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Geometric Measure Theory

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1. Introduction

The aim of these pages is to give a brief, self-contained introduction to that part of Geometric Measure Theory which is more directly related to the Calculus of Variations, namely the theory of currents and its applications to the solution of Plateau problem. (The theory of finite perimeter sets, which is closely related to currents and to the Plateau problem, is treated in the article "Free interfaces and free discontinuities: variational problems").

Named after the belgian physicist J.A.F. Plateau (1801-1883), this problem was originally formulated as follows: find the surface of minimal area spanning a given curve in the space. Nowadays, it is mostly intended in the sense of developing a mathematical framework where the existence of k-dimensional surfaces of minimal volume that span a prescribed boundary can be rigorously proved. Indeed, several solutions have been proposed in the last century, none of which is completely satisfactory.

One difficulty is that the infimum of the area among all smooth surfaces with a certain boundary may not be attained. More precisely, it may happen that all minimizing sequences (that is, sequences of smooth surfaces whose area approaches the infimum) converge to a singular surface. Therefore one is forced to consider a larger class of admissible surfaces than just smooth ones (in fact, one might want to do this also for modelling reasons—this is indeed the case with soap films, soap bubbles, and other capillarity problems). But what does it mean that a set "spans" a given curve? and what should we intend by area of a set which is not a smooth surface?

The theory of integral currents developed by H. Federer and W.H. Fleming [4] provides a class of generalized (oriented) surfaces with well-defined notions of boundary and area (called mass) where the existence of minimizers can

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be proved by direct methods. More precisely, this class is large enough to have good compactness properties with respect to a topology that makes the mass a lower semicontinuous functional. This approach turned out to be quite powerful and flexible, and in the last decade the theory of currents has found applications in several different areas, from dynamical systems (in particular, Mather theory), to the theory of foliations, to optimal transport problems.

2. Hausdorff measures, dimension, and rectifiability

The volume of a smooth d-dimensional surface in $\mathbb{R}^n$ is usually defined using parametrizations by subsets of $\mathbb{R}^d$. The notion of Hausdorff measure allows to compute the d-dimensional volume using coverings instead of parametrizations, and, what is more important, applies to all sets in $\mathbb{R}^n$, and makes sense even if d is not integer. Attached to Hausdorff measure is the notion of Hausdorff dimension. Again, it can be defined for all sets in $\mathbb{R}^n$ and is not necessarily integer. The last fundamental notion is rectifiability: k-rectifiable sets can be roughly understood as the largest class of k-dimensional sets for which it is still possible to define a k-dimensional tangent bundle, even if only in a very weak sense. They are essential to the construction of integral currents.

2.1. HAUSDORFF MEASURE. - Let $d \geq 0$ be a positive real number. Given a set E in $\mathbb{R}^n$, for every $\delta > 0$ we set

$$\mathscr{H}^d_\delta(E) := \frac{\omega_d}{2^d} \inf \left\{ \sum_j (\operatorname{diam}(E_j))^d \right\} , \tag{1}$$

where ω_d is the d-dimensional volume of the unit ball in $\mathbb{R}^d$ whenever d is an integer (there is no canonical choice for ω_d when d is not an integer; a convenient one is $\omega_d = 2^d$), and the infimum is taken over all countable families of sets $\{E_j\}$ that cover E and whose diameters satisfy $\operatorname{diam}(E_j) \leq \delta$. The d-dimensional *Hausdorff measure* of E is

$$\mathscr{H}^d(E) := \lim_{\delta \to 0} \mathscr{H}^d_\delta(E) \tag{2}$$

(the limit exists because $\mathscr{H}^d_\delta(E)$ is decreasing in δ).

Remarks. - (i) $\mathscr{H}^d$ is called d-dimensional because of its scaling behaviour: if E_λ is a copy of E scaled homothetically by a factor λ, then

$$\mathscr{H}^d(E_\lambda) = \lambda^d \, \mathscr{H}^d(E) \; .$$

Thus $\mathscr{H}^1$ scales like the length, $\mathscr{H}^2$ scales like the area, and so on.

(ii) The measure $\mathscr{H}^d$ is clearly invariant under rigid motions (translations and rotations). This implies that $\mathscr{H}^d$ agrees on $\mathbb{R}^d$ with the Lebesgue measure

up to some constant factor; the renormalization constant $\omega_d/2^d$ in (1) makes this factor equal to 1. Thus $\mathscr{H}^d(E)$ agrees with the usual d-dimensional volume for every set E in $\mathbb{R}^d$, and the area formula (§2.6) shows that the same is true if E is (a subset of) a d-dimensional surface of class C^1 in $\mathbb{R}^n$.

(iii) Besides Hausdorff measure, there are several other, less popular notions of d-dimensional measure: all of them are invariant under rigid motion, scale in the expected way, and agree with $\mathscr{H}^d$ for sets contained in $\mathbb{R}^d$ or in a d-dimensional surface of class C^1, and yet they differ for other sets (for further details, see [3], Section 2.10).

(iv) The definition of $\mathscr{H}^d(E)$ uses only the notion of diameter, and therefore makes sense when E is a subset of an arbitrary metric space. Note that $\mathscr{H}^d(E)$ depends only on the restriction of the metric to E, and not on the ambient space.

(v) The measure $\mathscr{H}^d$ is countably additive on the σ-algebra of Borel sets in $\mathbb{R}^n$, but not on all sets; to avoid pathological situations, we shall always assume that sets and maps are Borel measurable.

2.2. HAUSDORFF DIMENSION. - According to intuition, the length of a surface should be infinite, while the area of a curve should be null. These are indeed particular cases of the following implications:

$$\mathscr{H}^d(E) > 0 \Rightarrow \mathscr{H}^{d'}(E) = \infty \text{ for } d' < d,$$
$$\mathscr{H}^d(E) < \infty \Rightarrow \mathscr{H}^{d'}(E) = 0 \text{ for } d' > d.$$

Hence the infimum of all d such that $\mathscr{H}^d(E) = 0$ and the supremum of all d such that $\mathscr{H}^d(E) = \infty$ coincide. This number is called *Hausdorff dimension* of E, and denoted by $\dim_H(E)$. For surfaces of class C^1, the notion of Hausdorff dimension agrees with the usual one. Example of sets with non-integral dimension are described in §2.3.

Remarks. - (i) Note that $\mathscr{H}^d(E)$ may be 0 or ∞ even for $d = \dim_H(E)$.

(ii) The Hausdorff dimension of a set E is strictly related to the metric on E, and not just to the topology. Indeed, it is preserved under diffeomorphisms but not under homeomorphisms, and it does not always agree with the topological dimension. For instance, the Hausdorff dimension of the graph of a continuous function $f : \mathbb{R} \to \mathbb{R}$ can be any number between 1 and 2 (included).

(iii) For non-smooth sets, the Hausdorff dimension does not always conform to intuition: for example, the dimension of a cartesian product $E \times F$ of compact sets does not agree in general with the sum of the dimensions of E and F.

(iv) There are many other notions of dimension besides Hausdorff and topological one. Among these, packing dimension and box-counting dimension have interesting applications (see [2], Chapters 3,4).

2.3. SELF-SIMILAR FRACTALS. - Interesting examples of sets with non-integral dimension are self-similar fractals. We present here a simplified version of a construction due to J.E. Hutchinson (see [2], chapter 9). Let $\{\Psi_i\}$ be a finite set of similitudes of $\mathbb{R}^n$ with scaling factor $\lambda_i < 1$, and assume that there exists a bounded open set V such that the sets $V_i := \Psi_i(V)$ are pairwise disjoint and contained in V. The *self-similar fractal* associated with the system $\{\Psi_i\}$ is the compact set C which satisfies

$$C = \bigcup_i \Psi_i(C) \ . \tag{3}$$

The term "self-similar" follows by the fact that C can be written as union of scaled copies of itself. The existence (and uniqueness) of such C follows by a standard fixed-point argument applied to the map $C \mapsto \cup \Psi_i(C)$. The dimension d of C is the unique solution of the equation

$$\sum_i \lambda_i^d = 1 \ . \tag{4}$$

Formula (4) can be easily justified: if the sets $\Psi_i(C)$ are disjoint—and the assumption on V implies that this almost the case—then (3) implies $\mathscr{H}^d(C) = \sum \mathscr{H}^d(\Psi_i(C)) = \sum \lambda_i^d \mathscr{H}^d(C)$, and therefore $\mathscr{H}^d(C)$ can be positive and finite if and only if d satisfies (4).

An example of this constructions is the usual Cantor set in $\mathbb{R}$, which is given by the similitudes $\Psi_1(x) := \frac{1}{3}x$ and $\Psi_2(x) := \frac{2}{3} + \frac{1}{3}x$; by (4), its dimension is $d = \log 2/\log 3$. Other examples are described in Figures 1–3.

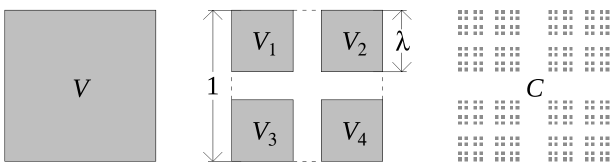

Fig. 1. The maps Ψ_i, $i = 1, \dots, 4$, takes the square V into the squares V_i at the corners of V. The scaling factor is λ for all i, hence $\dim_H(C) = \log 4/(-\log \lambda)$. Note that $\dim_H(C)$ can be any number between 0 and 2, including 1.

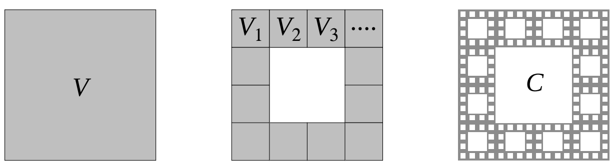

Fig. 2. A self-similar fractal with more complicated topology. The scaling factor is 1/4 for all twelve similitudes, hence $\dim_H(C) = \log 12/\log 4$.

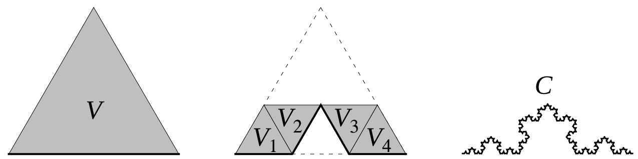

Fig. 3. The von Koch curve (or snowflake). The scaling factor is 1/3 for all four similitudes, hence $\dim_H(C) = \log 4/\log 3$.

2.4. RECTIFIABLE SETS. - Given an integer $k = 1, \dots, n$, we say that a set E in $\mathbb{R}^n$ is *k-rectifiable* if it can be covered by a countable family of sets $\{S_j\}$ such that S_0 is $\mathscr{H}^k$-negligible (that is, $\mathscr{H}^k(S_0) = 0$) and S_j is a k-dimensional surface of class C^1 for $j = 1, 2, \dots$ Note that $\dim_H(E) \le k$ because each S_j has dimension k.

A k-rectifiable set E bears little resemblance to smooth surfaces (it can be everywhere dense!), but it still admits a suitably weak notion of *tangent bundle*. More precisely, it is possible to associate with every $x \in E$ a k-dimensional subspace of $\mathbb{R}^n$, denoted by $\mathrm{Tan}(E, x)$, so that for every k-dimensional surface S of class C^1 in $\mathbb{R}^n$ there holds

$$\mathrm{Tan}(E, x) = \mathrm{Tan}(S, x) \quad \text{for } \mathscr{H}^k\text{-a.e. } x \in E \cap S, \tag{5}$$

where $\mathrm{Tan}(S, x)$ is the tangent space to S at x according to the usual definition.

It is not difficult to see that $\mathrm{Tan}(E, x)$ is uniquely determined by (5) up to an $\mathscr{H}^k$-negligible amount of points $x \in E$, and if E is a surface of class C^1, then it agrees with the usual tangent space for $\mathscr{H}^k$-almost all points of E.

Remarks. - (i) In the original definition of rectifiability, the sets S_j with $j > 0$ are Lipschitz images of $\mathbb{R}^k$, that is, $S_j := f_j(\mathbb{R}^k)$ where $f_j : \mathbb{R}^k \to \mathbb{R}^n$ is a Lipschitz map. It can be shown that this definition is equivalent to the one above.

(ii) The construction of the tangent bundle is straightforward: Let $\{S_j\}$ be a covering of E as in §2.4, and set $\mathrm{Tan}(E, x) := \mathrm{Tan}(S_j, x)$ where j is the smallest positive integer such that $x \in S_j$. Then (5) is an immediate corollary of the following lemma: if S and S' are k-dimensional surfaces of class C^1 in $\mathbb{R}^n$, then $\mathrm{Tan}(S, x) = \mathrm{Tan}(S', x)$ for $\mathscr{H}^k$-almost every $x \in S \cap S'$.

(iii) A set E in $\mathbb{R}^n$ is called *purely k-unrectifiable* if it contains no k-rectifiable subset with positive k-dimensional measure, or, equivalently, if $\mathscr{H}^k(E \cap S) = 0$ for every k-dimensional surface S of class C^1. For instance, every product $E := E_1 \times E_2$ where E_1 and E_2 are $\mathscr{H}^1$-negligible sets in $\mathbb{R}$ is a purely 1-unrectifiable set in $\mathbb{R}^2$ (it suffices to show that $\mathscr{H}^1(E \cap S) = 0$ whenever S is the graph of a function $f : \mathbb{R} \to \mathbb{R}$ of class C^1, and this follows by the usual formula for the length of the graph). Note that the Hausdorff dimension of such product sets can be any number between 0 and 2, hence rectifiability is not related to dimension. The self-similar fractals described in Figures 1 and 3 are both purely 1-unrectifiable.

2.5. RECTIFIABLE SETS WITH FINITE MEASURE. - If E is a k-rectifiable set with finite (or locally finite) k-dimensional measure, then $\mathrm{Tan}(E,x)$ can be related to the behaviour of E close to the point x.

Let $B(x,r)$ be the open ball in $\mathbb{R}^n$ with center x and radius r, and let $C(x,T,a)$ be the cone with center x, axis T—a k-dimensional subspace of $\mathbb{R}^n$—and amplitude $\alpha = \arcsin a$, that is,

$$C(x,T,a) := \big\{x' \in \mathbb{R}^n : \ \mathrm{dist}(x'-x,T) \le a\,|x'-x|\big\} .$$

For $\mathscr{H}^k$-almost every $x \in E$, the measure of $E \cap B(x,r)$ is asymptotically equivalent, as $r \to 0$, to the measure of a flat disk of radius r, that is,

$$\mathscr{H}^k\big(E \cap B(x,r)\big) \sim \omega_k r^k .$$

Moreover, the part of E contained in $B(x,r)$ is mostly located close to the tangent plane $\mathrm{Tan}(E,x)$, that is,

$$\mathscr{H}^k\big(E \cap B(x,r) \cap C(x,\mathrm{Tan}(E,x),a)\big) \sim \omega_k r^k \quad \text{for every } a>0.$$

When this condition holds, $\mathrm{Tan}(E,x)$ is called the *approximate tangent space* to E at x (see Figure 4).

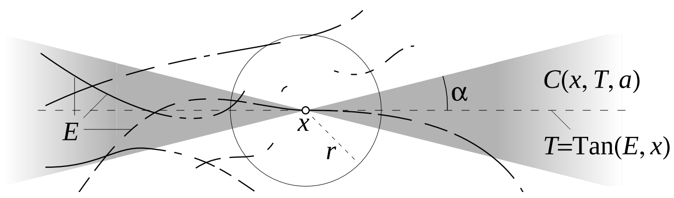

Fig. 4. A rectifiable set E close to a point x of approximate tangency. The part of E contained in the ball $B(x,r)$ but not in the cone $C(x,T,a)$ is not empty, but only small in measure.

2.6. THE AREA FORMULA. - The area formula allows to compute the measure $\mathscr{H}^k(\Phi(E))$ of the image of a set E in $\mathbb{R}^k$ as the integral over E of a suitably defined Jacobian determinant of Φ. When Φ is injective and takes values in $\mathbb{R}^k$, we recover the usual change of variable formula for multiple integrals.

We consider first the linear case. If L is a linear map from $\mathbb{R}^k$ to $\mathbb{R}^m$ with $m \ge k$, the volume ratio $\rho := \mathscr{H}^k(L(E))/\mathscr{H}^k(E)$ does not depend on E, and agrees with $|\det(PL)|$ where P is any linear isometry from the image of L into $\mathbb{R}^k$, and $\det(PL)$ is the determinant of the $k \times k$ matrix associated with PL. The volume ratio ρ can be computed using one of the following identities:

$$\rho = \sqrt{\det(L^*L)} = \sqrt{\sum (\det M)^2} , \tag{6}$$

where L^* is the adjoint of L (thus L^*L is a linear map from $\mathbb{R}^k$ into $\mathbb{R}^k$), and the sum in the last term is taken over all $k \times k$ minors M of the matrix associated with L.

Let $\Phi : \mathbb{R}^k \to \mathbb{R}^m$ be a map of class C^1 with $m \ge k$, and E a set in $\mathbb{R}^k$. Then

$$\int_{\Phi(E)} \#(\Phi^{-1}(y) \cap E)\, d\mathscr{H}^k(y) = \int_E J(x)\, d\mathscr{H}^k(x) , \tag{7}$$

where $\#A$ stands for the number of elements of A, and the Jacobian J is

$$J(x) := \sqrt{\det\big(\nabla\Phi(x)^* \,\nabla\Phi(x)\big)} . \tag{8}$$

Note that the left-hand side of (7) is $\mathscr{H}^k(\Phi(E))$ when Φ is injective.

Remark. - Formula (7) holds even if E is a k-rectifiable set in $\mathbb{R}^n$. In this case the gradient $\nabla\Phi(x)$ in (8) should be replaced by the tangential derivative of Φ at x (viewed as a linear map from $\mathrm{Tan}(E,x)$ into $\mathbb{R}^m$). No version of formula (7) is available when E is not rectifiable.

3. Vectors, covectors, and differential forms

In this section we review some basic notions of multilinear algebra. I have chosen a definition of k-vectors and k-covectors in $\mathbb{R}^n$, and of the corresponding exterior products, that is quite convenient for computations, even though not as satisfactory from the formal viewpoint. The main drawback is that it depends on the choice of a standard basis of $\mathbb{R}^n$, and therefore cannot be used to define forms (and currents) when the ambient space is a general manifold.

3.1. k-VECTORS AND EXTERIOR PRODUCT. - Let $\{e_1,\dots,e_n\}$ be the standard basis of $\mathbb{R}^n$. Given an integer $k \le n$, $I(n,k)$ is the set of all multi-indices $\mathbf{i} = (i_1,\dots,i_k)$ with $1 \le i_1 < i_2 < \ldots < i_k \le n$, and for every $\mathbf{i} \in I(n,k)$ we introduce the expression

$$e_{\mathbf{i}} = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} .$$

A *k-vector* in $\mathbb{R}^n$ is any formal linear combination $\sum \alpha_{\mathbf{i}}\, e_{\mathbf{i}}$ with $\alpha_{\mathbf{i}} \in \mathbb{R}$ for every $\mathbf{i} \in I(n,k)$. The space of k-vectors is denoted by $\wedge_k(\mathbb{R}^n)$; in particular, $\wedge_1(\mathbb{R}^n) = \mathbb{R}^n$. For reasons of formal convenience, we set $\wedge_0(\mathbb{R}^n) := \mathbb{R}$ and $\wedge_k(\mathbb{R}^n) := \{0\}$ for $k > n$.

We denote by $|\cdot|$ the euclidean norm on $\wedge_k(\mathbb{R}^n)$.

The *exterior product* $v \wedge w \in \wedge_{k+h}(\mathbb{R}^n)$ is defined for every $v \in \wedge_k(\mathbb{R}^n)$ and $w \in \wedge_h(\mathbb{R}^n)$, and is completely determined by the following properties: a) associativity, b) linearity in both arguments, c) $e_i \wedge e_j = -e_j \wedge e_i$ for every $i \ne j$ and $e_i \wedge e_i = 0$ for every i.

3.2. SIMPLE VECTORS AND ORIENTATION. - A *simple k-vector* is any v in $\wedge_k(\mathbb{R}^n)$ that can be written as a product of 1-vectors, that is,

$$v = v_1 \wedge v_2 \wedge \ldots \wedge v_k \ .$$

It can be shown that v is null if and only if the vectors $\{v_i\}$ are linearly dependent. If v is not null, then it is uniquely determined by the following objects: a) the k-dimensional space M spanned by $\{v_i\}$; b) the orientation of M associated with the basis $\{v_i\}$; c) the euclidean norm $|v|$. In particular, M does not depend on the choice of the vectors v_i. Note that $|v|$ is equal to the k-dimensional volume of the parallelogram spanned by $\{v_i\}$.

Hence the map $v \mapsto M$ is a one-to-one correspondence between the class of simple k-vectors with norm $|v| = 1$ and the Grassmann manifold of *oriented* k-dimensional subspaces of $\mathbb{R}^n$.

This remark paves the way to the following definition: if S is a k-dimensional surface of class C^1 in $\mathbb{R}^n$, possibly with boundary, an *orientation* of S is a continuous map $\tau_S : S \to \wedge_k(\mathbb{R}^n)$ such that $\tau_S(x)$ is a simple k-vector with norm 1 that spans $\mathrm{Tan}(S, x)$ for every x. To every orientation of S (if any exists) is canonically associated the orientation of the boundary ∂S that satisfies

$$\tau_S(x) = \eta(x) \wedge \tau_{\partial S}(x) \quad \text{for every } x \in \partial S, \tag{9}$$

where $\eta(x)$ is the *outer* unit normal to ∂S at x.

3.3. k-COVECTORS. - The standard basis of the dual of $\mathbb{R}^n$ is $\{dx_1, \ldots, dx_n\}$, where $dx_i : \mathbb{R}^n \to \mathbb{R}$ is the linear functional that takes every $x = (x_1, \ldots, x_n)$ into the i-th component x_i. For every $\mathbf{i} \in I(n, k)$ we set

$$dx_{\mathbf{i}} = dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k} \ ,$$

and the space $\wedge^k(\mathbb{R}^n)$ of *k-covectors* consists of all formal linear combinations $\sum \alpha_{\mathbf{i}} dx_{\mathbf{i}}$. The exterior product of covectors is defined as that for vectors. The space $\wedge^k(\mathbb{R}^n)$ is dual to $\wedge_k(\mathbb{R}^n)$ via the duality pairing $\langle \ ; \ \rangle$ defined by the relations $\langle dx_{\mathbf{i}}; e_{\mathbf{j}} \rangle := \delta_{\mathbf{ij}}$ (that is, 1 if $\mathbf{i} = \mathbf{j}$ and 0 otherwise).

3.4. DIFFERENTIAL FORMS AND STOKES THEOREM. - A *differential form* of order k on $\mathbb{R}^n$ is a map $\omega : \mathbb{R}^n \to \wedge^k(\mathbb{R}^n)$. Using the canonical basis of $\wedge^k(\mathbb{R}^n)$, we can write ω as

$$\omega(x) = \sum_{\mathbf{i} \in I(n,k)} \omega_{\mathbf{i}}(x)\, dx_{\mathbf{i}}$$

where the coordinates $\omega_{\mathbf{i}}$ are real functions on $\mathbb{R}^n$. The *exterior derivative* of a k-form ω of class C^1 is the $(k+1)$-form

$$d\omega(x) := \sum_{\mathbf{i} \in I(n,k)} d\omega_{\mathbf{i}}(x) \wedge dx_{\mathbf{i}} \ ,$$

where, for every scalar function f, df is the 1-form

$$df(x) := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\, dx_i \ .$$

If S is a k-dimensional oriented surface, the integral of a k-form ω on S is naturally defined by

$$\int_S \omega := \int_S \langle \omega(x); \tau_S(x) \rangle \, d\mathscr{H}^k(x) \ .$$

Stokes theorem states that for every $(k-1)$-form ω of class C^1 there holds

$$\int_{\partial S} \omega = \int_S d\omega \tag{10}$$

provided that ∂S is endowed with the orientation $\tau_{\partial S}$ that satisfies (9).

4. Currents

The definition of k-dimensional currents closely resembles that of distributions: they are the dual of smooth k-forms with compact support. Since every oriented k-dimensional surface defines by integration a linear functional on forms, currents can be regarded as generalized *oriented* surfaces. As every distribution admits a derivative, so every current admits a boundary. Indeed, many other basic notions of homology theory can be naturally extended to currents—this was actually one of the motivation behind the introduction of currents, due to G. de Rham.

For the applications to variational problems, smaller classes of currents are usually considered; the most relevant to the Plateau problem is that of integral currents. Note that the definitions of the spaces of normal, rectifiable and integral currents and the symbols used to denote them vary, sometimes more than slightly, depending on the author.

4.1. CURRENTS, BOUNDARY AND MASS. - Let n, k be integers with $n \geq k$. The space of *k-dimensional currents* on $\mathbb{R}^n$, denoted by $\mathscr{D}_k(\mathbb{R}^n)$, is the dual of the space $\mathscr{D}^k(\mathbb{R}^n)$ of smooth k-forms with compact support in $\mathbb{R}^n$. For $k \geq 1$, the *boundary* of a k-current T is the $(k-1)$-current ∂T defined by

$$\langle \partial T; \omega \rangle := \langle T; d\omega \rangle \quad \text{for every } \omega \in \mathscr{D}^{k-1}(\mathbb{R}^n), \tag{11}$$

while the boundary of a 0-current is set equal to 0. The *mass* of T is the number

$$\mathbb{M}(T) := \sup \big\{ \langle T; \omega \rangle : \ \omega \in \mathscr{D}^k(\mathbb{R}^n), \ |\omega| \leq 1 \big\} \ . \tag{12}$$

Fundamental examples of k-currents are *oriented* k-dimensional surfaces: to each oriented surface S of class C^1 is canonically associated the current $\langle T; d\omega\rangle := \int_S \omega$ (in fact, S is completely determined by the action on forms, that is, by the associated current). By Stokes theorem, the boundary of T is the current associated with the boundary of S, thus the notion of boundary for currents is compatible with the classical one for oriented surfaces. A simple computation shows that $\mathbb{M}(T) = \mathscr{H}^k(S)$, and therefore the mass provides a natural extension of the notion of k-dimensional volume to k-currents.

Remarks. - (i) Not all k-currents look like k-dimensional surfaces. For example, every k-vectorfield $v : \mathbb{R}^n \to \wedge_k(\mathbb{R}^n)$, defines by duality the k-current

$$\langle T; \omega\rangle := \int \langle \omega(x); v(x)\rangle \, d\mathscr{H}^n(x) \ .$$

The mass of T is $\int |v| \, d\mathscr{H}^n$, and the boundary is represented by a similar integral formula involving the partial derivatives of v (in particular, for 1-vectorfields, the boundary is the 0-current associated with the divergence of v). Note that the dimension of such T is k because k-vectorfields act on k-forms, and there is no relation with the dimension of the support of T, which is n.

(ii) To be precise, $\mathscr{D}^k(\mathbb{R}^n)$ is a locally convex topological vector-space, and $\mathscr{D}_k(\mathbb{R}^n)$ is its topological dual. As such, $\mathscr{D}_k(\mathbb{R}^n)$ is endowed with a *dual (or weak*) topology*. We say that a sequence of k-currents (T_j) converge to T if they converge in the dual topology, that is,

$$\langle T_j; \omega\rangle \to \langle T; \omega\rangle \quad \text{for every } \omega \in \mathscr{D}^k(\mathbb{R}^n). \tag{13}$$

Recalling the definition of mass, it is easy to show that it is lower semicontinuous with respect the dual topology, and in particular

$$\liminf \mathbb{M}(T_j) \geq \mathbb{M}(T) \ . \tag{14}$$

4.2. CURRENTS WITH FINITE MASS. - By definition, a k-current T with finite mass is a linear functional on k-forms which is bounded with respect to the supremum norm, and by Riesz theorem it can be represented as a bounded measure with values in $\wedge_k(\mathbb{R}^n)$. In other words, there exist a finite positive measure μ on $\mathbb{R}^n$ and a density function $\tau : \mathbb{R}^n \to \wedge_k(\mathbb{R}^n)$ such that $|\tau(x)| = 1$ for every x and

$$\langle T; \omega\rangle = \int \langle \omega(x); \tau(x)\rangle \, d\mu(x) \ .$$

The fact that currents are the dual of a separable space yields the following compactness result: A sequence of k-currents (T_j) with uniformly bounded masses $\mathbb{M}(T_j)$ admits a subsequence that converges to a current with finite mass.

4.3. NORMAL CURRENTS. - A k-current T is called *normal* if both T and ∂T have finite mass. The compactness result stated in the previous paragraph implies the following *compactness theorem for normal currents*: A sequence of normal currents (T_j) with $\mathbb{M}(T_j)$ and $\mathbb{M}(\partial T_j)$ uniformly bounded admits a subsequence that converges to a normal current.

4.4. RECTIFIABLE CURRENTS. - A k-current T is called *rectifiable* if it can represented as

$$\langle T; \omega\rangle = \int_E \langle \omega(x); \tau(x)\rangle \, \theta(x) \, d\mathscr{H}^k(x)$$

where E is a k-rectifiable set E, τ is an *orientation* of E—that is, $\tau(x)$ is a simple unit k-vector that spans $\mathrm{Tan}(E, x)$ for $\mathscr{H}^k$-almost every $x \in E$—and θ is a real function such that $\int_E |\theta| d\mathscr{H}^k$ is finite, called *multiplicity*. Such T is denoted by $T = [E, \tau, \theta]$. In particular, a rectifiable 0-current can be written as $\langle T; \omega\rangle = \sum \theta_i \, \omega(x_i)$ where $E = \{x_i\}$ is a countable set in $\mathbb{R}^n$ and $\{\theta_i\}$ is a sequence of real numbers with $\sum |\theta_i| < +\infty$.

4.5. INTEGRAL CURRENTS. - If T is a rectifiable current and the multiplicity θ takes integral values, T is called an *integer multiplicity rectifiable current*. If both T and ∂T are integer multiplicity rectifiable currents, then T is an *integral current*.

The first non-trivial result is the *boundary rectifiability theorem*: If T is an integer multiplicity rectifiable current and ∂T has finite mass, then ∂T is an integer multiplicity rectifiable current, too, and therefore T is an integral current.

The second fundamental result is the *compactness theorem for integral currents*: A sequence of integral currents (T_j) with $\mathbb{M}(T_j)$ and $\mathbb{M}(\partial T_j)$ uniformly bounded admits a subsequence that converges to an integral current.

Remarks. - (i) The point of the compactness theorem for integral currents is not the existence of a converging subsequence—that being already established by the compactness theorem for normal currents—but the fact that the limit is an integral current. In fact, this result is often referred to as a "closure theorem" rather than a "compactness theorem".

(ii) The following observations may clarify the role of assumptions in the compactness theorem. a) A sequence of integral currents (T_j) with $\mathbb{M}(T_j)$ uniformly bounded—but not $\mathbb{M}(\partial T_j)$—may converge to any current with finite mass, not necessarily a rectifiable one. b) A sequence of rectifiable currents (T_j) with rectifiable boundaries and $\mathbb{M}(T_j)$, $\mathbb{M}(\partial T_j)$ uniformly bounded may converge to any normal current, not necessarily a rectifiable one. Examples of both situations are described in Figure 5.

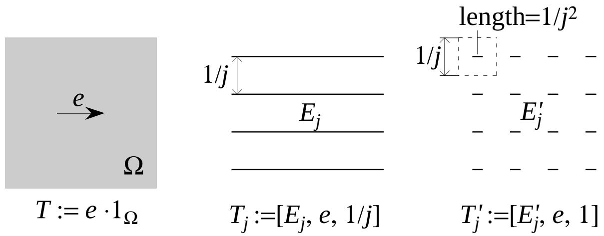

Fig. 5. T is the normal 1-current on $\mathbb{R}^2$ associated with the vectorfield equal to the unit vector e on the unit square Ω, and equal to 0 outside. T_j are the rectifiable currents associated with the sets E_j (middle) and the constant multiplicity $1/j$, and then $\mathbb{M}(T_j) = 1$, $\mathbb{M}(\partial T_j) = 2$. T'_j are the integral currents associated with the sets E'_j (left) and the constant multiplicity 1, and then $\mathbb{M}(T'_j) = 1$, $\mathbb{M}(\partial T'_j) = 2j^2$. Both (T_j) and (T'_j) converge to T.

4.6. Application to the Plateau problem. - The compactness result for integral currents implies the *existence of currents with minimal mass*: If Γ is the boundary of an integral k-current in $\mathbb{R}^n$, $1 \leq k \leq n$, then there exists a current T of minimal mass among those that satisfy $\partial T = \Gamma$.

The proof of this existence result is a typical example of the direct method: Let m be the infimum of $\mathbb{M}(T)$ among all integral currents with boundary Γ, and let (T_j) be a minimizing sequence (that is, a sequence of integral currents with boundary Γ such that $\mathbb{M}(T_j)$ converges to m). Since $\mathbb{M}(T_j)$ is bounded and $\mathbb{M}(\partial T_j) = \mathbb{M}(\Gamma)$ is constant, we can apply the compactness theorem for integral currents and extract a subsequence of (T_j) that converges to an integral current T. By the continuity of the boundary operator, $\partial T = \lim \partial T_j = \Gamma$, and by the semicontinuity of the mass $\mathbb{M}(T) \leq \lim \mathbb{M}(T_j) = m$ (cf. (14)). Thus T is the desired minimal current.

Remarks. - (i) Every integral $(k-1)$-current Γ with null boundary and compact support in $\mathbb{R}^n$ is the boundary of an integral current, and therefore is an admissible datum for the previous existence result.

(ii) A mass-minimizing integral current T is more regular than a general integral current. For $k = n-1$, there exists a closed singular set S with $\dim_H(S) \leq k-7$ such that T agrees with a smooth surface in the complement of S and of the support of the boundary. In particular T is smooth away from the boundary for $n \leq 7$. For general k it can only be proved that $\dim_H(S) \leq k-2$ Both results are optimal: In $\mathbb{R}^4 \times \mathbb{R}^4$, the minimal 7-current with boundary $\Gamma := \{|x| = |y| = 1\}$—a product of two 3-spheres—is the cone $T := \{|x| = |y| \leq 1\}$, and is singular at the origin. In $\mathbb{R}^2 \times \mathbb{R}^2$, the minimal 2-current with boundary $\Gamma := \{x = 0,\, |y| = 1\} \cup \{y = 0,\, |x| = 1\}$—a union of two disjoint circles—is the the union of the disks $\{x = 0,\, |y| \leq 1\} \cup \{y = 0,\, |x| \leq 1\}$, and is singular at the origin.

(iii) In certain cases, the mass-minimizing current T may not agree with the solution of the Plateau problem suggested by intuition. The first reason

is that currents do not include *non-orientable* surfaces, which sometimes may be more convenient (Figure 6). Another reason is that the mass of an integral current T associated with a k-rectifiable set E does not agree with the measure $\mathscr{H}^k(E)$—called *size* of T—because multiplicity must be taken into account, and for certain Γ the mass-minimizing current may be not size-minimizing (Figure 7). Unfortunately, proving the existence of size-minimizing currents is much more complicated, due to lack of suitable compactness theorems.

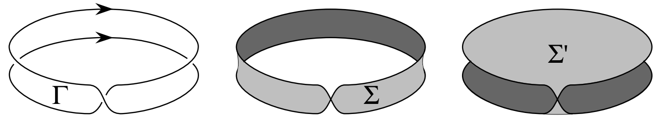

Fig. 6. The surface with minimal area spanning the (oriented) curve Γ is the Möbius strip Σ. However, Σ is not orientable, and cannot be viewed as a current. The mass-minimizing current with boundary Γ is Σ'.

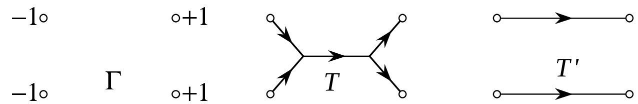

Fig. 7. The boundary Γ is a 0-current associated with four oriented points. The size (length) of T is smaller than that of T'. However, $\partial T = \Gamma$ implies that the multiplicity of T must be 2 on the central segment and 1 on the others; thus the mass of T is larger than its size. The size-minimizing current with boundary Γ is T, while the mass-minimizing one is T'.

(iv) For $k = 2$, the classical approach to the Plateau problem consists in parametrizing surfaces in $\mathbb{R}^n$ by maps f from a given 2-dimensional domain D into $\mathbb{R}^n$, and looking for minimizers of the area functional

$$\int_D \sqrt{\det(\nabla f^* \nabla f)}$$

(recall the area formula, §2.6) under the constraint $f(\partial D) = \Gamma$. In this framework, the choice of the domain D prescribes the topological type of admissible surfaces, and therefore the minimizer may differ substantially from the mass-minimizing current with boundary Γ (Figure 8).

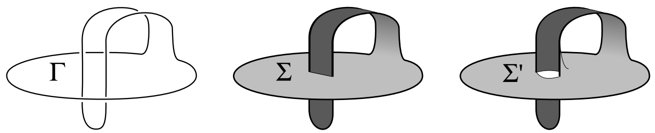

Fig. 8. The surface Σ minimizes the area among surfaces parametrized by the disc with boundary Γ. The mass-minimizing current Σ' can only be parametrized by a disc with a handle. Note that Σ is a singular surface, while Σ' is not.

(v) For some modeling problems, for instance those related to soap films and soap bubbles, currents do not provide the right framework (Figure 9). A possible alternative are integral varifolds (cf. [1]). However, it should be pointed out that this framework does not allow for "easy" application of the direct method, and the existence of minimal varifolds is in general quite difficult to prove.

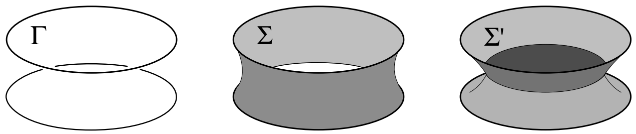

Fig. 9. Two possible soap films spanning the wire Γ: unlike Σ, Σ' cannot be viewed as a current with multiplicity 1 and boundary Γ.

4.7. MISCELLANEOUS RESULT AND USEFUL TOOLS. - (i) An important issue, related to the use of currents for solving variational problems, is to which extent integral currents can be approximated by regular objects. For many reasons, the "right" regular class to consider are not smooth surface, but integral polyhedral currents, that is, linear combinations with integral coefficients of oriented simplexes. The following *approximation theorem* holds: For every integral current T in $\mathbb{R}^n$ there exists a sequence of integral polyhedral currents (T_j) such that

$$T_j \to T,\ \partial T_j \to \partial T,\ \mathbb{M}(T_j) \to \mathbb{M}(T),\ \mathbb{M}(\partial T_j) \to \mathbb{M}(\partial T)\ .$$

The proof is based on a quite useful tool, called *polyhedral deformation*.

(ii) Many geometric operations for surfaces have an equivalent for currents. For instance, it is possible to define the image of a current in $\mathbb{R}^n$ via a smooth proper map $f : \mathbb{R}^n \to \mathbb{R}^m$. Indeed, to every k-form ω on $\mathbb{R}^m$ is canonically associated a k-form $f^{\#}\omega$ on $\mathbb{R}^n$, called *pull-back* of ω according to f. The adjoint of the pull-back is an operator, called *push-forward*, that takes every k-current T in $\mathbb{R}^n$ into a k-current $f_{\#}T$ in $\mathbb{R}^m$. If T is the rectifiable current associated with a rectifiable set E and a multiplicity θ, the push-forward $f_{\#}T$ is the rectifiable current associated with $f(E)$—and a multiplicity $\theta'(y)$ which is computed by adding up with the right sign all $\theta(x)$ with $x \in f^{-1}(y)$. As one might expect, the boundary of the push-forward is the push-forward of the boundary.

(iii) In general, it is not possible to give a meaning to the intersection of two currents, and not even of a current and a smooth surface. However, it is possible to define the intersection of a normal k-current T and a level surface $f^{-1}(y)$ of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^h$ (with $k \le h \le n$) *for almost every* y, resulting in a current T_y with the expected dimension $h-k$. This operation is called *slicing*.

(iv) When working with currents, a quite useful notion is that of *flat norm*:

$$\mathbb{F}(T) := \inf \big\{ \mathbb{M}(R) + \mathbb{M}(S) :\ T = R + \partial S \big\}$$

where T and R are k-currents, and S is a $(k+1)$-current. The relevance of this notion lies in the fact that a sequence (T_j) that converges with respect to the flat norm converges also in the dual topology, and the converse holds if the masses $\mathbb{M}(T_j)$ and $\mathbb{M}(\partial T_j)$ are uniformly bounded. Hence the flat norm metrizes the dual topology of currents (at least on sets of currents where the mass and the mass of the boundary are bounded).

Since $\mathbb{F}(T)$ can be explicitly estimated from above, it can be quite useful in proving that a sequence of currents converges to a certain limit. Finally, the flat norm gives a (geometrically significant) measure of how far apart two currents are: for instance, given the 0-currents δ_x and δ_y (the Dirac masses at x and y, respectively), then $\mathbb{F}(\delta_x - \delta_y)$ is exactly the distance between x and y.

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