DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **3**, Number **2**, June **2010**

doi:10.3934/dcdss.2010.3.231

pp. **231–236**

A CHALLENGING OPEN PROBLEM: THE INVISCID LIMIT UNDER SLIP-TYPE BOUNDARY CONDITIONS

Hugo Beirão da Veiga

Dipartimento di Matematica Applicata "U. Dini"
Via Buonarroti, 1/C -56127 Pisa, Italy

Dedicated to Vsevolod A. Solonnikov on the occasion of his 75th birthday.

Abstract. In these notes we present some results proved in the forthcoming paper [3]. We consider the $3-D$ evolutionary Navier-Stokes equations with a Navier slip-type boundary condition, and study the problem of the *strong convergence* ($k > 1 + \frac{3}{p}$, see below) of the solutions, as the viscosity goes to zero, to the solution of the Euler equations under the zero-flux boundary condition. This problem is still open, except in the case of flat boundaries. However, if we drop the convective terms (Stokes problem), the inviscid, strong limit result holds. The cause of this different behavior is quite subtle.

1. **Introduction.** Consider the Navier-Stokes equations

$$\begin{cases} \partial_t u^\nu + (u^\nu \cdot \nabla)\, u^\nu - \nu\, \Delta\, u^\nu + \nabla\, \pi = 0, \\ div\, u^\nu = 0\,, \\ u^\nu(0) = u_0\,, \end{cases} \tag{1.1}$$

under the slip-type boundary condition

$$\begin{cases} (u^\nu \cdot \underline{n})_{|\Gamma} = 0, \\ \omega^\nu \times \underline{n} = 0\,. \end{cases} \tag{1.2}$$

We are mainly interested in studying the strong convergence up to the boundary, as $\nu \to 0$, of the solutions u^ν to the solution u of the Euler equations

$$\begin{cases} \partial_t\, u + (u \cdot \nabla)\, u + \nabla\, \pi = 0, \\ div\, u = 0\,, \\ u(0) = u_0\,, \end{cases} \tag{1.3}$$

under the zero-flux boundary condition $(u \cdot \underline{n})_{|\Gamma} = 0$. *Strong inviscid limit results* means here results obtained from an uniform estimate (i.e., independent of ν, and of some $T > 0$) of the solutions of problem (1.1), (1.2) in an $L^\infty(0, T; W^{k,p}(\Omega))$ space, for some k and p such that $k > 1 + \frac{3}{p}$.

The following two theorems are known.

Theorem 1.1. *(see [9]). Assume that $u_0 \in W^{3,2}(\Omega)$, is solenoidal and satisfies (1.2). Then*

$$\begin{cases} u^\nu \to u \quad in \quad L^p(0, T_0; W^{3,2}(\Omega))\,, \quad \textit{for each } p \in [1, \infty[\,, \\ u^\nu \to u \quad in \quad C([0, T_0]; W^{2,2}(\Omega))\,. \end{cases} \tag{1.4}$$

2000 *Mathematics Subject Classification.* 35Q30, 76D05, 76D09.

Key words and phrases. Stokes and Navier-Stokes equations, inviscid limit, slip boundary conditions.

In the next two theorems Ω is a cubic domain, and the boundary condition (1.2) is imposed only on two opposite faces. On the other faces space-periodicity is assumed, as a device to avoid unessential technical difficulties. The following result holds.

Theorem 1.2. *(see [2]). Let Ω be the above "cubic domain". Assume that $p > \frac{3}{2}$, and that $u_0 \in W^{3,\,p}(\Omega)$ is solenoidal and satisfies (1.2). Then*

$$\begin{cases} u^\nu \rightharpoonup u \quad in \quad L^\infty(0,\,T_0;\,W^{3,\,p}(\Omega)) \quad weak-*\,, \\ u^\nu \to u \quad in \quad C([0,\,T_0];\,W^{s,\,p}(\Omega))\,, \quad for\ each\ s < 3\,, \\ \partial_t\,u^\nu \to \partial_t\,u \quad in \quad L^\infty(0,\,T_0;\,W^{1,\,p}(\Omega))\,, \\ \partial_t\,u^\nu \to \partial_t\,u \quad in \quad L^p(0,\,T_0;\,W^{1,\,3\,p}(\Omega))\,. \end{cases} \tag{1.5}$$

In reference [3] we extend the above theorem 1.2 to arbitrarily large values of k and p. We prove the following result:

Theorem 1.3. *Let Ω be the above "cubic domain". Assume that $p \geq 2$ and that the initial data u_0 belongs to $W^{k,\,p}(\Omega)$, is divergence free, and satisfies the boundary conditions (1.2). Then*

$$\begin{cases} u^\nu \rightharpoonup u \quad in \quad L^\infty(0,\,T_0;\,W^{k,\,p}(\Omega)) \quad weak-*\,, \\ \\ u^\nu \to u \quad in \quad C([0,\,T_0];\,W^{s,\,p}(\Omega))\,, \quad for\ each\ s < k\,. \end{cases} \tag{1.6}$$

Further,

$$\partial_t\,u^\nu \to \partial_t\,u \quad in \quad L^\infty(0,\,T_0;\,W^{k-\,2,\,p}(\Omega)) \tag{1.7}$$

and, if $p \geq 2$,

$$\partial_t\,u^\nu \to \partial_t\,u \quad in \quad L^p(0,\,T_0;\,W^{k-\,2,\,3\,p}(\Omega))\,. \tag{1.8}$$

The proofs presented in references [9], [2] and [3] require flat-boundaries. Hence the strong limit problem remains open in the presence of smooth boundaries. The main obstacle consists of the boundary integrals resulting from an integration by parts related to the viscous term (these integrals vanish on flat portions of the boundary). We present here an attempt to extend our approach from flat to non-flat boundaries. We partially succeed in this attempt, insofar as our approach works well for Stokes problems. The reason for these two distinct behaviors is subtle. It suffices to say that the obstacle resulting from the addition of the convective term is not due to the related volume integral (the classical "trilinear form", when $p = 2$), but to the destabilizing effect of the convective term on boundary integrals, which already occur in the Stokes framework. In other words, the main obstacle is due to the combination of viscosity with convection, near the boundary. Concerning the Stokes inviscid limit we show the following result (note that here the solution of the limit problem is time independent, more precisely, $u(t,\,x) = u_0(x)$).

Theorem 1.4. *(*Stokes inviscid limit*). Drop the convective terms in equations (1.1) and (1.3), and assume that Ω is of class C^2. Then*

$$\begin{cases} u^\nu \rightharpoonup u \quad in \quad L^\infty(0,\,T_0;\,W^{2,\,p}(\Omega)) \quad weak-*\,, \\ u^\nu \to u \quad in \quad C([0,\,T_0];\,W^{s,\,p}(\Omega))\,, \quad for\ each\ s < 2\,, \\ \\ \partial_t\,u^\nu \to 0 \quad in \quad L^\infty(0,\,T_0;\,L^p(\Omega)) \\ \partial_t\,u^\nu \to 0 \quad in \quad L^p(0,\,T_0;\,L^{3\,p}(\Omega))\,. \end{cases} \tag{1.9}$$

It is worth noting that, in the presence of the convective term, we are able to lower the order of the derivatives occurring in the above troublesome boundary integrals, see (3.5). However this seems insufficient to prove a strong inviscid limit result.

In [3] we also show, as a simple by-product of our estimates, some regularity results for the solutions to the Navier-Stokes equations under the boundary conditions (1.2) as, for instance, the following theorem.

Theorem 1.5. *(Regularity, Navier-Stokes). Let Ω be a regular open, bounded, set in $\mathbb{R}^3$ and let $\nu > 0$ be given. Assume, for convenience, that $p \geq 2$. Let $u_0 \in W^{2,\,p}(\Omega)$ be a given divergence free vector field satisfying (1.2). Then there is a $T > 0$ such that a unique solution $u = u_\nu$ to the problem (1.1), (1.2) exists in $[0,\,T]$. Moreover,*

$$u \in L^\infty(0,\,T;\,W^{2,\,p}(\Omega)) \tag{1.10}$$

and

$$\begin{cases} |\,\Delta\, u\,|^{\frac{p-2}{2}}\,|\,\nabla\,(\Delta\, u)\,| \in L^2(0,\,T;\,L^2(\Omega)) \\ \\ |\,\Delta\, u\,|^{\frac{p}{2}} \in L^2(0,\,T;\,W^{1,\,2}(\Omega))\,. \end{cases} \tag{1.11}$$

In particular $u \in L^2(0,\,T;\,W^{3,\,2}(\Omega))$.

The above results hold in presence of an external force $f \in L^p(0,\,T;\,W^{2,\,p}(\Omega))$.

The proof of this last theorem can be easily extended to the usual slip boundary condition, see [8] and [1], since this last boundary condition differs from (1.2) only by lower order terms. Note that these two conditions coincide on flat portions of the boundary.

Remark 1.1. *It is worth noting that in reference [4] G. Grubb proves very general, strong, and complete, regularity results in L^p spaces for solutions to the Navier-Stokes equations under nonhomogeneous, boundary conditions. The above reference follows previous work in collaboration with V.A. Solonnikov (see, for instance, [5], and references in [4]). This theory has its very beginning in some classical papers by V.A. Solonnikov as, for instance, [7]. However the proofs in reference [4] are particularly involved. Hence, a simpler approach is desirable, even if less general. In reference [3] we just give a contribution in this direction.*

2. **A basic estimate.**

In order to show the main problem in a simple situation, we consider here the $W^{2,\,p}$ approach, $p > 3$. By setting $\omega = curl\, u$, it follows that

$$\partial_t\, \omega - \nu\, \Delta\, \omega + (u \cdot \nabla)\, \omega - (\omega \cdot \nabla)\, u = 0\,. \tag{2.1}$$

Further, by setting $\zeta = curl\, \omega$, it follows that

$$\partial_t\, \zeta - \nu\, \Delta\, \zeta + (u \cdot \nabla)\, \zeta + \sum c\,(\,D\, u\,)\,(\,D\, \omega\,) = 0\,. \tag{2.2}$$

Multiplication by $|\,\zeta\,|^{p-2}\,\zeta$, integration in Ω, yields (see [6] and [1])

$$\begin{aligned} &\tfrac{1}{p}\,\tfrac{d}{d\,t}\|\,\zeta\|_p^p + \tfrac{\nu}{2}\int_\Omega |\,\zeta\,|^{p-2}\,|\,\nabla\,\zeta|^2\,dx + c\,\nu\int_\Omega \left|\,\nabla\,|\,\zeta\,|^{\frac{p}{2}}\,\right|^2 dx \\ &\leq c\int_\Omega |\,\nabla\, u\,|\,|\,\nabla\,\omega\,|\,|\,\zeta\,|^{p-1}\,dx + c\,\nu\left|\,\int_\Gamma |\,\zeta\,|^{p-2}\,(\partial_i\,\zeta_j)\,n_i\,\zeta_j\,d\Gamma\,\right|. \end{aligned} \tag{2.3}$$

On the other hand

$$(\partial_i \zeta_j)\, n_i \zeta_j = (curl\, \zeta) \times n \cdot \zeta + \nabla (\zeta \cdot n) \cdot \zeta - (\partial_j n_i)\, \zeta_i \zeta_j\, .$$

Moreover,

$$\omega \times \underline{n} = 0 \Rightarrow \zeta \cdot \underline{n} = 0 \Rightarrow \nabla (\zeta \cdot n) \cdot \zeta = 0$$

on Γ. Hence,

$$\begin{aligned} &\frac{1}{p} \frac{d}{dt} \| \zeta \|_p^p + c\, \nu \int_\Omega \left| \nabla | \zeta |^{\frac{p}{2}} \right|^2 dx \\ &\leq c \int_\Omega |\nabla u|\, |\nabla \omega|\, |\zeta|^{p-1}\, dx + c\, \nu \left| \int_\Gamma |\zeta|^{p-2} (curl \zeta) \times \underline{n} \cdot \zeta\, d\Gamma \right| \\ &+ c\, \nu \left| \int_\Gamma |\zeta|^{p-2} (\partial_j n_i)\, \zeta_i \zeta_j\, d\Gamma \right| . \end{aligned} \tag{2.4}$$

The second boundary integral on the right hand side of (2.4) can be estimated by terms in the left hand side, *uniformly* with respect to ν, by appealing to the following result.

Lemma 2.1. *To each $\epsilon > 0$ there corresponds a positive C_ϵ such that*

$$\|\zeta\|_{p,\Gamma}^p \leq \epsilon \, \|\nabla |\zeta|^{\frac{p}{2}} \|_{2;\Omega}^2 + C_\epsilon \, \|\zeta\|_{p,\Omega}^p \, . \tag{2.5}$$

Hence, by fixing a sufficient small, positive, ϵ, it follows that

$$\begin{aligned} &\frac{1}{p} \frac{d}{dt} \| \zeta \|_p^p + c\, \nu\, \| \nabla | \zeta |^{\frac{p}{2}} \|_2^2 \\ &\leq \int_\Omega |\nabla u|\, |\nabla \omega|\, |\zeta|^{p-1}\, dx + \nu \left| \int_\Gamma |\zeta|^{p-2} (curl\, \zeta) \times \underline{n} \cdot \zeta\, d\Gamma \right| \\ &+ c\, \nu \| \zeta \|_{p,\Omega}^p \, . \end{aligned} \tag{2.6}$$

Control of terms of the form $c \| \zeta \|_p^p$ is obvious. Control of the “convective integral” is proved as in [2]. So, the obstacle to overcome, is just the boundary integral on the right hand side of equation (2.6).

3. **The inviscid limit.** In the Stokes case (recall that $-\Delta \omega = curl\, \zeta$), it follows that

$$\partial_t \omega + \nu\, curl\, \zeta = 0\, .$$

Hence, due to the boundary condition $\omega \times \underline{n} = 0$, one shows that

$$\underline{n} \times curl\, \zeta = 0 \tag{3.1}$$

on Γ. So, the boundary integral

$$\int_\Gamma |\zeta|^{p-2} (curl\, \zeta) \times \underline{n} \cdot \zeta\, d\Gamma$$

vanishes. This leads to the results stated in theorem 1.4.

In the Navier-Stokes framework, equations $\underline{n} \times curl\, \zeta = 0$ does not hold in general. However, the following result is available. To prove it, we express all the differential operators and equations in a suitable local system of curvilinear coordinates. See [3].

Theorem 3.1. *Let the boundary Γ be a surface of class C^k, $k \geq 2$. Then*

$$\left| curl\, (u \times \omega) \times \underline{n} \right| \leq H\, |u|\, |\omega|\, , \tag{3.2}$$

on Γ, where H depends, pointwisely, at most on the scale functions h_j, and on their first and second order derivatives.

Equation (2.1) can be written in the form

$$\partial_t \omega + \nu \, curl \zeta + curl\,(u \times \omega) = 0 . \tag{3.3}$$

Since $(\partial_t \omega) \times \underline{n} = 0$ on Γ, it follows that

$$- \nu\, (curl\, \zeta\,) \times \underline{n} = curl\,(u \times \omega) \times \underline{n} . \tag{3.4}$$

So, we have

Corollary 3.1. *The estimate*

$$\nu \left| (curl\, \zeta\,) \times \underline{n} \right| \leq H\,|u|\,|\,\omega\,| \tag{3.5}$$

holds, pointwisely, on Γ*. In particular*

$$\nu \Big| \int_\Gamma |\,\zeta\,|^{p-2}\, (curl\,\zeta) \times n \cdot \zeta\, d\Gamma \Big| \leq c \int_\Gamma |\,\zeta\,|^{p-1}\, |\,u\,|\,|\,\omega\,|\, d\Gamma . \tag{3.6}$$

Roughly speaking, equation (3.6) allows us to lower by two the order of the higher derivative occurring in the boundary integral, however, at the *high* cost of losing multiplication by ν. From equations (2.6) and (3.6) one has

$$\begin{aligned} &\tfrac{1}{p} \; \tfrac{d}{dt} \|\,\zeta\|_p^p + c\,\nu\, \|\,\zeta\,\|_{3\,p}^p + c\,\nu \int_\Omega |\zeta|^{p-2}\, |\,\nabla\,\zeta|^2\, dx \\ &\quad + c\,\nu \int_\Omega \left|\, \nabla\,|\,\zeta\,|^{\frac{p}{2}}\, \right|^2 dx \leq \int_\Omega |\,\nabla\, u\,|\,|\,\nabla\,\omega\,|\,|\,\zeta\,|^{p-1}\, dx \\ &\quad + c\,\nu\, \|\,\zeta\,\|_p^p + c \int_\Gamma |\,\zeta\,|^{p-1}\, |\,u\,|\,|\,\omega\,|\, d\Gamma . \end{aligned} \tag{3.7}$$

Unfortunately, this estimate seems not sufficient to prove the desired Navier-Stokes strong inviscid limit result in the case of non-flat boundaries.

4. **A regularity result.** We prove Theorem 1.5 by appealing to (3.7). For convenience, we consider the case in which the external forces vanish. It is sufficient to show that

$$\int_\Gamma |\,\zeta\,|^{p-1}\, |\,u\,|\,(\,|\,u\,| + |\nabla\, u\,|\,)\, d\Gamma \leq \epsilon\, \|\, \nabla\, |\zeta\,|^{\frac{p}{2}}\, \|_2^2 + c\, \|\,\zeta\,\|_{p,\,\Omega}^{2\,p} + C_\epsilon\, \|\,\zeta\,\|_{p,\,\Omega}^{p+1} . \tag{4.1}$$

Since $p > \frac{3}{2}$ it follows that

$$\|\,u\,\|_{\infty,\,\Gamma} \leq c\, \|\,u\,\|_{2,\,\Omega} \leq c\, \|\,\zeta\,\|_{p,\,\Omega} .$$

On the other hand,

$$\|\,\nabla\,u\,\|_{p,\,\Gamma} \leq c\, \|\nabla\,u\,\|_{1,\,p\,\Omega} \leq \|\,\zeta\,\|_{p,\,\Omega} .$$

Hence, by appealing to Hőlder's inequality, we show that

$$\int_\Gamma |\,\zeta\,|^{p-1}\, |\,u\,|\,(\,|\,u\,| + |\nabla\, u\,|\,)\, d\Gamma \leq c\, \|\,\zeta\,\|_{p,\,\Gamma}^{p-1}\, \|\,\zeta\,\|_{p,\,\Omega}^2 .$$

By taking into account (2.5), (4.1) follows.

REFERENCES

[1] H. Beirão da Veiga, *Existence and asymptotic behaviour for strong solutions of the Navier–Stokes equations in the whole space*, Indiana University Math. Journal, **36** (1987), 149–166.

[2] H. Beirão da Veiga and F. Crispo, *Sharp inviscid limit results under Navier type boundary conditions. An L^p theory*, J. Math. Fluid Mech., (2009), DOI 10.1007/s00021-009-0295-4.

[3] H. Beirão da Veiga and F. Crispo, *Concerning the $W^{k,p}-$inviscid limit for $3-D$ flows under a slip boundary condition*, J. Math. Fluid Mech., DOI 10.1007/s00021-009-0012-3.

[4] G. Grubb, *Nonhomogeneous time-dependent Navier-Stokes problems in L_p Sobolev spaces*, Diff. Int. Eq., **8** (1995), 1013–1046.

[5] G. Grubb and V. A. Solonnikov, *boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods*, Math. Scand., **69** (1991), 217–290.

[6] S. Rionero and G. P. Galdi, *The weight function approach to uniqueness of viscous flows in unbounded domains*, Arch. Rat. Mech. Anal., **69** (1979), 37–52.

[7] V. A. Solonnikov, *Estimates for solutions of nonstationary Navier-Stokes systems*, Zap. Nauch. Sem. LOMI, **38** (1973), 213-317; J. Soviet Math., **8** (1977), 467–529.

[8] V. A. Solonnikov and V. E. Ščadilov, *On a boundary value problem for a stationary system of Navier-Stokes equations*, Proc. Steklov Inst. Math., **125** (1973), 186–199.

[9] Y. Xiao and Z. Xin, *On the vanishing viscosity limit for the 3-D Navier-Stokes equations with a slip boundary condition*, Comm. Pure Appl. Math., **60** (2007), 1027–1055.

Received for publication June 2009.

E-mail address: bveiga@dma.unipi.it