Lecture 2

Remarks on quasi-static evolutions

1. What drives evolution?

In this setting, what makes things change is the fact that $\mathcal{E}$ is not constant in time.

If $\mathcal{E}$ is constant in time, nothing moves.

This can be seen looking at discretized evolutions: if $\mathcal{E}(t,x) = \mathcal{E}(t+\delta, x)$, then $x(t+\delta) = x(t) \cdots$

On the other hand, if $\mathcal{E}(t,x) \neq \mathcal{E}(t+\delta, x)$, then $x(t)$ may be no longer a minimizer of $\mathcal{E}(t+\delta, x) + \mathcal{D}(x, x(t))$.

2. General framework

$x \in X$ subset of a linear space V (or a manifold)

$\mathcal{E} : [0,T] \times X \longrightarrow \mathbb{R}$ energy

$\mathcal{R} : [0,T] \times X \times V \longrightarrow \mathbb{R}$ dissipation rate (convex and 1-homogeneous in V)

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instead of R one can give

$\mathcal{D} : [0,T] \times X \times X \to \mathbb{R}$ dissipation potential (or dissipation distance)

and then set

$$R(t,x,v) := \lim_{h \to 0} \frac{\mathcal{D}(t,x,x+hv)}{h}$$

3. The infinite dimensional case

Additional difficulties, related to the search of minimizers.

The following version of Helly's theorem is sometimes useful:

Let X be a compact metric space.
Let (x^δ) be maps from $[0,T] \to X$ with uniformly bounded variations.
Then, after passing to a suitable subsequence, $x^\delta(t)$ converge to some $x(t)$ for every t,
and .....

4. Why "rate-independent" ?

The energy dissipated by friction to go from state x to state $x' = x + v\,\Delta t$ is

$$\partial R(v) \cdot \Delta x = R(\Delta x) = R(v) \cdot \Delta t = R(v \cdot \Delta t)$$

and therefore depends only on Δx (and not on v).

Assuming the existence of a dissipation potential $\mathcal{D}$ automatically plugs this property into the system...

5. Invariance under re-parametrization of time

Let $x(t)$ be a solution of q.s. evolution associated to $\mathcal{E}$ and $\mathcal{R}$.

$$\mathcal{E} = \mathcal{E}(t,x) \qquad \mathcal{R} = \mathcal{R}(v)$$

Let $t = t(\tau)$ be any increasing change of variable.

Then $\tilde{x}(\tau) := x(t(\tau))$ is a solution of q.s. evolution associated to $\tilde{\mathcal{E}}(\tau, x) := \mathcal{E}(t(\tau), x)$ and $\mathcal{R}(v)$.

6. What is not rate-independent?

Example: a solid ball moving in the air is subject to a resistence force proportional to velocity:

$$f_r = -cv$$

(at least in a certain range of velocity)

Thus the energy dissipated from x to $x' = x + v\Delta t$ is

$$-f_r \cdot \Delta x = c\,v \cdot \Delta x = c\,\frac{\Delta x}{\Delta t} \cdot \Delta x = c\,\frac{|\Delta x|^2}{\Delta t} \cdots$$

In this case the balance of forces is $-cv = -\partial_x \mathcal{E}(t,x)$. Assuming $c=1$ and $\mathcal{E} = \mathcal{E}(x)$

$$\dot{x} = -\nabla \mathcal{E}(x).$$

This is a gradient flow, and has a completely different structure, and meaning.

Note that gradient flows are not invariant under (nonlinear) time re-parametrization.

(Example: the heat equation $u_t = \Delta u$ as gradient flow of the Dirichlet energy $\frac{1}{2}\int |\nabla u|^2$ w.r.t. the L^2-scalar product.)

Time-discretization of gradient flows is also completely different: replacing $\dot{x}(t)$ by $\frac{x(t)-x(t-h)}{h}$ we get

$$\frac{x(t)-x(t-h)}{h} = -\nabla \mathcal{E}(x(t))$$

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that is

$$x(t) \in \operatorname{argmin}\left\{\mathcal{E}(x) + \frac{1}{2h}|x-x(t-h)|^2\right\}$$

7. References.

The abstract framework I described is essentially due to Mielke and coauthors.

See his lecture notes for more "advanced" examples.

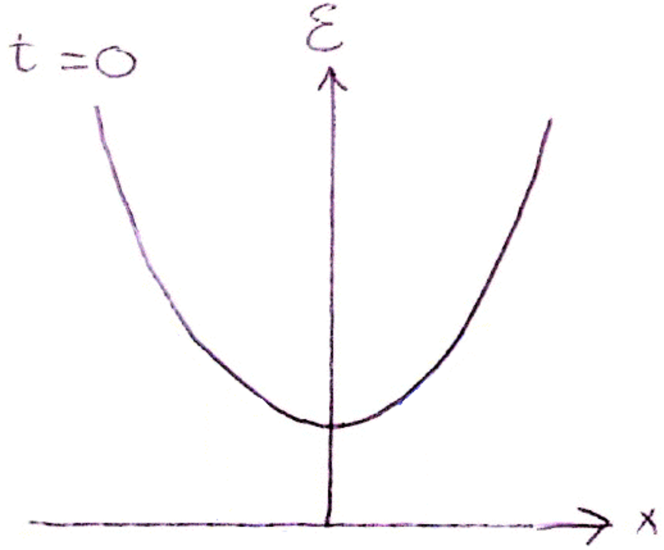
Francfort and Marigo set up a similar framework in the specific context of q.s. evolution of fractures in brittle materials.

8. Regularity and uniqueness.

If $\mathcal{E}$ is convex in x (and R is reasonable) then the solution of q.s. evolution with initial condition x_0 is unique and Lipschitz.

In particular one can make sense of the differential inclusion $0 \in \partial_x \mathcal{E}(t,x) + \partial R(\dot{x})$.

9. Example of non uniqueness



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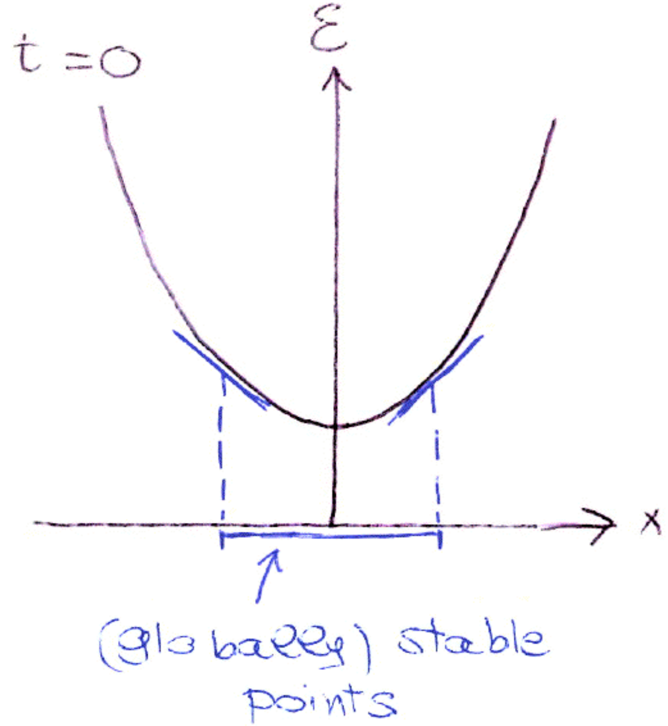

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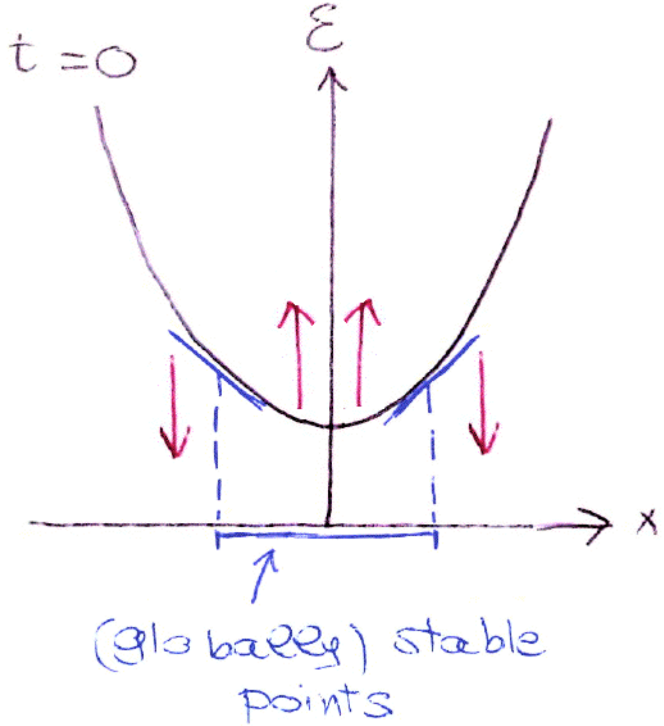

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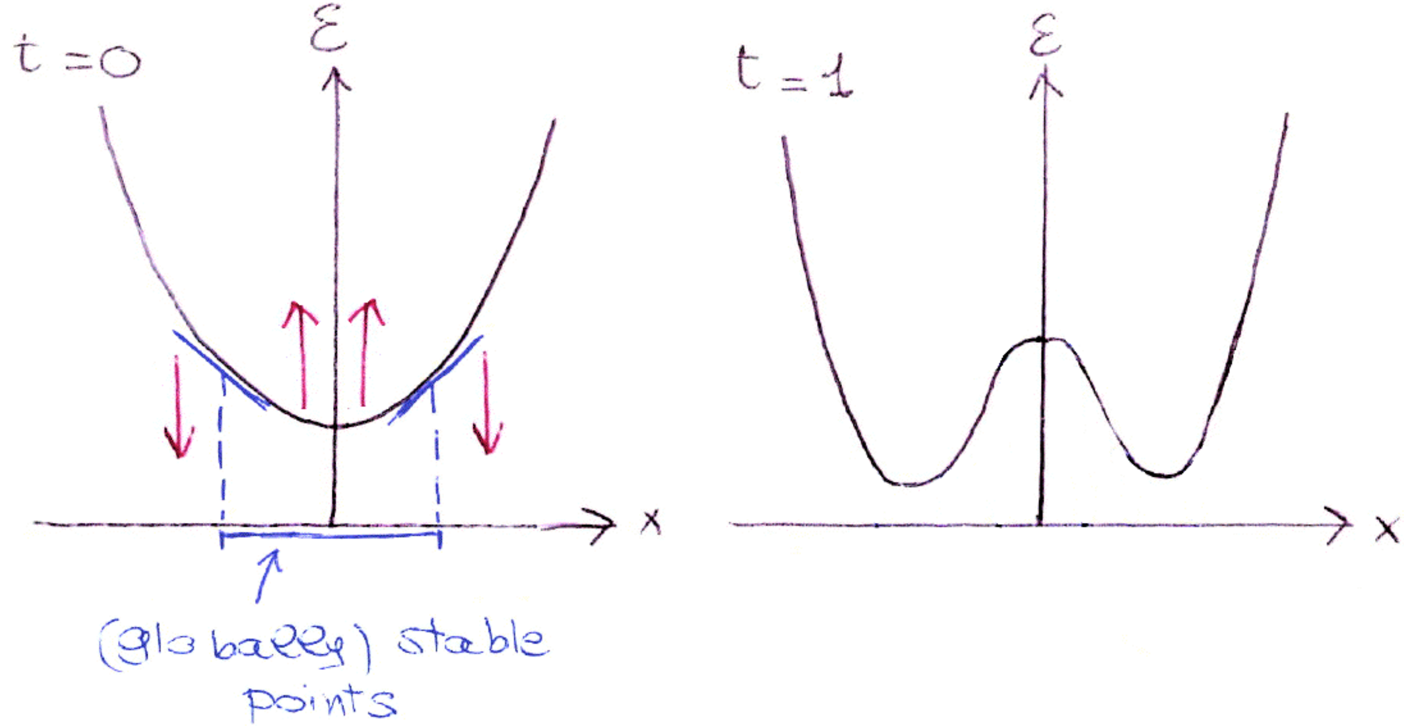

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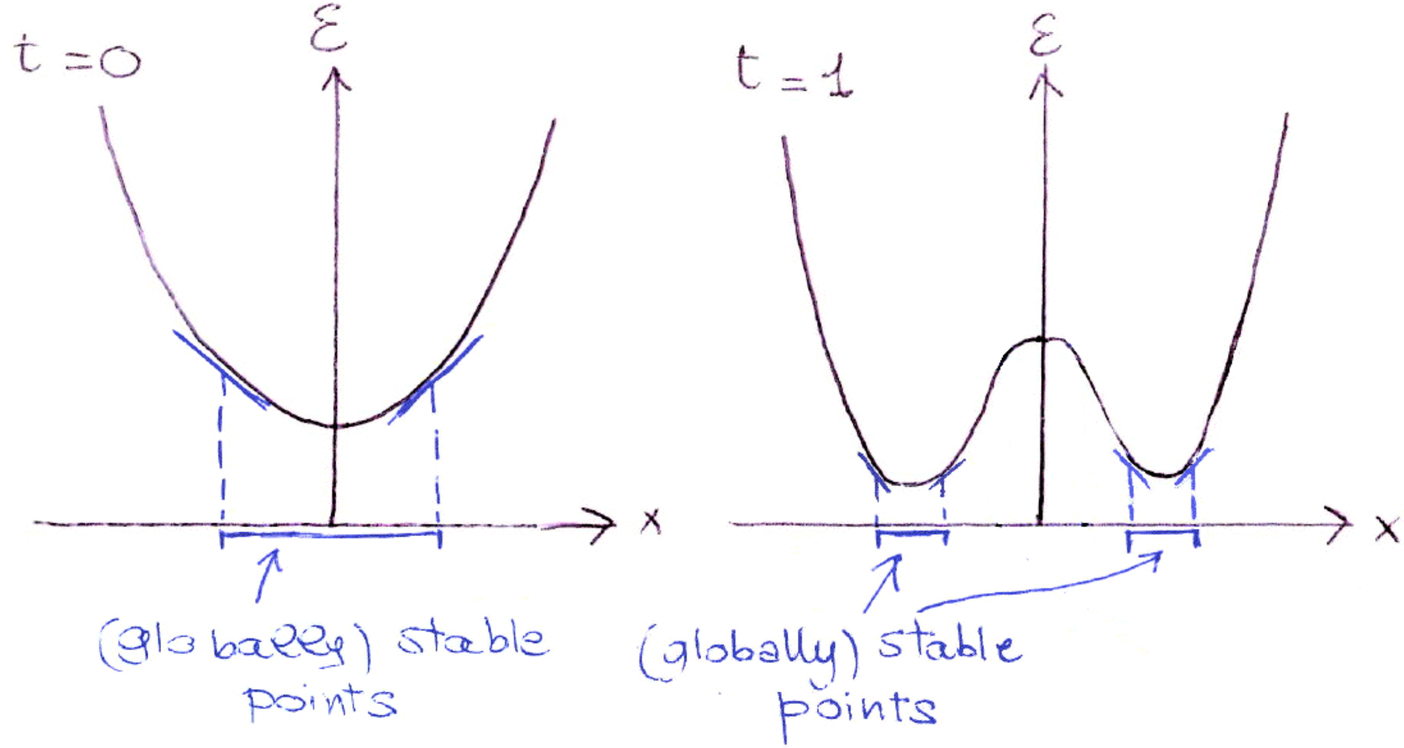

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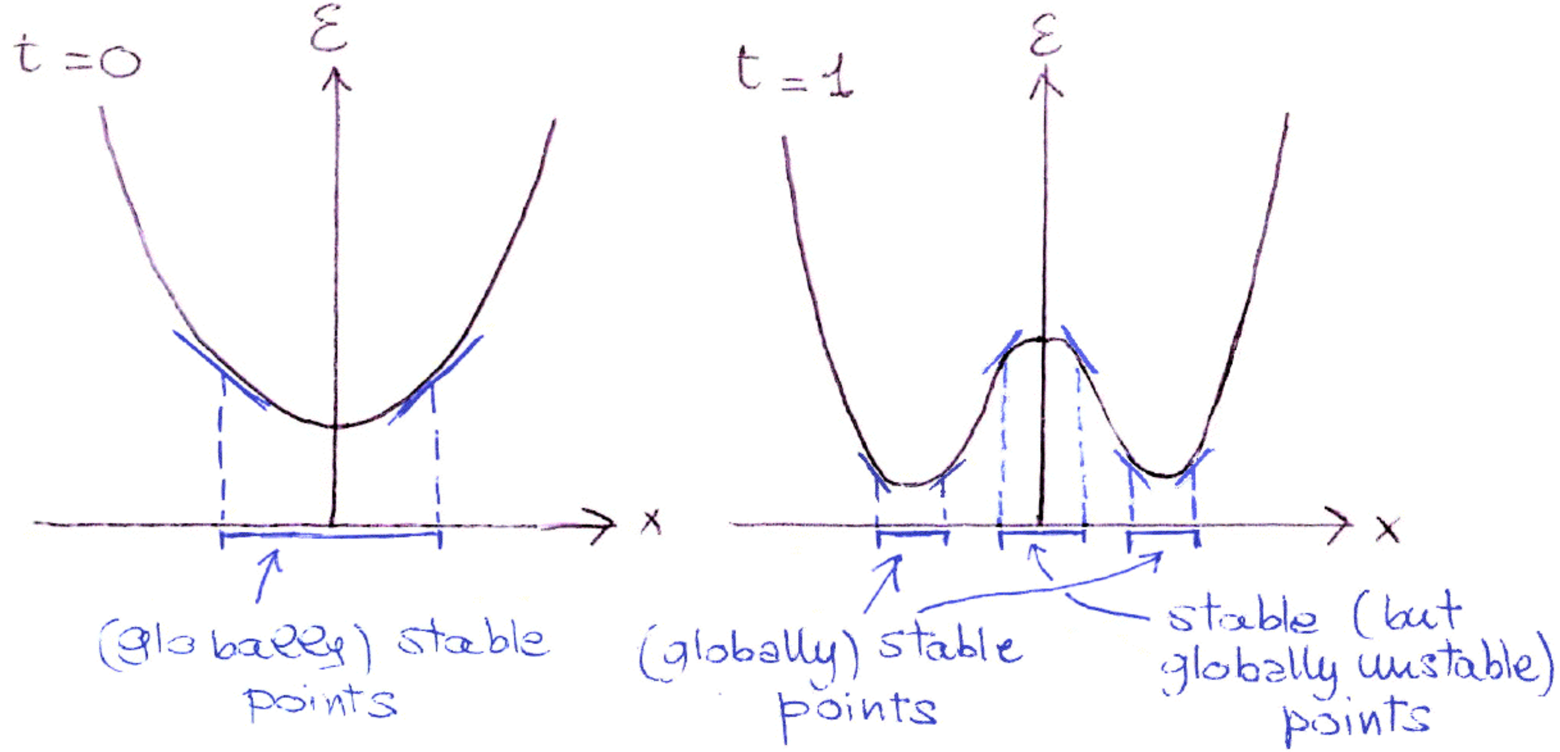

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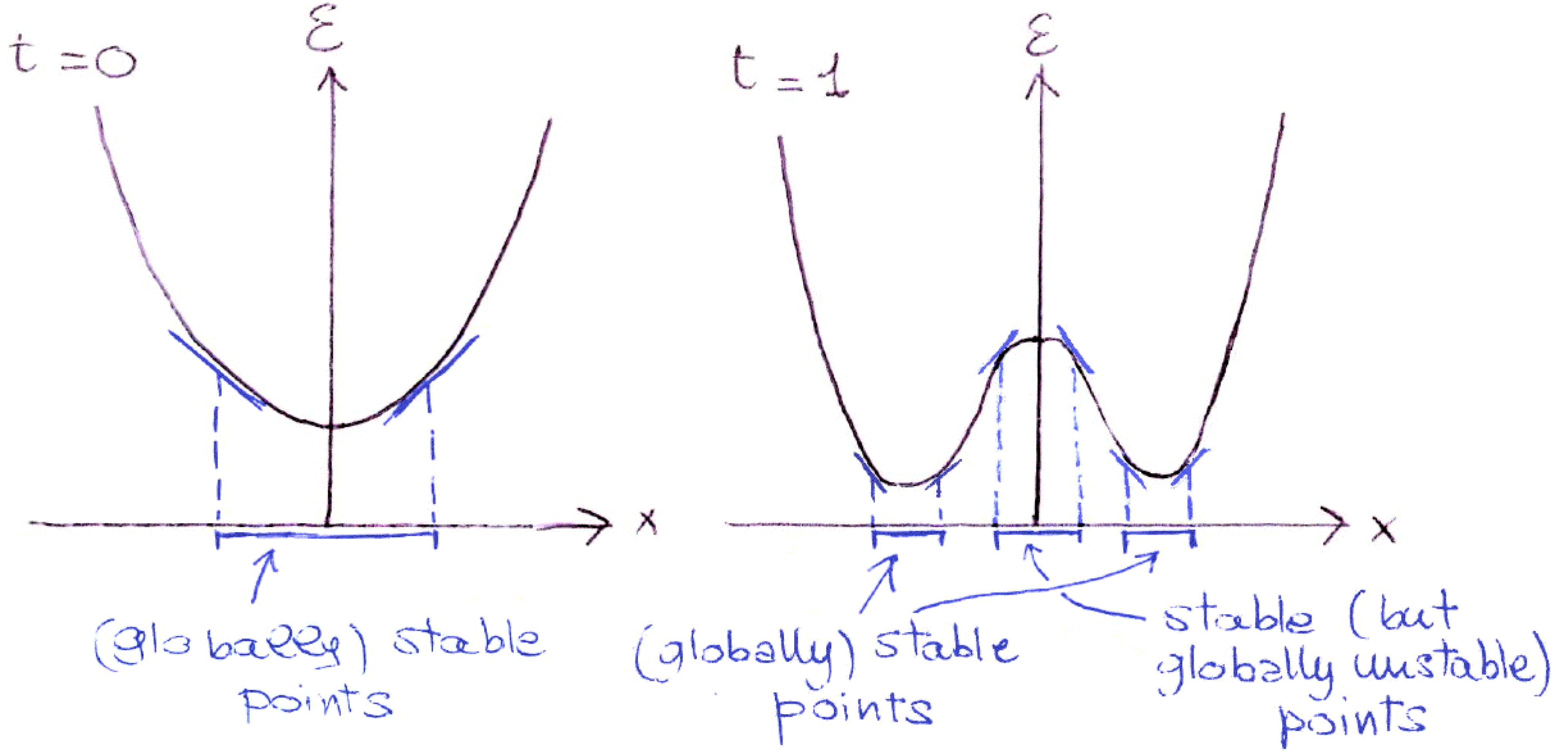


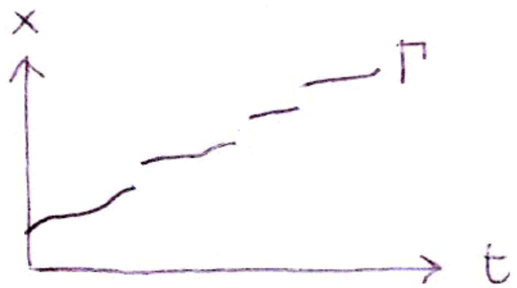
Symmetry breaking: the (energetic) solution with initial condition $x(0)=0$ is not unique.

10. (Lack of) Regularity

We know solutions can be discontinuous.
Are they piecewise $\mathcal{C}^1$ or piecewise Lipschitz?
No!

Let $x : [0,T] \to \mathbb{R}$ be bounded and increasing.
Let $R(v) = |v|$.
Then there exists $\mathcal{E} : [0,T] \times \mathbb{R} \to \mathbb{R}$ <u>smooth</u>
such that x is an (energetic) solution of the q.s. evolution associated to $\mathcal{E}$ and R.

Let Γ be the graph of x



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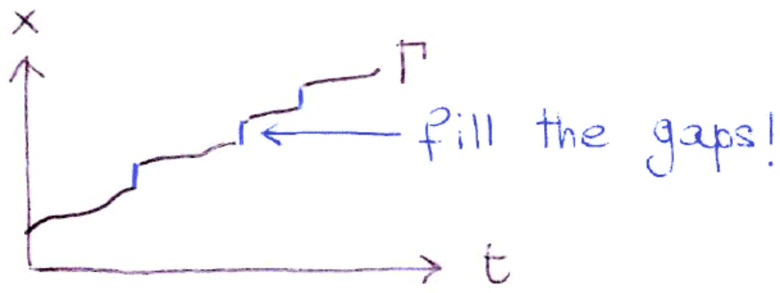

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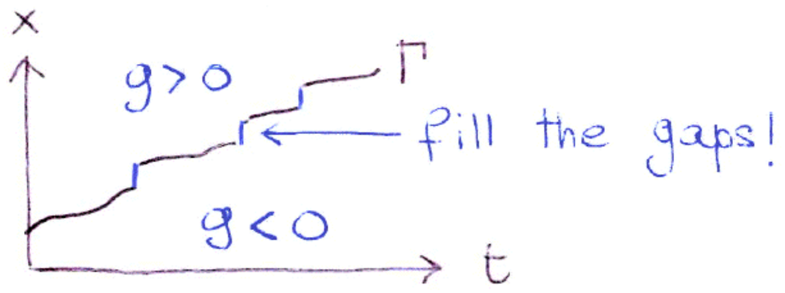
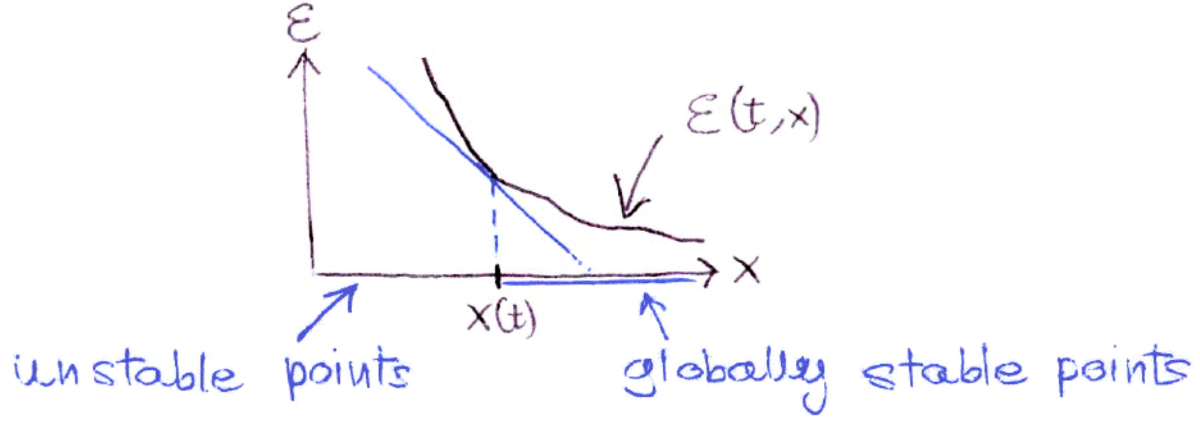


Let $g : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function s.t. $g<0$ below Γ, $g=0$ on Γ, $g>0$ above Γ.

Define $\mathcal{E}$ by $\partial_x \mathcal{E} = -1 + g$. Then $\forall t$



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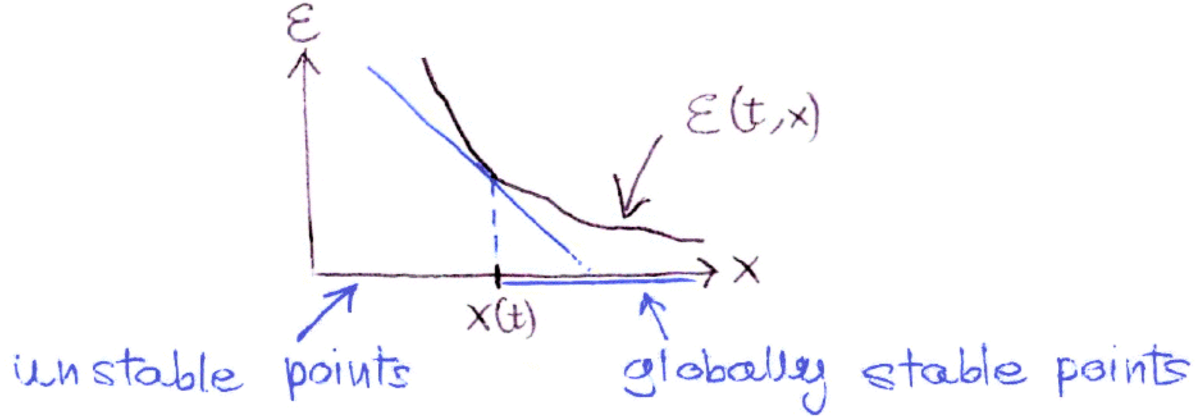

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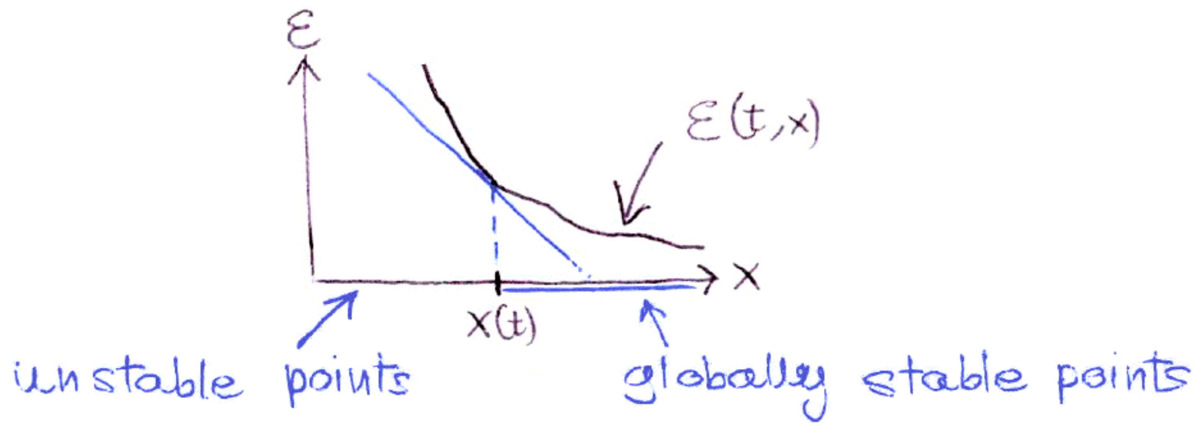
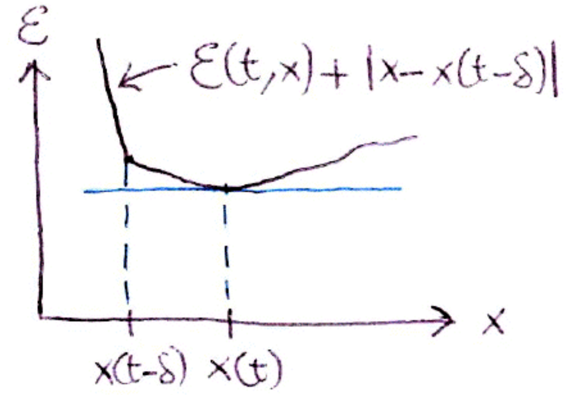
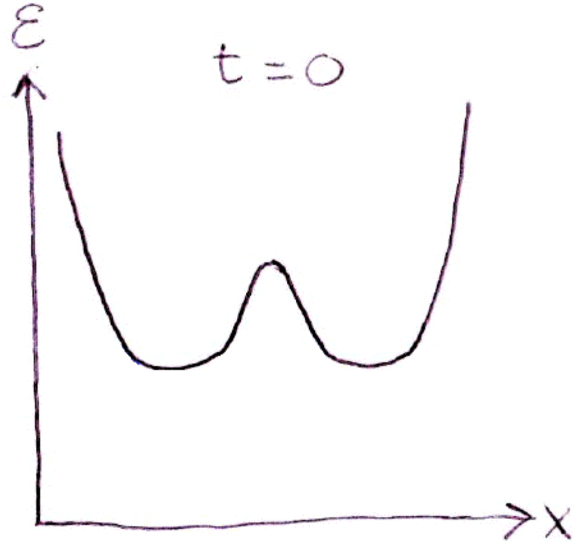


Then $x(t)$ is an energetic solution.
The one obtained by time discretization!



ii. Problems with jumps (discont. sol's)

I: unphysical energetic solutions



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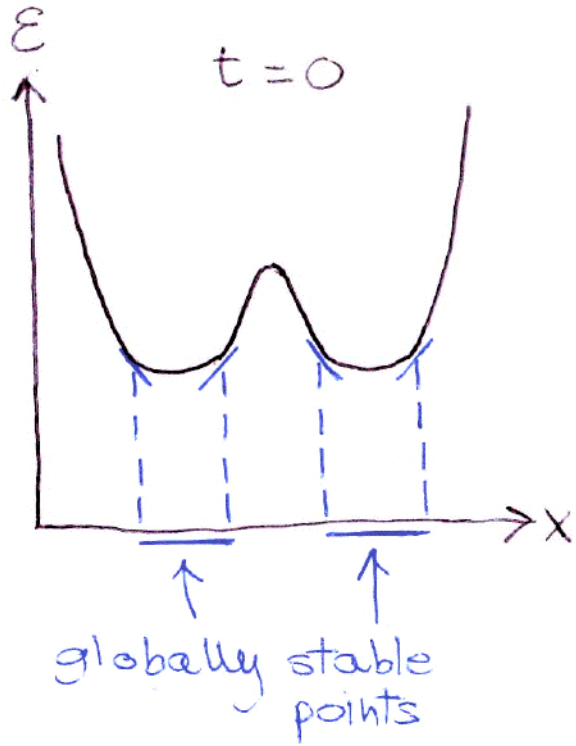

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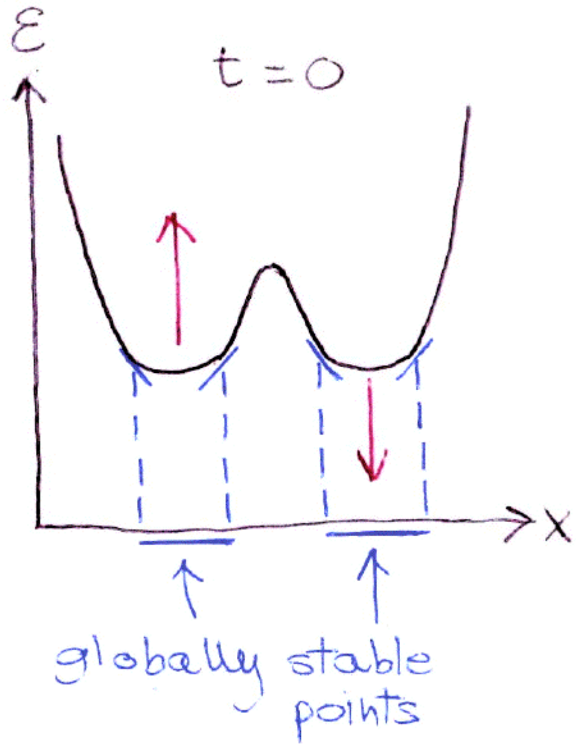

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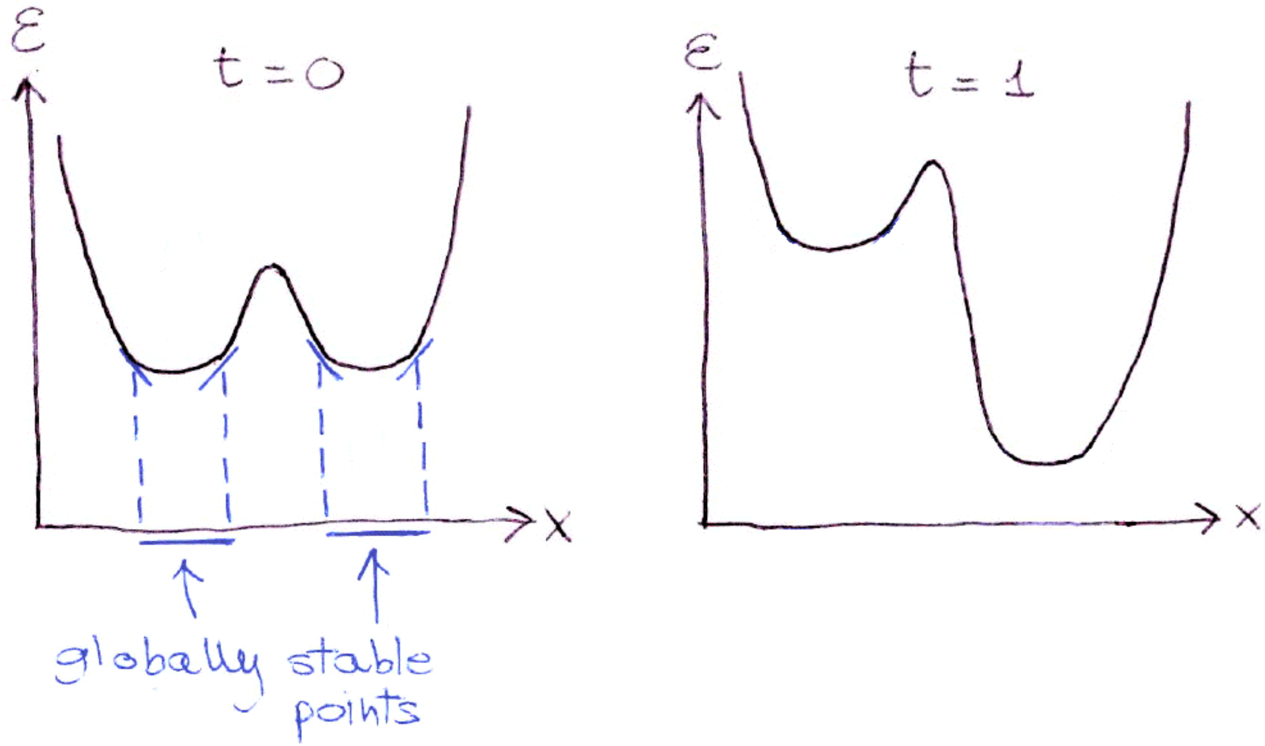

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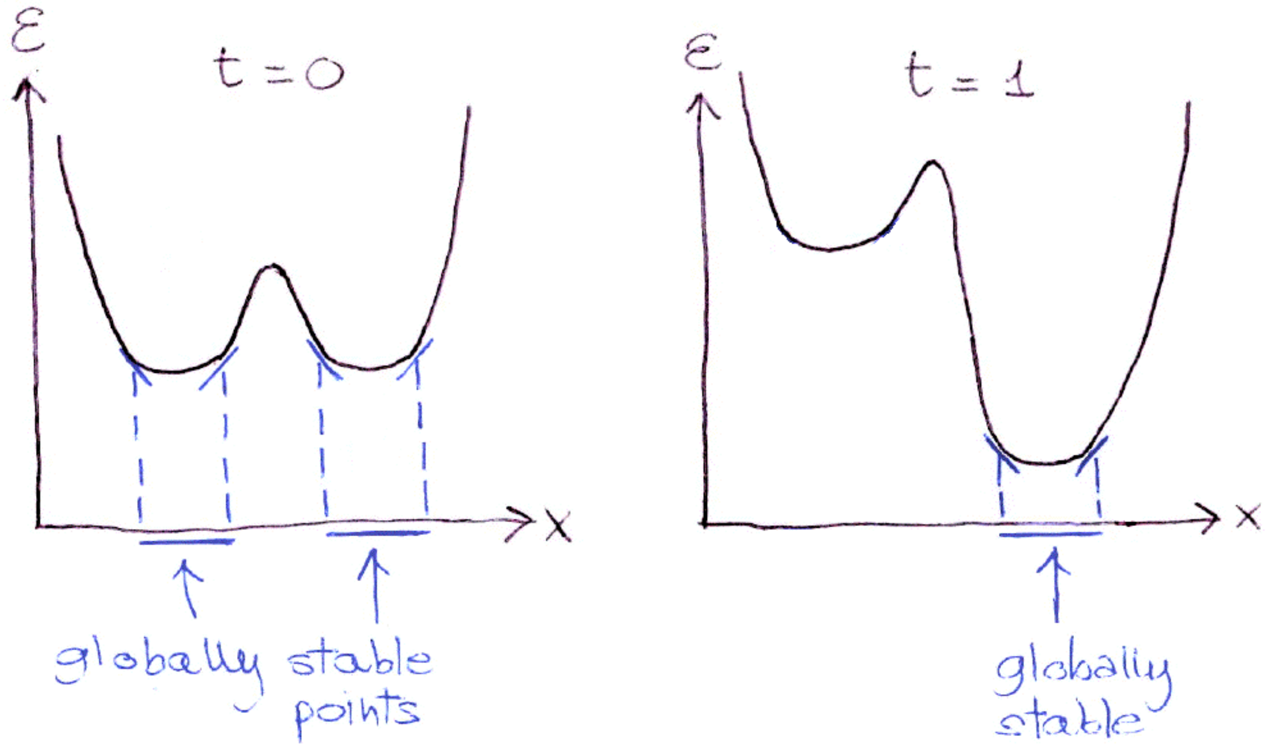

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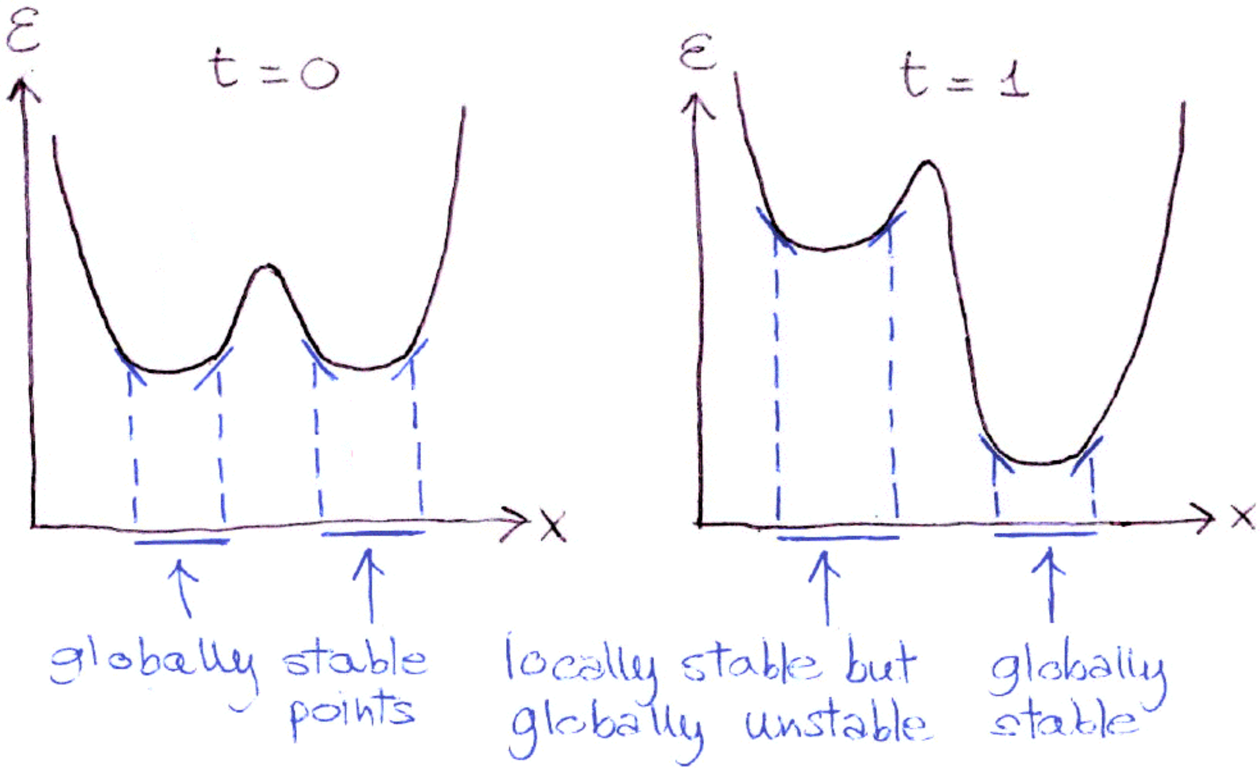

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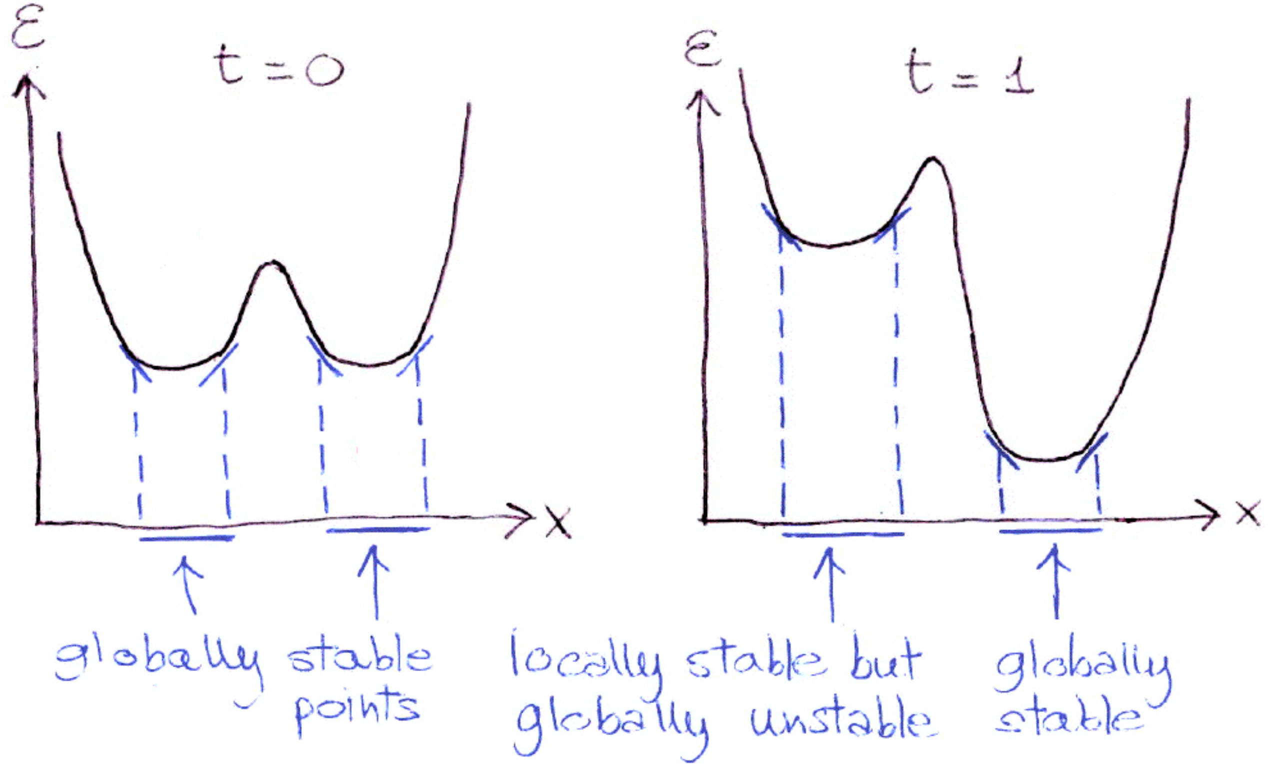

An energetic solution starting in the first interval will jump to the second one....

12. Problems with jumps II: "philosophical"

Can we claim that we are dealing with a quasi-static evolution if a solution jumps (infinite velocity)?

Sometimes YES!

Consider for example our favourite example: a box sliding on a "variable carpet", and assume it has small but positive mass, and inertia is not neglected....

13. Vanishing viscosity approach.

Mielke and coauthors → see Mielke's <u>recent</u> (2009) lecture notes.

Add a "small" (artificial) viscosity to the balance equation:

$$0 \in f_\varepsilon + f_a + \varepsilon V \qquad \varepsilon \ll 1$$

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In term of time-discretization:

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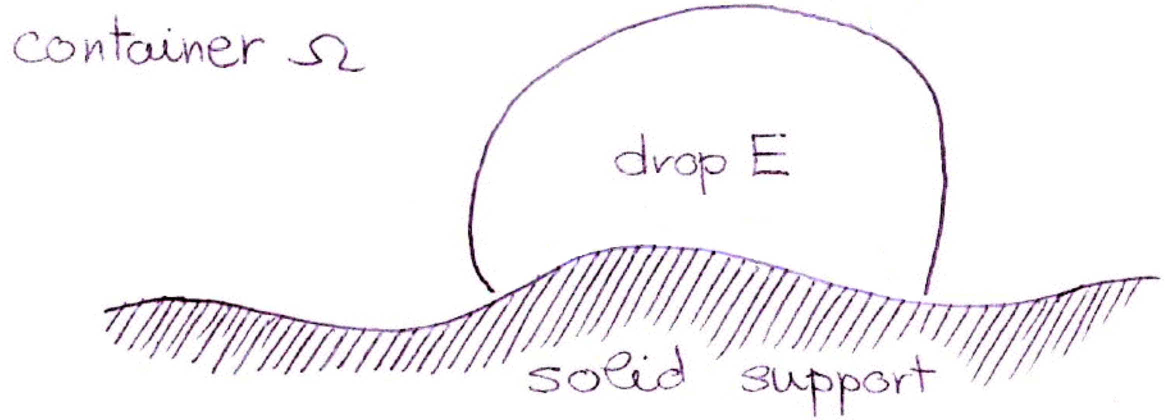
Close locally stable points are preferable w.r.t. far globally stable ones. Here is why....

2. The classical variational model for equilibrium capillary drops

Through the rest of this lecture we consider a drop (of water) a t r e s t on a solid surface.

So there is no velocity, no Stokes equation.

2.1. Terminology



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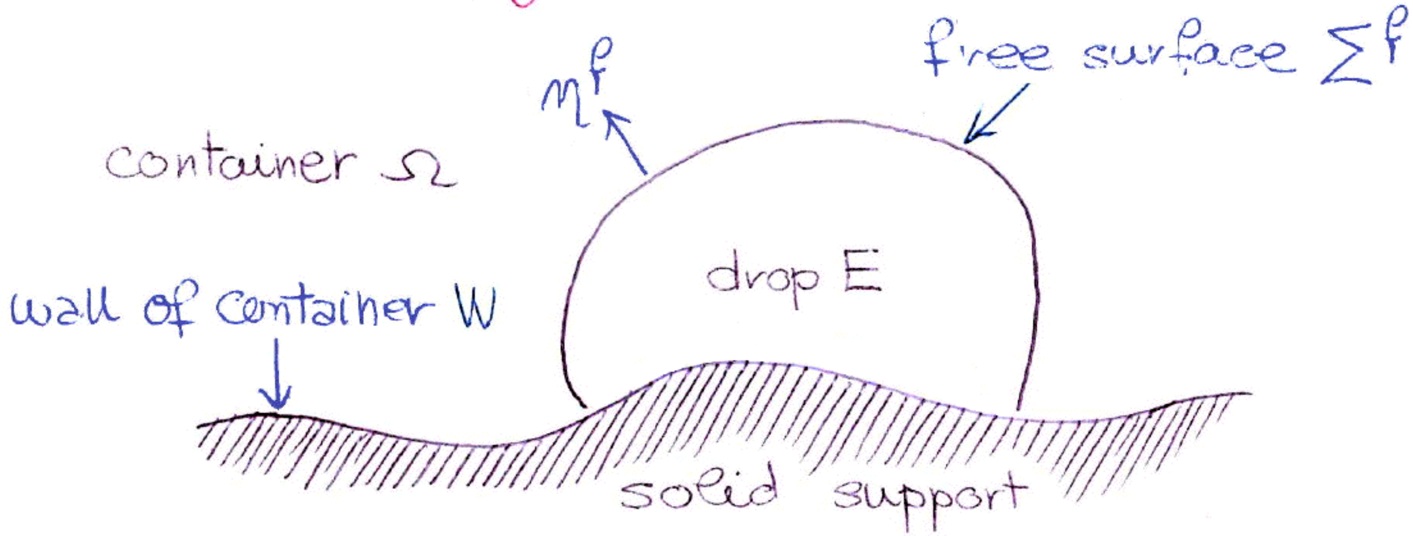

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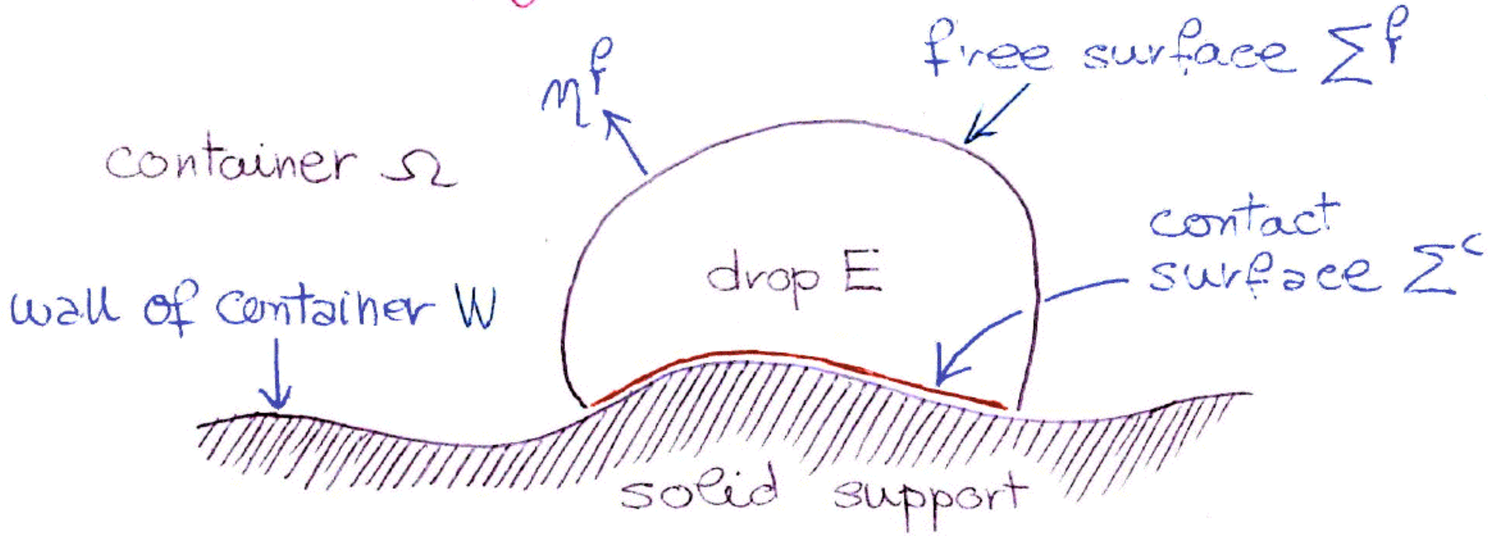

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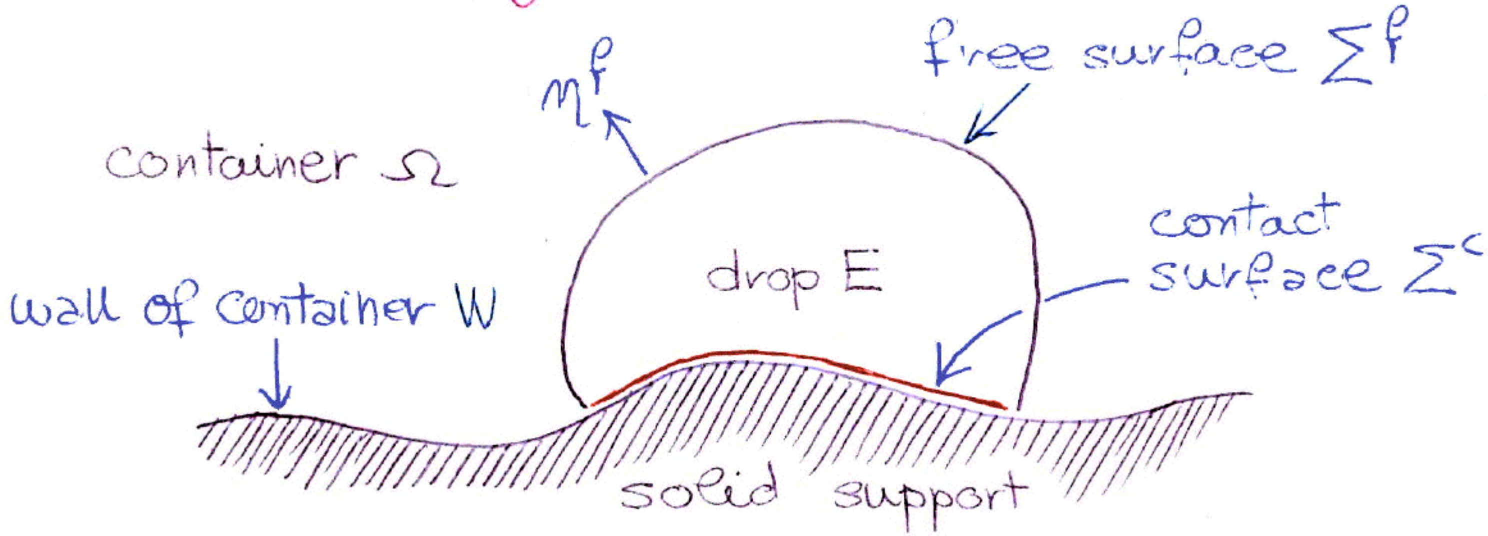

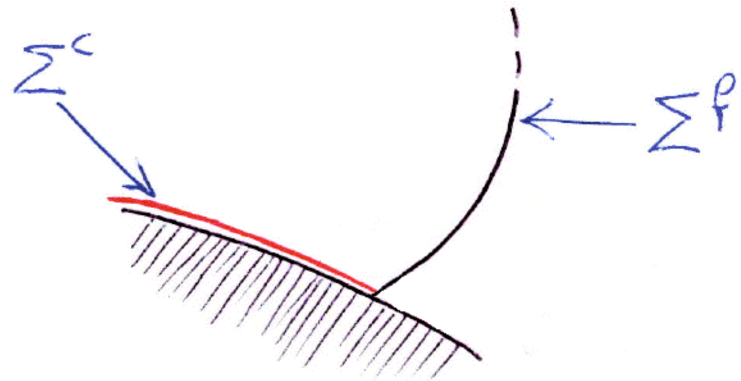

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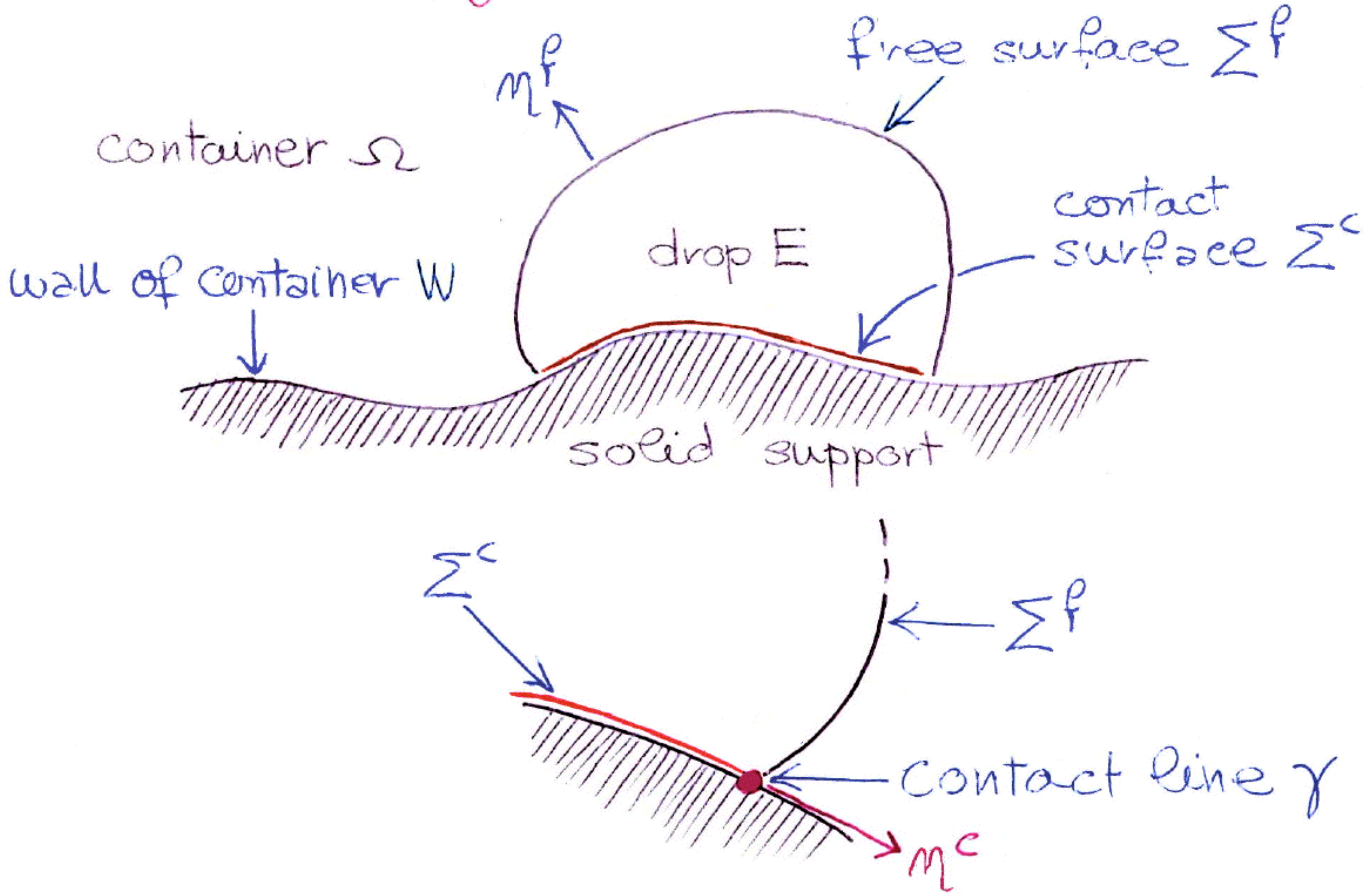

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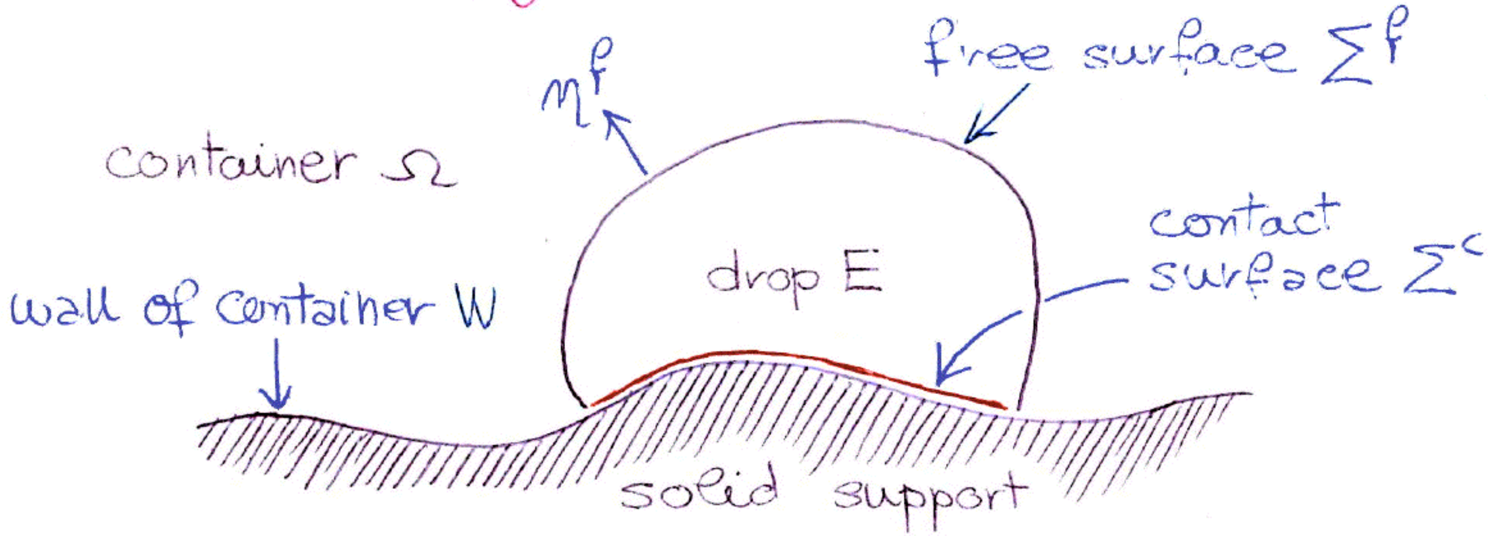

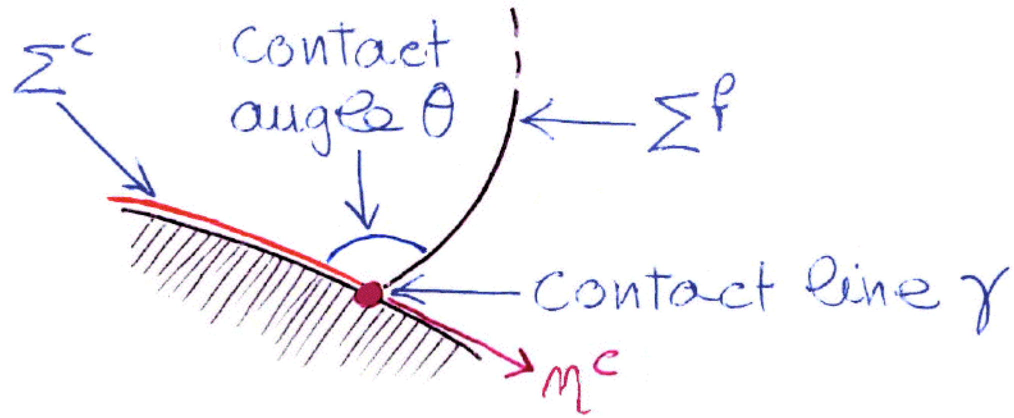

2.2. Capillary energy

$$\mathcal{E} = \sigma_{LV} |\Sigma^{\ell}| + \sigma_{LS} |\Sigma^{c}| + \sigma_{SV} |W \setminus \Sigma^{c}| + V$$

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liquid-vapour surface tension coefficient

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liquid-vapour surface tension coefficient

$$V = \int_E \rho(t,x)\, dx$$

volume energy (e.g. gravitational en.)

2.3. Wetting conditions

It is assumed that

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Physical interpretation: $\sigma_{LS} \leqslant \sigma_{SV} + \sigma_{VL}$ means that is never convenient to interpose a thin layer of air between the solid surface and the drop; conditions $\sigma_{SV} \leq \sigma_{SL} + \sigma_{LV}$ means that is never convenient to interpose a thin layer of water between the solid surface and the air.

2.4. Definition of Young's angle

Define the Young's angle $\theta_Y \in [0, \pi]$ by

$$\cos\theta_Y = \frac{\sigma_{SV} - \sigma_{LS}}{\sigma_{LV}}$$

The energy can be rewritten as

$$\mathcal{E} = \sigma_{LV}\Big(|\Sigma^f| - \cos\theta_Y |\Sigma^c|\Big) + V$$

2.5. Equilibrium conditions

If the drop E is at equilibrium (e.g. E is a local minimizer of $\mathcal{E}$ among drops with same volume)

then Laplace law applies:

$$-2\sigma_{LV} H + p = \text{constant} \qquad \text{on } \Sigma^f$$

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mean curvature of Σ^f volume energy density   pressure

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mean curvature of Σ^f volume energy density pressure

and Young's law:

$$\theta = \theta_Y \text{ on } \gamma.$$

2.6. Derivation of equilibrium conditions

Consider an arbitrary "variation" of a given drop E. That is a (decent) 1-parameter family of sets $E(h)$ s.t. $E(0) = E$.

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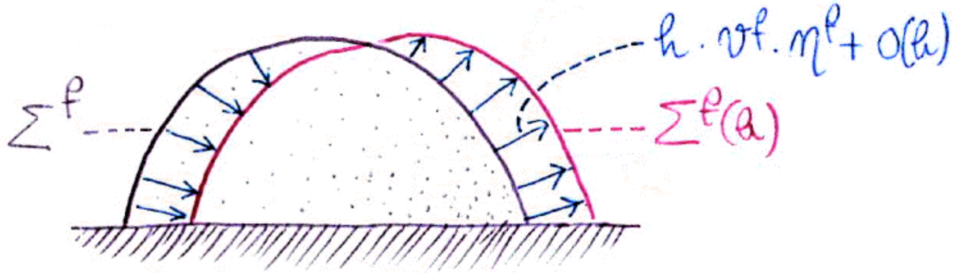

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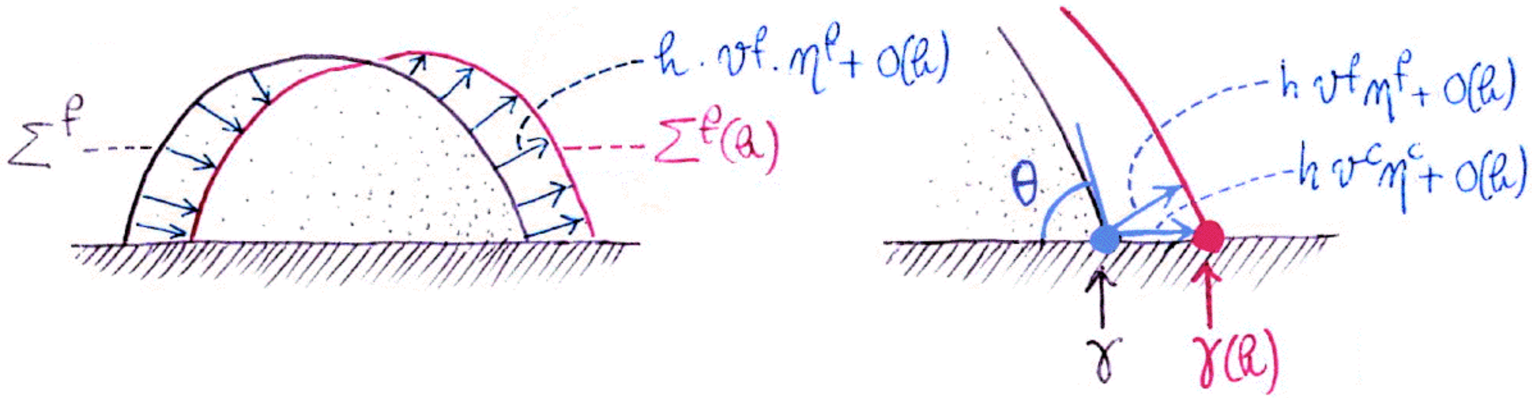

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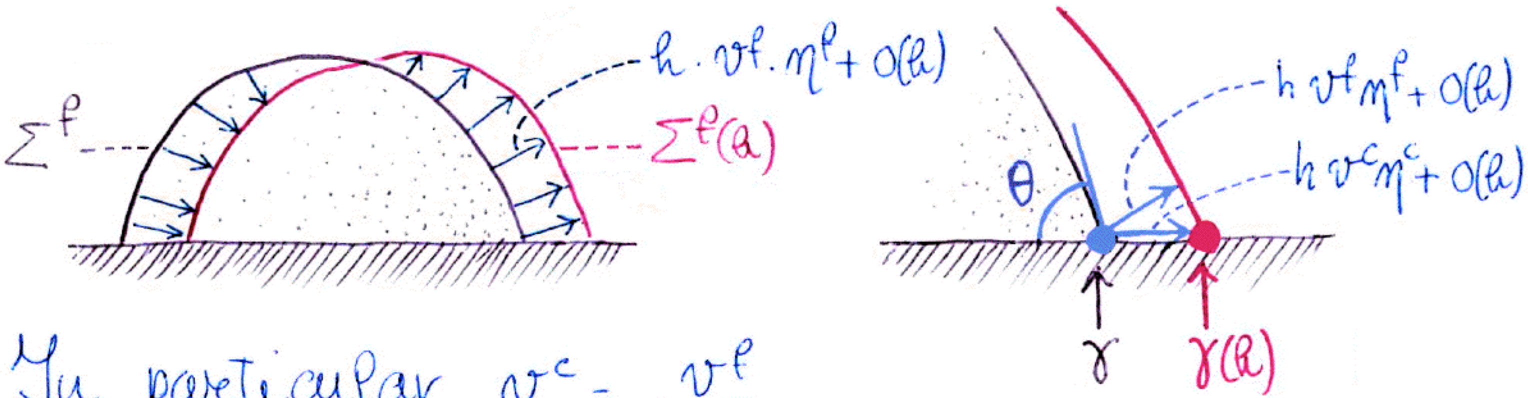

In particular $v^c = \dfrac{v^f}{\operatorname{sen}\theta}$.

Useful variational formulas

$$\frac{d}{dh} \mathrm{vol}(E(h))\Big|_{h=0} = \int_{\Sigma^f} v^f$$

$$\frac{d}{dh} V(E(h))\Big|_{h=0} = \int_{\Sigma^f} p \cdot v^f$$

$$\frac{d}{dh} |\Sigma^c(h)|\Big|_{h=0} = \int_{\gamma} v^c$$

$$\frac{d}{dh} |\Sigma^f(h)|\Big|_{h=0} = \int_{\Sigma^f} -2H v^f + \int_{\gamma} \cos\theta \cdot v^c$$

Later about the proof....

If E is at equilibrium among configurations with same volume (a constrained equilibrium) then there exists $\lambda \in \mathbb{R}$ (a Lagrange multiplier) s.t. for every variation $E(h)$

$$\frac{d}{dh}\mathcal{E}(E(h))\Big|_{h=0} - \lambda \frac{d}{dh}\mathrm{Vol}(E(h))\Big|_{h=0} = 0 .$$

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$$\frac{d}{dh}\mathcal{E}(E(h))\Big|_{h=0} - \lambda \frac{d}{dh}\mathrm{Vol}(E(h))\Big|_{h=0} = 0 .$$

And by the previous formulas

$$\int_{\Sigma^f} (-2\sigma_{LV} H + \rho - \lambda)\, v^f + \int_{\gamma} \sigma_{LV}(\cos\theta - \cos\theta_Y)\, v^c = 0 \qquad (1)$$

If $v^f = 0$ on γ then $v^c = 0$ on γ (the two velocities are related by the formula $v^c = v^f / \sin\theta$).
Hence (1) becomes

$$\int_{\Sigma^f} \left(-2\sigma_{LV} H + p - \lambda \right) v^f = 0 .$$

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$$\int_{\Sigma^f} (-2\sigma_{LV} H + p - \lambda)\, v^f = 0 .$$

Since v^f is an arbitrary (decent) function (we do not assume that the variation is volume preserving!) then the only possibility is

$$-2\sigma_{LV} H + p - \lambda \equiv 0 \quad \text{on } \Sigma^f$$

which is Laplace law.

Thus (1) becomes

$$\int_{\gamma} \sigma_{LV}(\cos\theta - \cos\theta_Y)\, v^c = 0$$

and since v^c is an arbitrary (decent) function

$$\cos\theta - \cos\theta_Y \equiv 0 \quad \text{on } \gamma$$

which is Young's law.

Final remarks

1. A more correct approach to the deviation of the stability conditions would be considering variations $E(\ell)$ obtained by flowing the free-surface Σ^f by the flow associated to a vectorfield which is tangent to the solid surface.

This way the velocity of Σ^f is not necessary normal.

However computations (and intuition) show that by splitting this velocity in a normal and a tangent component we get the same result as above....

2. The variational formula

$$\frac{d}{d\ell} \mathrm{Vol}(E(\ell)) \Big|_{\ell=0} = \int_{\Sigma f} v f$$

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The simplest consists in assuming Σ^f piecewise flat and note that in this case

$$\mathrm{Vol}(E(h)) = \mathrm{Vol}(E) + h \int_{\Sigma^f} v^f + O(h^2)$$

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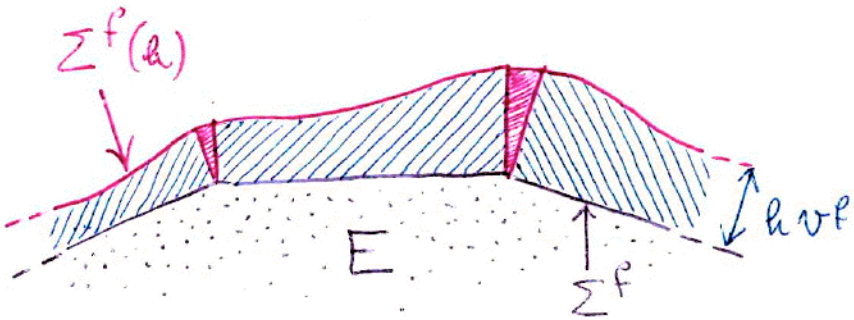

$\mathrm{Vol}(\text{blue}) = h \int_{\Sigma^f} v^f$

$\mathrm{Vol}(\text{pink}) = O(h^2)$

A more rigorous alternative consists in constructing a parametrization of $E(h) \Delta E$ starting from a parametrization $g : D \to \Sigma^f$,

[something of the form

$$(s,t) \in D \times [0,h] \longmapsto g(s) + t\, \vartheta^f(g(s))\, \eta^f(g(s)) + O(h) \ ]$$

and using it to compute

$$\mathrm{Vol}\,(E(h)) = \mathrm{Vol}\,(E) + h \int_{\Sigma^f} \vartheta^f + o(h).$$

3. The variational formulas

$$\frac{d}{dh} V(E(h))\Big|_{h=0} = \int_{\Sigma^f} \rho\, v^f$$

and

$$\frac{d}{dh} \left|\Sigma^c(h)\right|\Big|_{h=0} = \int_{\gamma} v^c$$

can be proved (essentially) in the same way.

4. Formula

$$\frac{d}{dh}\left|\Sigma^f(h)\right|\Big|_{h=0} = \int_{\Sigma^f} -2H\,v^f + \int_\gamma \cos\theta\, v^c$$

is more complicated.

4. Formula

$$\frac{d}{dh}\left|\Sigma^f(h)\right|\Big|_{h=0} = \int_{\Sigma^f} -2H\,v^f + \underbrace{\int_{\gamma} \cos\theta\, v^c}_{(\mathrm{I})}$$

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Contribution (I) accounts for the fact that $\gamma = \partial\Sigma^f$ does not move in the normal direction n^f

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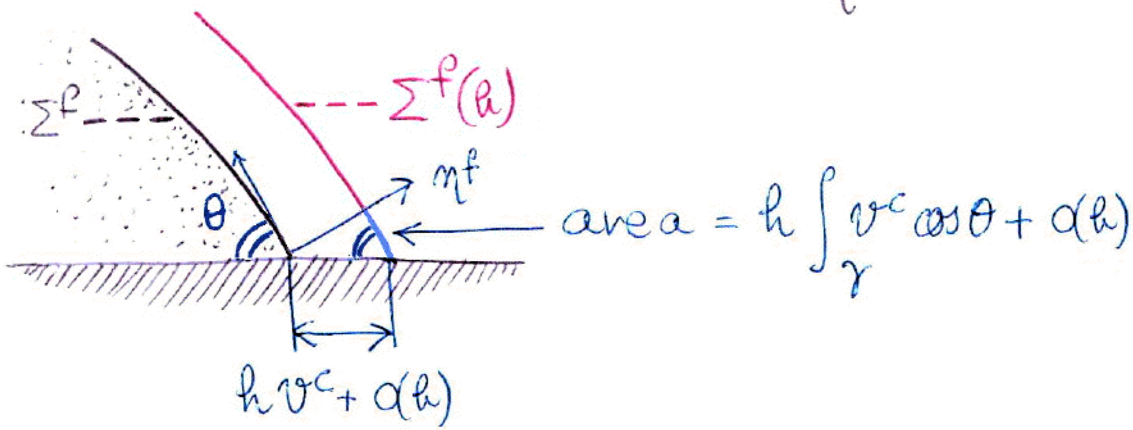
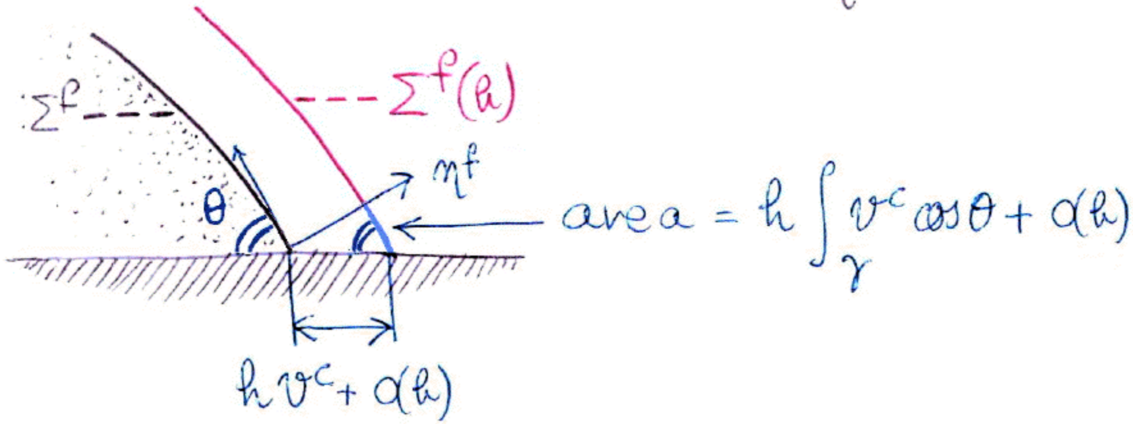


4. Formula

$$\frac{d}{dh}\left|\Sigma^f(h)\right|\Big|_{h=0} = \underbrace{\int_{\Sigma^f} -2H\,v^f}_{(\mathrm{II})} + \underbrace{\int_{\gamma} \cos\theta\, v^c}_{(\mathrm{I})}$$

is more complicated.

Contribution (I) accounts for the fact that $\gamma = \partial\Sigma^f$ does not move in the normal direction n^f



Contribution (II) is due to curvature.

The term (II) can be easily justified when:

a) Σ^f is one-dim (a curve) and piecewise linear,

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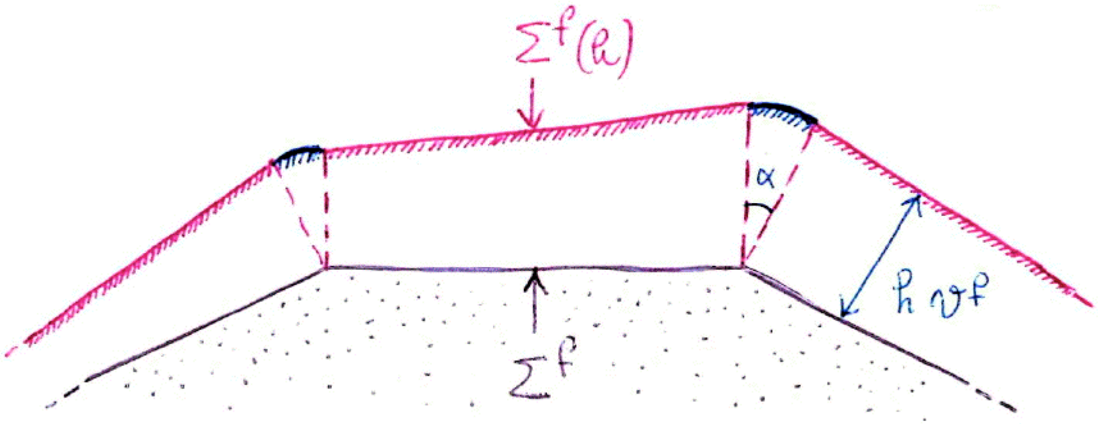

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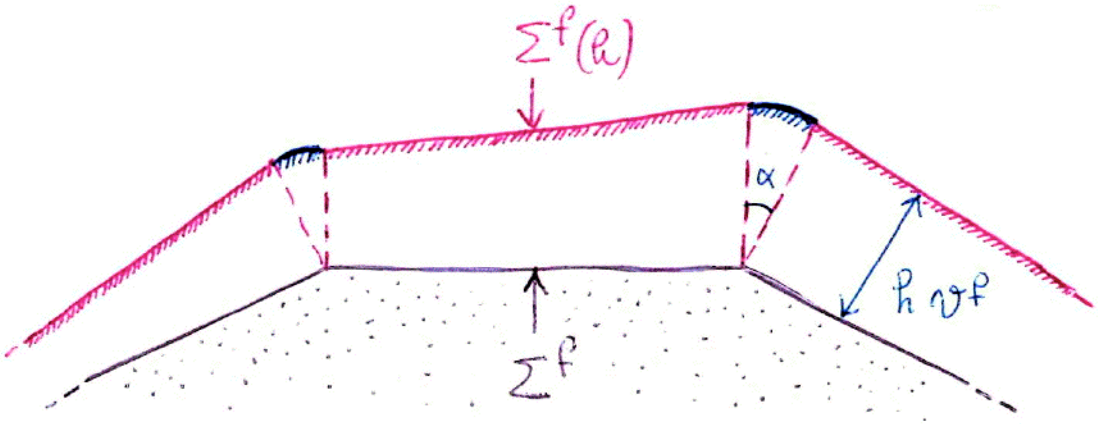

Then

$$\text{length}(\text{—}) = \int_{\Sigma^f} \sqrt{1+(h\dot{\vartheta}^f)^2} = \text{length}(\Sigma^f) + O(h^2)$$

$$\text{length}(\text{—}) = h \sum \vartheta^f.\alpha \sim h \int \vartheta^f \varkappa^f \leftarrow \text{curvature of } \Sigma^f$$