribes the adherence of the fluid to the solid wall. It is well known that the mismatch between $\underline{u}^\nu$ and $\underline{u}$ at the boundary results in boundary layer effects. For the study of boundary layer problems we refer, for instance, to the books [13, 32], to the survey paper [2] on instability and turbulence, and to [16]. For a wide review on singular limits in fluid mechanics, and bibliography, we refer to [28]. Although the analysis of the boundary layer phenomena and inviscid limit behaviour, in the case of Dirichlet conditions, has a long history, it is still far from being complete.

In particular, in the last few years, considerable interest of mathematicians involved in inviscid limit problems has been reserved to the Navier-boundary condition, see (6.1). Actually, in the last century several studies have shown that the no-slip condition might not always hold, and that there are flows for which they must be relaxed to allow some slip. Moreover, many papers show the limits of the no-slip conditions, and evidence for slip conditions from simulation studies.

We are interested in *strong convergence, up to the boundary* (see definition 2.3), as $\nu \to 0$, of the solutions $\underline{u}^\nu$ of the Navier–Stokes equations (1.1), under the boundary condition (introduced, for the first time, in [1])

$$\begin{cases} \underline{u}^\nu \cdot \underline{n} = 0, \\ \underline{\omega}^\nu \times \underline{n} = 0, \end{cases} \tag{1.4}$$

to the solution $\underline{u}$ of the Euler equations (1.2) under the boundary condition (1.3). We use the notation $\underline{\omega} = \operatorname{curl} \underline{u}$. We show that, in the three-dimensional case, strong inviscid limit results under the above boundary condition (1.4) are false in general. However, strong inviscid limit results hold in 3D for flat boundaries; see [5, 7, 35] (for the magnetohydrodynamic system see [37]). In particular, in [7], convergence is proved in arbitrarily strong topologies. It is worth noting that, on flat portions of the boundary, the Navier slip-type boundary condition (1.4) coincides with the classical Navier-boundary condition (see section 6). Actually, the arbitrarily strong convergence results proved in [7], some estimates proved for non-flat boundaries in [5, 6], and strong convergence results in the two-dimensional case (see, for instance, [5, 14, 24, 26]) strengthened the opposite conviction, at least in 'moderately strong' topologies. After some attempts to prove this conjecture, as, for instance, in [6, 10, 35], we realized that it is false. Indeed, in [8] we show that the result is false when Ω is the unit sphere, and for $C^\infty(\overline{\Omega})$ divergence-free initial data which satisfy the slip boundary conditions (1.4). For instance, the solutions to the Navier–Stokes equations do not converge in general to the solution of the Euler equations in $L^1(0, t_0; W^{s,q})$, for any arbitrarily small $t_0 > 0$, any $q \geqslant 1$ and any $s > 1 + \frac{1}{q}$. Solutions may be classical (even infinitely differentiable). In this paper we extend the above result to general domains, by presenting a different, much simpler proof.

Clearly, our negative result does not exclude inviscid limit results under additional assumptions on the data (see [11, 36]), or in weaker spatial norms, such as $H^1(\Omega)$. In [21], the authors show that a weak solution of the 3D Navier–Stokes equations converges to a solution of the Euler equations in $L^\infty(0, T; L^2(\Omega))$, under the Navier-boundary conditions with friction (6.1). For similar results also see [12]. Convergence in $L^2(0, T; H^1(\Omega))$ is obtained in the recent and very interesting paper [22], where the rigorous construction of a boundary layer expansion has been performed and the following orders of convergence are proved:

$$\|\underline{u}^\nu - \underline{u}\|_{L^2(0,T;L^2(\Omega))} \leqslant c\nu^{3/4}, \qquad \|\underline{u}^\nu - \underline{u}\|_{L^2(0,T;H^1(\Omega))} \leqslant c\nu^{1/4}.$$

Furthermore, in the recent paper [29], by appealing to anisotropic conormal Sobolev spaces, the authors prove the existence of strong solutions of the Navier–Stokes equations with uniform bounds on a time interval independent of viscosity. This allows one to obtain the following space-time uniform vanishing viscosity limit result:

$$\sup_{[0,T]} \|\underline{u}^\nu - \underline{u}\|_\infty \to 0.$$

All these results could be extended to our Navier slip-type boundary condition (1.4). In this regard we also refer to [20], where the authors get back the results of [22, 29], under some 'generalized' Navier-boundary conditions, which in particular include our condition (1.4).

Remark 1.1. In a first version of this paper, see [9], the proof of our results was considerably longer because, as remarked by a referee, we do not make use of the basic facts in differential geometry, used now in section 4, and basically we show everything again. Actually, our previous proof came out step by step, and this clouded the final overall view. However, we advise the interested reader to have a look at the proof in [9], even if the valuable simplification suggested by the referee leads to a much shorter, and elegant proof. We take this occasion to thank the referee for their welcome contribution.

2. Results

In the following Ω is a bounded open, connected set in $\mathbb{R}^3$, locally situated on one side of its boundary, a smooth manifold Γ. The boundary Γ may consist of a finite number of disjoint, connected components, Γ_j, $j = 0, 1, \ldots, m$, $m \geqslant 0$. Γ_0 denotes the 'external boundary'. If Γ is not connected we assume the typical existence of N mutually disjoint and transversal (regular) cuts, after which Ω becomes simply connected. See [18, 34] for details.

We start by the following definition.

Definition 2.1. *We say that a vector field* $\underline{u}_0$ *is* admissible *if it is smooth, divergence free in* $\overline{\Omega}$, *and satisfies the boundary condition*

$$\begin{cases} \underline{u}_0 \cdot \underline{n} = 0, \\ \underline{\omega}_0 \times \underline{n} = 0, \end{cases} \tag{2.1}$$

where $\underline{\omega}_0 = \operatorname{curl} \underline{u}_0$.

Next we define the *persistence property* with respect to the boundary conditions $\underline{\omega} \times \underline{n} = 0$.

Definition 2.2. *Let the initial data* $\underline{u}_0$ *be admissible, and consider the corresponding solution* $u(t)$ *to the Euler equations (1.2), under the boundary condition (1.3). Set* $\underline{\omega}(t) = \operatorname{curl} u(t)$. *We say that the persistence property holds for the initial data* $\underline{u}_0$, *if there is some* $t_0 > 0$ *(which may depend on* $\underline{u}_0$*) such that* $\underline{\omega}(t) \times \underline{n} = 0$ *on* Γ, *for each* $t \in (0, t_0)$.

For the existence of strong, unique, local in time solutions to the Euler equations see, for instance, [33].

We denote by κ_j, $j = 1, 2$, the principal curvatures at boundary points. Hence, $K = \kappa_1\kappa_2$, is the Gaussian curvature.

We set

$$\Sigma = \{s \in \Gamma : K \neq 0\}. \tag{2.2}$$

It is worth noting that, for Ω as above, Σ is never empty. Mostly, Σ coincides with Γ itself.

The following are the main results of the paper.

Theorem 2.1. *Let $\underline{u}_0$ be admissible, and assume that there is some point $x_0 \in \Sigma$ such that*

$$\underline{\omega}_0(x_0) \neq 0. \tag{2.3}$$

Then, for the initial data $\underline{u}_0$, the persistence property fails.

The following result, together with theorem 2.1, shows that the persistence property fails in general.

Proposition 2.1. *Let $\gamma_j \subset \Gamma_j$, $j = 0, 1, ..., m$, be arbitrary closed, regular lines. There are admissible vector fields $\underline{u}_0$ such that (2.3) holds everywhere in Γ except, at most, on the above lines.*

Finally, theorem 2.2 shows that strong inviscid limit results, in arbitrary three-dimensional domains, fail in general. We introduce the following preliminary definition.

Definition 2.3. *By* strong convergence *we mean any (sufficiently strong) convergence in $(0, T) \times \Omega$ such that if $\underline{u}^\nu$ converges to $\underline{u}$ with respect to this convergence, and if $\underline{\omega}^\nu \times \underline{n} = 0$ on Γ, then necessarily $\underline{\omega} \times \underline{n} = 0$ on $(0, T) \times \Gamma$. The* strong inviscid limit *is defined accordingly.*

Examples of strong convergence in the above sense are, for instance, convergence in $L^1(0, T; W^{s,q})$, for some $q > 1$, and some $s > 1 + \frac{1}{q}$, and convergence in $L^1(0, T; W^{2,1})$. Recall that $L^1(0, T)$ convergence implies a.e. convergence in $(0, T)$, for suitable 'subsequences'.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, regular, open connected set. Let the admissible initial data $\underline{u}_0$ be given. If the persistence property fails then strong convergence of the solutions of the Navier–Stokes equations (1.1), (1.4) to the solution of the Euler equations (1.2), (1.3) fails in any positive time interval.*

3. A preliminary result: proof of theorem 2.2

In [8], section 2, the following result was proved.

Theorem 3.1. *Let the admissible initial data $\underline{u}_0$ be given. Then:*

(a) If a strong inviscid limit result holds in $(0, t_0)$, then the persistence property holds in the same time interval.

(b) If the persistence property holds, then necessarily

$$[(\underline{\omega}_0 \cdot \nabla)\underline{u}_0 - (\underline{u}_0 \cdot \nabla)\underline{\omega}_0] \times \underline{n} = 0 \tag{3.1}$$

everywhere on Γ.

It is clear that a strong inviscid limit result in $(0, t_0)$ immediately implies the persistence property in $(0, t_0)$, as already remarked in [35, corollary 8.3]. This proves the above statement (a), which is just theorem 2.2.

Assume now the persistence property. The application of the operator curl to the first equation (1.2) leads to the well-known Euler vorticity evolution equation

$$\partial_t \underline{\omega} + (\underline{u} \cdot \nabla)\underline{\omega} - (\underline{\omega} \cdot \nabla)\underline{u} = 0. \tag{3.2}$$

External multiplication of (3.2) by $\underline{n}$ gives

$$\partial_t(\underline{\omega} \times \underline{n}) + [(\underline{u} \cdot \nabla)\underline{\omega} - (\underline{\omega} \cdot \nabla)\underline{u}] \times \underline{n} = 0. \tag{3.3}$$

Since the persistency property holds, the time derivative in the above equation must vanish on Γ, at time $t = 0$. Hence, the second term must verify this same property at time $t = 0$, which means that $\underline{u}_0$ must satisfy (3.1). This proves statement (b).

It follows from theorem 3.1 that, in order to prove the failure of the persistence property and, *a fortiori*, that of strong inviscid limit results, it is sufficient to show the existence of admissible vector fields $\underline{u}_0$ for which (3.1) fails.

4. Proof of theorem 2.1

We recall some basic definitions and results in differential geometry. The Gauss map

$$\underline{n} : \Gamma \to S^2$$

maps a regular surface Γ in $\mathbb{R}^3$, with an orientation $\underline{n}$, to the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. It associates with any point P on Γ the normal vector to Γ at P (and translates it to the origin of $\mathbb{R}^3$).

The differential $\nabla \underline{n}_P$ of the Gauss map at a point P is called the *shape operator*. It is a linear map acting from the tangent space to Γ in P, $T_P(\Gamma)$, to the tangent space to S^2 in $\underline{n}(P)$. Since these two planes are parallel, $\nabla \underline{n}_P$ can be looked on as an application from $T_P(\Gamma)$ into itself:

$$\nabla \underline{n}_P : T_P(\Gamma) \to T_P(\Gamma).$$

We define the *Gaussian curvature* K of Γ at P as the determinant of the shape operator. See [17, chapter III, definition 6]. Note that $\frac{\partial \underline{n}}{\partial \underline{\tau}_j} = (\nabla \underline{n})\underline{\tau}_j = \kappa_j \underline{\tau}_j$, if $\underline{\tau}_j$, $j = 1, 2$, are the (unit) principal directions and κ_j the corresponding principal curvatures. Hence, with respect to the axis $\{\underline{\tau}_1, \underline{\tau}_2\}$, the matrix $(\nabla \underline{n})_P$ has the representation $\mathrm{diag}\{\kappa_1, \kappa_2\}$. From the above representation it follows that the shape operator is a self-adjoint linear map and its determinant is given by $K = \kappa_1\kappa_2$ (see [17, chapter III, section 2]). Finally, recall that Σ denotes the subset of boundary points where the product $\kappa_1\kappa_2$ does not vanish (see (2.2)).

In the following we use the following notation:

$$(\nabla \underline{a})\underline{b} =: b_i \frac{\partial a_j}{\partial x_i}, \qquad (\nabla \underline{a})^{\mathrm{T}}\underline{b} =: b_i \frac{\partial a_i}{\partial x_j}.$$

In particular, $(\nabla \underline{n})\underline{u}_0 = (\underline{u}_0 \cdot \nabla)\underline{n}$.

Proposition 4.1. *Let* $\underline{u}_0$ *be an admissible vector field and let* $\underline{\omega}_0 = \alpha \underline{n}$ *denote* curl $\underline{u}_0$ *on* Γ. *Then, the identity*

$$[(\underline{u}_0 \cdot \nabla)\underline{\omega}_0 - \underline{\omega}_0 \cdot \nabla)\underline{u}_0] \times \underline{n} = 2\alpha[(\nabla \underline{n})\underline{u}_0] \times \underline{n} \tag{4.1}$$

holds on Γ.

Proof. Note that, since $\underline{\omega}_0$ is parallel to $\underline{n}$ on Γ, one has $\underline{\omega}_0 = \alpha \underline{n}$ on Γ, for some scalar function α. In the following we assume that the normal $\underline{n}$ is smoothly extended to a neighbourhood of Γ. Further, since in equation (4.1) the operator $(\underline{u}_0 \cdot \nabla)$ is a tangential derivative, only the trace of $\underline{\omega}_0$ on Γ is significant here. Hence we assume that $\underline{\omega}_0 = \alpha \underline{n}$, in a neighbourhood of Γ.

On the other hand, a straightforward computation shows that for each $h = 1, 2, 3$, $(\underline{\omega} \times \underline{n})_h = (\partial_j u_h - \partial_h u_j) n_j$. Hence,

$$\underline{\omega}_0 \times \underline{n} = \left(\nabla \underline{u}_0 - (\nabla \underline{u}_0)^{\mathrm{T}}\right) \underline{n}. \tag{4.2}$$

By appealing to the assumption $\underline{\omega}_0 = \alpha \underline{n}$ and to (4.2) it follows that

$$\begin{aligned}(\underline{\omega}_0 \cdot \nabla)\underline{u}_0 &= \alpha(\underline{n} \cdot \nabla)\underline{u}_0 = \alpha(\nabla \underline{u}_0)\underline{n} = \alpha[\nabla \underline{u}_0 - (\nabla \underline{u}_0)^{\mathrm{T}}]\underline{n} \\ &\quad + \alpha(\nabla \underline{u}_0)^{\mathrm{T}}\underline{n} = \alpha \underline{\omega}_0 \times \underline{n} + \alpha(\nabla \underline{u}_0)^{\mathrm{T}}\underline{n} = \alpha(\nabla \underline{u}_0)^{\mathrm{T}}\underline{n},\end{aligned} \tag{4.3}$$

where, in the last step, we have used that $\underline{\omega}_0 \times \underline{n} = 0$. By appealing to

$$\nabla(\underline{u}_0 \cdot \underline{n}) = (\nabla \underline{u}_0)^{\mathrm{T}} \underline{n} + (\nabla \underline{n})^{\mathrm{T}} \underline{u}_0 \tag{4.4}$$

and to $(\nabla \underline{n})^{\mathrm{T}} \underline{u}_0 = (\nabla \underline{n}) \underline{u}_0$, we obtain

$$(\underline{\omega}_0 \cdot \nabla) \underline{u}_0 = \alpha [\nabla(\underline{u}_0 \cdot \underline{n}) - (\nabla \underline{n}) \underline{u}_0].$$

As $(\underline{u}_0 \cdot \nabla) \underline{\omega}_0 = (\underline{u}_0 \cdot \nabla \alpha) \underline{n} + \alpha (\underline{u}_0 \cdot \nabla) \underline{n}$, it follows that

$$(\underline{u}_0 \cdot \nabla) \underline{\omega}_0 - (\underline{\omega}_0 \cdot \nabla) \underline{u}_0 = \alpha [2 (\nabla \underline{n}) \underline{u}_0 - \nabla(\underline{u}_0 \cdot \underline{n})] + (\underline{u}_0 \cdot \nabla \alpha) \underline{n}. \tag{4.5}$$

Since the first and third terms on the right-hand side of (4.5) are parallel to $\underline{n}$, the thesis follows. □

Next we prove the following result.

Proposition 4.2. *Let $\underline{u}_0$ be admissible, and assume that*

$$[(\underline{u}_0 \cdot \nabla) \underline{\omega}_0 - \underline{\omega}_0 \cdot \nabla) \underline{u}_0] \times \underline{n} = 0 \qquad \text{on } \Sigma. \tag{4.6}$$

Then,

$$\underline{\omega}_0 = 0 \qquad \text{on } \Sigma. \tag{4.7}$$

Proof. Suppose, by contradiction, that $\underline{\omega}_0 \neq 0$ in some point $x_0 \in \Sigma$. Since Σ is open, $\underline{\omega}_0 \neq 0$ in a neighbourhood $U \subset \Sigma$ of x_0. From $\underline{\omega}_0 = \alpha \underline{n}$ on Γ, it follows that $\alpha \neq 0$ on U. Hence, by assumption (4.6) and the identity (4.1), we have $[(\nabla \underline{n}) \underline{u}_0] \times \underline{n} = 0$. On the other hand, $(\nabla \underline{n}) \underline{u}_0$ is tangential since $\underline{u}_0$ is tangential. Hence $(\nabla \underline{n}) \underline{u}_0 = 0$. Recalling that the determinant of the shape operator does not vanish on Σ, it follows necessarily $\underline{u}_0 = 0$ on U. Then, in particular, $\underline{u}_0$ is normal to Γ on U, hence its curl is tangential on U. Taking into account that $\underline{\omega}_0$ is parallel to $\underline{n}$, this would imply that $\underline{\omega}_0 = 0$ on U. □

Theorem 2.1 follows from part (b) of theorem 3.1 together with proposition 4.2.

5. Proof of theorem 2.1

Let $\beta(s)$ be a smooth real function on Γ such that

$$\int_{\Gamma_j} \beta(s) \, \mathrm{d}s = 0, \qquad j = 0, \ldots, m, \tag{5.1}$$

and define

$$\underline{b}(s) = \beta(s) \underline{n}. \tag{5.2}$$

Clearly,

$$\int_{\Gamma_j} \underline{b}(s) \cdot \underline{n} \, \mathrm{d}s = 0, \qquad j = 0, \ldots, m. \tag{5.3}$$

It is well known that, under assumption (5.3), there exists in $\overline{\Omega}$ an extension $\underline{b}(x)$ of $\underline{b}(s)$ such that

$$\operatorname{div} \underline{b}(x) = 0. \tag{5.4}$$

See, for instance, [19, chapter III] and [25, problem I.2.1, chapter III]. On the other hand, it is well known that, under the constraints (5.3) and (5.4), the linear problem

$$\begin{cases} \operatorname{div} \underline{a} = 0, & \\ \operatorname{curl} \underline{a} = \underline{b}, & \text{in} \quad \Omega, \\ \underline{a} \cdot \underline{n} = 0, & \text{on} \quad \Gamma \end{cases} \tag{5.5}$$

is always solvable. See for instance [18] or [34]. This existence result is sufficient for our purposes. However, we recall that the solution is unique if Ω is simply connected and that, in general, the kernel of the related linear map (set $\underline{b} = 0$ in (5.5)) has dimension N (see section 2). For a very clear and complete treatment of this and related problems we refer the reader to section 1 in [18].

The above approach leads immediately to the following result.

Lemma 5.1. *For each $\beta(s)$ as above, the vector field $\underline{a}$ is admissible.*

In fact (5.2), together with the second equation (5.5), implies the second boundary condition (2.1).

Finally, to prove proposition 2.1 we fix a real function $\beta(s)$ satisfying (5.1), and different from zero everywhere outside the lines γ_j, we construct $\underline{a}$ as above, and finally we set $\underline{u}_0 = \underline{a}$.

6. On Navier-boundary conditions

On flat portions of the boundary, the slip boundary condition (1.4) coincides with the classical Navier-boundary condition. This condition already appeared in the paper of Navier [31], and was derived by Maxwell [30] in the kinetic theory of gases. A very interesting discussion on this, and related problems, can be found in the recent paper [3].

Denote by $\underline{t}$ the stress vector, defined by $\underline{t} := \mathcal{T} \cdot \underline{n}$, where $\mathcal{T}$ is the stress tensor given by

$$\mathcal{T} = -\pi I + \nu(\nabla\underline{u} + \nabla\underline{u}^{\mathrm{T}}),$$

and $\underline{n}$ is the exterior unit normal to $\partial\Omega$. We also denote by $\underline{\tau}$ a generic unit tangential vector. Then the Navier-boundary condition reads as

$$\begin{cases} \underline{u} \cdot \underline{n} = 0, \\ \underline{t} \cdot \underline{\tau} + \beta\nu\underline{u} \cdot \underline{\tau} = 0, \end{cases}$$

where the constant $\beta \geqslant 0$ is the friction parameter. The above condition is usually written in a more explicit form as

$$\begin{cases} \underline{u} \cdot \underline{n} = 0, \\ 2[(\mathcal{D}\underline{u})\underline{n}] \cdot \underline{\tau} + \beta\underline{u} \cdot \underline{\tau} = 0, \end{cases} \tag{6.1}$$

using the obvious identity

$$\underline{t} \cdot \underline{\tau} = 2\nu[(\mathcal{D}\underline{u})\underline{n}] \cdot \underline{\tau}.$$

If one neglects the friction coefficient, the above condition reduces to

$$\begin{cases} \underline{u} \cdot \underline{n} = 0, \\ \underline{t} \cdot \underline{\tau} = 0. \end{cases} \tag{6.2}$$

It prescribes no penetration and the vanishing of the shear stress on the boundary. In the literature, this last condition is often referred to as the slip or free-slip or, even, stress-free boundary condition. It is important to note the following identity:

$$\underline{t} \cdot \underline{\tau} = \nu(\underline{\omega} \times \underline{n}) \cdot \underline{\tau} - 2\nu\underline{u} \cdot \frac{\partial\underline{n}}{\partial\underline{\tau}}. \tag{6.3}$$

If $\underline{\tau}$ is a principal direction, then $\frac{\partial\underline{n}}{\partial\underline{\tau}} = \kappa_\tau\underline{\tau}$, where κ_τ is the principal curvature in the $\underline{\tau}$ direction, positive if the corresponding centre of curvature lies inside Ω. Relation (6.3) follows by a direct computation. Relation (6.3) implies that, on flat portions of the boundary, the boundary conditions (1.4) coincide with (6.2) (hence it coincides with (6.1) for $\beta = 0$).

Note that our counter-example in [8] and the results presented here do not exclude that strong inviscid limit results hold under the Navier-boundary condition (6.2) in the non-flat boundary case.

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