# Variational convergence for functionals of Ginzburg-Landau type.

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*This paper is dedicated to the memory of Ennio De Giorgi*

Abstract. In the first part of this paper we prove that certain functionals of Ginzburg-Landau type for maps from a domain in $\mathbb{R}^{n+k}$ into $\mathbb{R}^k$ converge in a suitable sense to the area functional for surfaces of dimension $n$ (Theorem 1.1). In the second part we modify this result in order to include Dirichlet boundary condition (Theorem 5.5), and, as a corollary, we show that the rescaled energy densities and the Jacobians of minimizers converge to minimal surfaces of dimension $n$ (Corollaries 1.2 and 5.6). Some of these results were announced in [2].

## 1. Introduction

In this paper we investigate the asymptotic behaviour as $\varepsilon \to 0$ of the functionals

$$F_\varepsilon(u) := \int_\Omega \frac{1}{k}|Du|^k + \frac{1}{\varepsilon^2}W(u) \tag{1.1}$$

where $\Omega$ is a regular domain in $\mathbb{R}^{n+k}$, with $n \geq 0$, $k \geq 2$, the map $u : \Omega \to \mathbb{R}^k$ belongs to the Sobolev space $W^{1,k}$, and $W$ is a positive continuous potential on $\mathbb{R}^k$ which vanishes only on the unit sphere $S^{k-1}$.

In particular, for $k = 2$, $n = 0$, and $W(u) = \frac{1}{4}(1-|u|^2)^2$, we recover the complex Ginzburg-Landau functional in the simplified form considered by F. Bethuel, H. Brezis, and F. Hélein [9]. This functional has been widely studied in recent years because it shares many of the relevant features of more complex functionals used to model vortices in superfluidity and superconductivity (see [26, 11]).

We focus our attention on some asymptotic properties of the minimizers $u_\varepsilon$ of $F_\varepsilon$ with prescribed boundary datum $v$ in the trace space $W^{1-1/k,k}(\partial\Omega, S^{k-1})$. Note that this boundary condition has no physical meaning, but is rather a replacement for the additional terms which appear in the physical functionals (e.g., accounting for the interaction with an external magnetic field). If $n \geq 1$ and the Jacobian of the boundary datum $v$ is non-trivial (cf. Section 5.2), the energy of minimizers $F_\varepsilon(u_\varepsilon)$ is of order $|\log\varepsilon|$ as $\varepsilon \to 0$, and the energy densities

$$e_\varepsilon(u_\varepsilon) := \frac{1}{k}|Du_\varepsilon|^k + \frac{1}{\varepsilon^2}W(u_\varepsilon) \tag{1.2}$$

concentrate on a minimal surface of dimension $n$ in $\Omega$.

From a variational viewpoint, this energy-concentration result can be viewed as a corollary of a more general fact: as $\varepsilon \to 0$, the functionals $F_\varepsilon$ approximate in a suitable sense the area functional for surfaces of dimension $n$ (and codimension $k$) in $\Omega$. This fact is well-known for the *scalar* Ginzburg-Landau functionals, namely those defined as in (1.1) for $k = 1$, $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$, and $W$ a double-well potential on $\mathbb{R}$; a rigorous justification of their convergence to the area functional for codimension-one surfaces was given in terms of $\Gamma$-convergence by L. Modica and S. Mortola [32, 33] (see also [31, 5, 7, 1]), who developed an early suggestion of E. De Giorgi.

In this paper we prove a $\Gamma$-convergence result for the functionals $F_\varepsilon$ with $k \geq 2$ and $n \geq 0$, which is very close in spirit to that of Modica and Mortola.[1]

The underlying idea is that the energy densities of the maps $u_\varepsilon$ are closely connected to their Jacobians—in other words the Jacobian is the right tool to track energy concentration in (vector) Ginzburg-Landau functionals. We recall that the Jacobian of a map $u \in W^{1,k}(\Omega, \mathbb{R}^k)$ is the $k$-form $Ju := du_1 \wedge \ldots \wedge du_k$, where each $du_i$ is the differential of the $i$-th component of $u$ (in particular, for $n = 0$, $Ju = \det(Du)\, dx_1 \wedge \cdots \wedge dx_k$). If we denote by $\star$ the standard identification of $k$-covectors and $n$-vectors in $\mathbb{R}^{n+k}$, then $\star Ju$ is an $L^1$ map on $\Omega$ valued in $n$-vectors, and defines an $n$-dimensional current in $\Omega$; moreover $Ju$ is a differential, and this means that $\star Ju$ is a boundary (see Section 2.10). Thus the Jacobian is naturally endowed with a structure of oriented $n$-dimensional surface in a generalized sense; of course, being an $L^1$ function, it brings little resemblance to the usual notion of $n$-dimensional surface, and should be rather understood as a sort of *diffuse* surface.

However, given maps $u_\varepsilon$ such that $F_\varepsilon(u_\varepsilon)$ is of order $|\log\varepsilon|$, the corresponding Jacobians $\star Ju_\varepsilon$ converge, up to a subsequence, to a $n$-dimensional *integral boundary*, that is, a current supported on an $n$-dimensional rectifiable set $M$, which is also a boundary. We prove indeed the following compactness and $\Gamma$-convergence result:[2]

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[1] Our results do not quite fit in the standard definition of $\Gamma$-convergence as given by E. De Giorgi and T. Franzoni [18] (see also [1, 14, 17]), and we do not rely on any abstract result from that theory.

[2] In what follows, $\alpha_k$ is the Lebesgue measure of the unit ball of $\mathbb{R}^k$, $\beta_k := (k-1)^{k/2}\alpha_k$, and $\|M\|$

**Theorem 1.1.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^{n+k}$ with $n \geq 0$ and $k \geq 2$. Then the following statements hold.*

(i) Compactness and lower-bound inequality. *Given a (countable) sequence of maps $(u_\varepsilon)$ such that $F_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$ for some finite $C$, we can extract a subsequence (not relabelled) such that the Jacobians $\star Ju_\varepsilon$ converge in the flat norm $\mathbf{F}_\Omega$ to $\alpha_k M$, where $M$ is an $n$-dimensional integral boundary in $\Omega$, and*

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \geq \beta_k \|M\| . \tag{1.3}$$

(ii) Upper bound inequality. *For every $n$-dimensional integral boundary $M$ in $\Omega$ there exist maps $u_\varepsilon$ such that $\mathbf{F}_\Omega(\star Ju_\varepsilon - \alpha_k M) \to 0$ and*

$$\lim_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} = \beta_k \|M\| . \tag{1.4}$$

**Remarks.** (i) It is important to notice that the constant $\beta_k$ in the upper and lower bound inequality does not depend on the choice of the potential $W$. Thus, if we replace $W$ by $\sigma W$ in (1.1) and let $\sigma$ tend to 0 in the lower bound inequality (1.3), we obtain the stronger estimate

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_\Omega \frac{1}{k} |Du_\varepsilon|^k \geq \beta_k \|M\| .$$

In particular, if the sequence $(u_\varepsilon)$ satisfies the upper bound inequality (1.4), then the potential part of the energy is asymptotically negligible, that is

$$\int_\Omega \frac{1}{\varepsilon^2} W(u_\varepsilon) = o(|\log \varepsilon|) . \tag{1.5}$$

(ii) As shown in Lemma 3.7 below, the sequence $\star Ju_\varepsilon$ is asymptotically equivalent in the flat norm $\mathbf{F}_\Omega$ to the sequence $\rho(u_\varepsilon) \star Ju_\varepsilon$ for every bounded function $\rho \in C^1(\mathbb{R}^k)$ whose integral on the unit ball of $\mathbb{R}^k$ is $\alpha_k$, and therefore they converge to the same limit $\alpha_k M$.

(iii) For $k = 2$ and $n$ arbitrary, i.e., for maps valued in $\mathbb{R}^2$, the compactness and lower bound inequality for the functionals $F_\varepsilon$ have been proved independently by R.L. Jerrard and H.M. Soner in [29].

(iv) The convergence of $\star Ju_\varepsilon$ to $\alpha_k M$ in the flat norm implies strong convergence in the dual of forms of class $C_0^1$ (cf. Section 2.5). By a simple interpolation

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is the mass of $M$; see Section 2 for further details and the precise definitions of Lipschitz domain, integral boundary and flat norm.

argument (cf. [29, Theorem 3.5]), one can derive strong convergence of $\star J u_\varepsilon$ in the dual of forms of class $C_0^{0,\beta}$ for every $\beta > 0$. However, the Jacobians $\star J u_\varepsilon$ are not necessarily uniformly bounded in mass,[3] and therefore convergence may not extend to forms of class $C_0^0$.

(v) Theorem 1.1 should hold also when $\Omega$ is an $(n+k)$-dimensional, smooth, compact manifold with boundary. However, while statement (i) can be easily deduced from the flat case via a localization argument, the proof of statement (ii) requires a careful (even though not difficult) extension of Theorem 5.10 of [3].

(vi) While the exponent $k$ in (1.1) has a relevant geometric meaning—cf. Remark (iv) after Corollary 1.2—the exponent 2 has none: replacing the term $\varepsilon^{-2}$ in front of $W(u)$ by any negative power of $\varepsilon$ would only change the right-hand side of (1.3) and (1.4) by a constant factor.

(vii) A different way of stating the convergence of $F_\varepsilon$ to the area functional is the following: for a certain $k - 1 < p < k$, the rescaled functionals $|\log \varepsilon|^{-1} F_\varepsilon$ $\Gamma$-converge on $W^{1,p}$, endowed with the weak topology, to the functional which agrees with $\|Ju\|$ for $u \in W^{1,p}(\Omega, S^{k-1})$ and with $+\infty$ otherwise. For $n = 0$ and $k = 2$ this statement was actually proved in [29, Theorem 4.1]. For general $n$, the proof of the upper bound inequality would require a much more delicate construction than Theorem 1.1. Note that such a result makes sense only if coupled with the compactness of minimizers in $W^{1,p}$, which is rather difficult to prove (see remark (vi) after Corollary 1.2). On the other hand, the $\Gamma$-limit in weaker topologies is usually trivial: for instance, one easily checks that for every vanishing sequence $\sigma_\varepsilon$ and every $1 \le p < \infty$ the rescaled functionals $\sigma_\varepsilon F_\varepsilon$ $\Gamma$-converge in the strong topology of $L^p$ to the functional which is 0 on $L^p(\Omega, S^{k-1})$ and $+\infty$ elsewhere.

As stated, Theorem 1.1 has little usefulness, because the minimizers of $F_\varepsilon$ on $W^{1,k}(\Omega, \mathbb{R}^k)$ are all constant functions with value in $S^{k-1}$, and there is nothing more to be said about them. However, suitable variants of this result can be used to understand the asymptotics of different problems with non-trivial minimizer. For instance, similar ideas were used in [34] to find non-trivial *local minimizers* for the Ginzburg-Landau functional in 3d. Another interesting example is the minimization of the Ginzburg-Landau functional for sections of suitable fiber bundles on a given manifold, in the spirit of [6, 7].

Following the work of J. Bourgain, H. Brezis, and P. Mironescu [13], in the second half of this paper we consider the minimization of $F_\varepsilon$ with a prescribed

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[3] For $n = 0$, $k = 2$, and $\Omega = (0,1) \times (0,1)$, set $u_\varepsilon(x_1, x_2) := [1 + \varepsilon \cos(x_1/\delta)] \cdot \tilde{u}(x_2)$ where $\delta := \varepsilon |\log \varepsilon|^{-1/4}$ and $\tilde{u}$ is a smooth map of $(0,1)$ onto $S^1$. Then $F_\varepsilon(u_\varepsilon) \sim (\varepsilon/\delta)^2 = |\log \varepsilon|^{1/2} = o(|\log \varepsilon|)$. On the other hand, $u_\varepsilon^{-1}(y)$ has cardinality of order $1/\delta$ for every $y \in \mathbb{R}^2$ such that $1 - \varepsilon < |y| < 1 + \varepsilon$, and the area formula yields $\int_\Omega |Ju_\varepsilon| \sim \varepsilon/\delta = |\log \varepsilon|^{1/4}$, which diverges as $\varepsilon \to 0$.

boundary datum $v$ in the trace space $W^{1-1/k,k}(\partial\Omega, S^{k-1})$. We prove a modification of Theorem 1.1 that includes these Dirichlet boundary conditions (Theorem 5.5) and immediately derive the concentration of Jacobians and energy densities for minimizing sequences (Corollary 5.6) to area-minimizing surfaces of codimension $k$. A particular case of this result is the following:

**Corollary 1.2.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^{n+k}$, with $n \geq 1$ and $k \geq 2$, whose $n$-th homology group $H_n(\overline{\Omega}, \mathbb{Z})$ is trivial. Let $v$ be a map in the trace space $W^{1-1/k,k}(\partial\Omega, S^{k-1})$, and for every $\varepsilon > 0$ let $u_\varepsilon$ be a minimizer of $F_\varepsilon$ with trace $v$ on $\partial\Omega$.[4] Then*

$$\min_{u=v \text{ on } \partial\Omega} F_\varepsilon(u) = F_\varepsilon(u_\varepsilon) = O(|\log\varepsilon|) \ .$$

*Possibly passing to a subsequence, the Jacobians $\star Ju_\varepsilon$ converge in the flat norm $\mathbf{F}_{\mathbb{R}^{n+k}}$ to $\alpha_k M$, where $M$ minimizes the mass among all $n$-dimensional rectifiable currents supported in $\overline{\Omega}$ with boundary $\partial M = \alpha_k^{-1} \star Jv$. Moreover the potential part of the energy is asymptotically negligible in the sense that (1.5) holds, and the energy densities $e_\varepsilon(u_\varepsilon)$ and $\frac{1}{k}|Du_\varepsilon|^k$, rescaled by $1/|\log\varepsilon|$, converge in the sense of measures to $\beta_k|M|$, where $|M|$ is the variation of the vector measure $M$.*

**Remarks.** (i) The boundary of $\star Ju_\varepsilon$, viewed as a current on the entire $\mathbb{R}^{n+k}$, is $\star Jv$ (see Section 5.2). Moreover $\star Jv$ is of the form $\alpha_k \partial N$ where $N$ is rectifiable $n$-dimensional current supported in $\overline{\Omega}$ (see Remark 5.4(ii)) and therefore the minimum problem that defines $M$ is well-posed.

(ii) When $H_n(\overline{\Omega}, \mathbb{Z})$ is non-trivial, then $M$ solves the Plateau's problem with boundary condition $\partial M = \alpha_k^{-1} \star Jv$ in a prescribed cobordism class determined by a generic level set of $u$ (see Corollary 5.6 for a precise statement).

(iii) For $n = 0$ it is usually assumed (cf. [9]) that $\partial\Omega$ is connected and the boundary datum $v : \partial\Omega \to S^{k-1}$ is a smooth map with degree $d \neq 0$. In this case the Jacobians of the minimizers $u_\varepsilon$ converge to $\alpha_k M$ where $M$ is a 0-dimensional integral current—a sum of Dirac masses with integer multiplicities $d_i$—which minimizes the mass $\|M\| = \sum |d_i|$ under the constraint $\sum d_i = d$. Notice that this minimization problem only implies that the integers $d_i$ have equal signs, while the locations of the Dirac masses are determined by a lower-order term in the formal expansion of the energies $F_\varepsilon$ (see [9] for the case $k = 2$). Thus the situation is very different from the case $n > 0$, where the limit currents $M$ are (almost) completely determined by their minimality property.

(iv) If the term $|Du|^k$ in $F_\varepsilon$ is replaced by $|Du|^p$, then energy-concentration occurs only for $p \geq k$. Indeed, for every $p > 1$ the functionals $F_\varepsilon$ are

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[4] The existence of such minimizers is ensured by standard semicontinuity and compactness results.

equicoercive and $\Gamma$-converge on $L^1$ to the functional $F$ equal to $\int_\Omega |Du|^p$ for $u \in W^{1,p}(\Omega, S^{k-1})$, and $+\infty$ elsewhere. As pointed out in [24], for $p < k$ every map $v$ in $W^{1-1/p,p}(\partial\Omega, S^{k-1})$ is the trace of a map in $W^{1,p}(\Omega, S^{k-1})$, and therefore the minimum of $F_\varepsilon$ with Dirichlet boundary datum $v$ is of order $O(1)$. Hence the minimizers of $F_\varepsilon$ converge weakly in $W^{1,p}$ to minimizers of $F$—that is, $p$-harmonic maps from $\Omega$ into $S^{k-1}$—and convergence carries over to Jacobians and energy densities. On the other hand, for $p \geq k$ not all maps $v$ in $W^{1-1/p,p}(\partial\Omega, S^{k-1})$ are traces of maps in $W^{1,p}(\Omega, S^{k-1})$; thus the previous argument does not apply, and in fact we expect a situation very similar to the case $p = k$, albeit with few essential differences: the scaling factor $1/|\log\varepsilon|$ in Theorem 1.1 should be replaced by $\varepsilon^{p-k}$, and the constant $\beta_k$ should be replaced by a different one, determined by a suitable optimal-profile problem.[5] The fact that for $p = k$ the constant $\beta_k$ is not determined by an optimal-profile problem and the $\Gamma$-limit of the functionals $F_\varepsilon$ does not depend on the choice of the potential $W$ is related to the fact that $p = k$ is in some sense a critical case (as shown in [4], a similar situation occurs also for scalar functionals).

(v) Corollary 1.2 is very close to many concentration results proved in recent years for minimizers of the Ginzburg-Landau functional. For $n = 0$ and $k = 2$, that is, for maps from $\Omega \subset \mathbb{R}^2$ into $\mathbb{R}^2$, an exhaustive description of the asymptotic behaviour of minimizers $u_\varepsilon$ was given by Bethuel, Brezis, and Hélein [9] (see also [10, 40]). Some of their results about concentration on points have been later extended in [22, 25] to maps from $\Omega \subset \mathbb{R}^k$ into $\mathbb{R}^k$ with $k$ arbitrary.[6] For $n = 1$, concentration results on curves of minimal length were obtained by T. Rivière [35] in the case $k = 2$, and by E. Sandier [38] in the case $k$ arbitrary. Concentration on minimal surfaces of arbitrary dimension $n$ and codimension $k = 2$ was proved by F.-H. Lin and T. Rivière [30].

(vi) In the papers mentioned above, it is also shown that the minimizers $u_\varepsilon$ are weakly compact in some Sobolev space. For $k = 2$ and $n$ arbitrary, an elegant and relatively simple proof of the compactness of critical points of $F_\varepsilon$ in $W^{1,p}$ for $p < n/(n-1)$ has been given in [8]. We underline that such compactness results cannot be obtained only by energy methods, but require the full strength of the Euler-Lagrange equation of $F_\varepsilon$ (cf. [16]). To our knowledge, no results are available for general $k$ and $n$.

To better explain the relation between Jacobians and energy concentration, we briefly sketch the idea of the proof of statement (i) of Theorem 1.1 for maps from $\Omega \subset \mathbb{R}^k$ into $\mathbb{R}^k$ (i.e., $n = 0$).

Let maps $u_\varepsilon$ be given such that $F_\varepsilon(u_\varepsilon)$ is of order $|\log\varepsilon|$, and let $e_\varepsilon = e_\varepsilon(u_\varepsilon)$

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[5] The proof of the $\Gamma$-convergence for $p > k$ has been recently carried out in [19].

[6] In [9] it is shown that, for every $\delta > 0$, the sets $\{|u_\varepsilon| \leq 1 - \delta\}$ converge in the sense of Kuratowski to a finite set $S$, where also energy densities concentrate. For $k > 2$, energy concentration in a finite set $S$ follows by the weak convergence of $u_\varepsilon$ in $W^{1,k}_{\text{loc}}(\Omega \setminus S, \mathbb{R}^k)$ proved in [25] (see also [27]).

denote the corresponding energy densities. For every $\varepsilon > 0$ we choose a cubic grid of size $\ell = \ell(\varepsilon)$, and denote its $(k-1)$-dimensional skeleton by $R_{\varepsilon,k-1}$ (see Figure 3.1 in Section 3). If $\ell$ is chosen so that $\varepsilon|\log\varepsilon| = o(\ell)$ and the grid is suitably positioned, the functions $|u_\varepsilon|$ converge to 1 uniformly on $R_{\varepsilon,k-1}$ as $\varepsilon \to 0$, and then we can slightly perturb each $u_\varepsilon$ so that $|u_\varepsilon| = 1$ on $R_{\varepsilon,k-1}$. Now, for every $k$-dimensional cell $Q$ in the grid we denote by $d_Q$ the degree of the restriction of $u_\varepsilon$ to $\partial Q$ (as a map valued in $S^{k-1}$). Then

(i) $\int_Q e_\varepsilon \geq \beta_k |d_Q| \cdot |\log\varepsilon|$ by a fundamental estimate of R.L. Jerrard [27];[7]

(ii) $\int_Q \star J u_\varepsilon = \alpha_k d_Q$. Indeed $\star J u_\varepsilon = \det(Du_\varepsilon)$, and by the area formula its integral on $Q$ is equal to the integral of the degree $\deg(u_\varepsilon, Q, y)$ over all $y \in \mathbb{R}^k$; on the other hand, $\deg(u_\varepsilon, Q, y)$ is equal to $d_Q$ for $|y| < 1$ and to 0 for $|y| > 1$.

We consider now the measure $\mu_\varepsilon := \sum d_Q \delta_Q$, where $\delta_Q$ is the Dirac mass at the center of $Q$ and the sum is taken over all cells $Q$ in the grid. By (i) we obtain that $|\log\varepsilon|^{-1} F_\varepsilon(u_\varepsilon) \geq \beta_k \sum |d_Q| = \beta_k \|\mu_\varepsilon\|$. Hence the measures $\mu_\varepsilon$ converge (up to a subsequence) to a measure $\mu$ which is also a sum of Dirac masses with integer multiplicity, i.e., a 0-dimensional integral current, and moreover

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \geq \beta_k \|\mu\| \ .$$

Statement (ii) above shows that $\star J u_\varepsilon$ gets closer and closer to $\alpha_k \mu_\varepsilon$, in the sense that their integrals over all cells in the grid are the same, and the size of these cells tends to 0. Indeed $\star J u_\varepsilon$ and $\alpha_k \mu_\varepsilon$ are asymptotically equivalent in the flat norm $\mathbf{F}_\Omega$ if $\ell = o(|\log\varepsilon|^{-1})$, and therefore $\star J u_\varepsilon$ converge to $\alpha_k \mu$.

Let us briefly comment on the passage to higher dimension ($n > 0$). The key step in the proof sketched above consists in approximating the Jacobians $\star J u_\varepsilon$ by the measures $\alpha_k \mu_\varepsilon$ which are obtained by "projecting" all the mass of $\star J u_\varepsilon$ contained in each cell of the grid into its center. Now we realize that this is just Federer-Fleming polyhedral deformation theorem applied to the 0-dimensional current $\star J u_\varepsilon$ and to the dual grid, in the sense that $\alpha_k \mu_\varepsilon$ is the push-forward of $\star J u_\varepsilon$ according to a retraction of $\mathbb{R}^2 \setminus R_{\varepsilon,k-1}$ onto the 0-dimensional dual skeleton $R'_{\varepsilon,0}$, namely the set of centers of the cells in the original grid (see Figure 3.1 in Section 3).

Thus, for general $n$ and $k$, we apply a suitable version of the deformation theorem to the $n$-dimensional current $\star J u_\varepsilon$, or, more precisely, to a slight perturbation of $\star J u_\varepsilon$, and obtain a current $\alpha_k M_\varepsilon$, where $M_\varepsilon$ is a polyhedral boundary with integral multiplicity. Then we show that $\alpha_k M_\varepsilon$ get closer and closer to

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[7]The estimate actually contains an additional error term that depends on the restriction of $u_\varepsilon$ on $\partial Q$. However, the proof can be adjusted so to control this error by the integral of $e_\varepsilon$ on $\partial Q$, and this happens to be enough for our purposes. Results along this line were also obtained in earlier work by E. Sandier [37].

$\star Ju_\varepsilon$ as $\varepsilon$ tends to 0, and its mass can be controlled by $|\log \varepsilon|^{-1} F_\varepsilon(u_\varepsilon)$. Now the compactness of $M_\varepsilon$, and therefore that of $\star Ju_\varepsilon$, follows from the closure theorem for integral currents, while the lower-bound inequality (1.3) is a by-product of an estimate obtained in the process and a standard localization argument.

The proof of statement (ii) of Theorem 1.1 is a direct application of the construction of $S^{k-1}$-valued maps with prescribed Jacobians described in [3], inspired to the dipole construction of [15]. In the case of prescribed boundary values (Theorem 5.5) compactness and lower bound inequality can be derived (with some work) from statement (i) of Theorem 1.1, but the upper bound inequality must be proved almost from scratch, and requires a non-trivial modification of the construction of [3] (see Appendix B).

The paper is organized as follows:

## Contents

## 2. Preliminaries and Notation

In this paper we use the term sequence also for families $(x_\varepsilon)$ (of points, functions, or else) parametrized by the continuous positive parameter $\varepsilon$, and call countable subsequence any subfamily $(x_{\varepsilon_i})$, with $i$ integer and $\lim \varepsilon_i = 0$ as $i \to +\infty$, In order to simplify the notation, we often omit to relabel subsequences, and simply write $x_\varepsilon$ instead of $x_{\varepsilon_i}$.

Given functions $f$ and $g$, we write as usual $f = o(g)$ and $f = O(g)$ to mean that $\lim |f|/|g|$ vanishes and $\limsup |f|/|g|$ is finite, respectively. We use the letter $K$, with no index, to denote all universal constants, that is, constants that depend only on $k$, $n$, and on the potential $W$, but not on $\Omega$, $\varepsilon$ or any other parameter; $K$ may take different values even within the same expression. We use the letter $C$, possibly indexed, for constants that are not universal.

We write $a \vee b$ for $\max\{a, b\}$, and $a \wedge b$ for $\min\{a, b\}$. The symbol $\wedge$ is also used for the wedge product of vectors and covectors, the difference being clear from the context. We write $U \Subset V$ to mean that $U$ is relatively compact in $V$.

We denote by $B_r = B_r^d$ the open ball in $\mathbb{R}^d$ with center in the origin and radius $r$; for every integer $k \geq 2$, $S^{k-1} := \partial B_1^k$ is the unit sphere in $\mathbb{R}^k$, and

$$\alpha_k := \mathscr{L}^k(B_1^k) , \quad \beta_k := (k-1)^{k/2} \alpha_k , \tag{2.1}$$

where $\mathscr{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$. $\mathscr{H}^h$ is the $h$-dimensional Hausdorff measure. All other measures $\mu$ are Borel, locally bounded, and possibly vector-valued; $\mu \llcorner E$ is the restriction of the measure $\mu$ to the Borel set $E$, while $f \cdot \mu$ is the measure associated to a density function $f \in L^1_{\text{loc}}(\mu)$. Sets and functions are always assumed Borel measurable. When no doubts may arise, we omit any explicit mention of the measure in integrals.

Given $h$-dimensional Lipschitz manifolds $N$ and $N'$ with $N$ compact and $\partial N' = \varnothing$, and a continuous map $f : N \to N'$, we denote by $\deg(f, N, N', y)$ the degree of $f$ at $y \in N' \setminus f(\partial N)$. When $N$ has no boundary and $N'$ is connected the degree does not depend on $y$, and we simply write $\deg(f, N, N')$.

For the rest of this paper, $n \geq 0$ and $k \geq 2$ are given integers, and $\Omega$ is a *bounded Lipschitz domain* in $\mathbb{R}^{n+k}$. This means that $\partial \Omega$ can be covered by open sets $U$ such that $\Omega$ agrees inside each $U$ with the epigraph of a Lipschitz function (with respect to a suitable choice of the axes). Concerning the regularity of the boundary of $\partial \Omega$, we do not aim to achieve the maximal generality, but rather to show that the concentration phenomena that we investigate—unlike others—are not affected by a mild lack of smoothness in the boundary, including the presence of corner points.

**2.1. Ginzburg-Landau functionals.** The potential $W$ is a continuous function on $\mathbb{R}^k$ which vanishes on $S^{k-1}$, is strictly positive elsewhere, and has growth of order at least two around the zero-set, and at least $k$ at infinity,[8] that is

$$\liminf_{|y| \to 1} \frac{W(y)}{(1-|y|)^2} > 0 \quad \text{and} \quad \liminf_{|y| \to \infty} \frac{W(y)}{|y|^k} > 0 . \tag{2.2}$$

Given $\varepsilon > 0$, $u \in W^{1,k}(\Omega, \mathbb{R}^k)$, and $A \subset \Omega$, we set (cf. (1.1))

$$F_\varepsilon(u, A) := \int_A \frac{1}{k} |Du|^k + \frac{1}{\varepsilon^2} W(u)$$

where $|Du|$ stands for the euclidean norm of the matrix $Du$.[9] The $\varepsilon$-energy density associated to $u$ is

$$e_\varepsilon(u) := \frac{1}{k} |Du|^k + \frac{1}{\varepsilon^2} W(u) .$$

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[8] The growth of order two around the zero set is needed only for a direct application of Jerrard's estimate in the proof of Lemma 3.10, and could be removed with a little extra work. However, a growth of order $k$ around the zero set and at infinity is essential in the proof of Lemma 3.7.

[9] The choice of the norm is related to the value of the constant $\beta_k$ in Theorem 1.1. It plays an effective role only in estimate (3.16) and computation (4.6).

**2.2. Currents.** For currents and forms we follow, with few exceptions, the standard notation (see [39, Chapter 6], or [21, Chapter 2], to which we also refer for a more detailed account of the theory). We just recall here some basic definitions.

For every integer $h = 0, \dots, n + k$, an $h$-form on $\Omega$ is a map from $\Omega$ into the space of $h$-covectors $\bigwedge^h(\mathbb{R}^{n+k})$, while an $h$-dimensional current is a map, or more generally a distribution, valued into the space of $h$-vectors $\bigwedge_h(\mathbb{R}^{n+k})$. Using the standard duality product of vectors and covectors, the space $\mathscr{D}_h(\Omega)$ of all $h$-currents is identified with the dual of the space $\mathscr{D}^h(\Omega)$ of all smooth $h$-forms with compact support. In particular, the integration of $h$-forms on a given $h$-dimensional oriented surface of class $C^1$ defines an $h$-current (often identified with the surface itself).

The boundary of an $h$-current $T$ is the $(h-1)$-current defined by the identity $\partial T[\omega] := T[d\omega]$ for every $\omega \in \mathscr{D}^{h-1}(\Omega)$, where $d\omega$ is the differential of $\omega$. We call boundary any current which is also a boundary.

A current $T$ is said to have (locally) finite mass when it can be represented as a (locally) bounded Borel measure valued in $\bigwedge_h(\mathbb{R}^{n+k})$; in this case $|T|$ is the variation of the measure $T$ and the mass of $T$ is the total variation $\|T\| := |T|(\Omega)$.[10] We write $\|T\|_U := |T|(U)$ for the mass of $T$ in the open set $U \subset \Omega$. The restriction of $T$ to a set $E$ is denoted by $T \llcorner E$, as for measures.

**2.3. Rectifiable and polyhedral currents.** A set $M$ in $\Omega$ is $h$-rectifiable if it can be covered, up to an $\mathscr{H}^h$-negligible subset, by countably many $h$-surfaces of class $C^1$. As such, it admits at $\mathscr{H}^h$-a.e. $x \in M$ a tangent space $\mathrm{Tan}(M,x)$ in a measure theoretic sense. An orientation of $M$ is a map which associates to $\mathscr{H}^h$-a.e. $x \in M$ a simple unitary $h$-vector which spans $\mathrm{Tan}(M,x)$.[11] Given a rectifiable set $M$, an orientation $\tau_M$, and a multiplicity $\sigma_M$—namely a real function in $L^1_{\mathrm{loc}}(\mathscr{H}^h \llcorner M)$—we define the current

$$T[\omega] := \int_M \sigma_M \, (\omega \cdot \tau_M) \, d\mathscr{H}^h \quad \text{for every } \omega \in \mathscr{D}^h(\Omega), \tag{2.3}$$

that is, $T := \sigma_M \, \tau_M \cdot \mathscr{H}^h \llcorner M$. In this case, the mass of $T$ is just the measure of $M$ counted with multiplicity, that is, $\|T\| = \int_M |\sigma_M| \, d\mathscr{H}^h$.

A current $T$ is *rectifiable* if it can be represented as in (2.3) for some integer-valued multiplicity $\sigma_M$, and it is *integral* if both $T$ and $\partial T$ are rectifiable. Notice

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[10] The spaces $\bigwedge_h(\mathbb{R}^{n+k})$ and $\bigwedge^h(\mathbb{R}^{n+k})$ are endowed with the euclidean norms induced by the standard bases. If $v$ is a *simple $h$-vector*, that is, if $v$ can be written as $v_1 \wedge \cdots \wedge v_h$, then the norm $|v|$ is the $h$-dimensional volume of the parallelogram spanned by the vectors $v_1, \dots, v_h$, and agrees with the mass of $v$, as defined in [20, Section 1.8.1]. Note that Jacobians and rectifiable currents are valued in simple vectors, and then the difference between mass and euclidean norm is irrelevant.

[11] If $M$ is a smooth $h$-surface with boundary, the orientation $\tau_M$ is always assumed continuous, and the orientation of the boundary satisfies $\eta(x) \wedge \tau_{\partial M}(x) = \tau_M(x)$ for every $x \in \partial M$, where $\eta(x)$ is the outer unit normal of $\partial M$ at $x$.

that a rectifiable boundary $T$ is also integral, because $\partial T = 0$. A sum of finitely many $k$-dimensional simplices in $\mathbb{R}^{n+k}$, endowed with constant orientations and constant real (resp., integral) multiplicities, is a real (resp., integral) *polyhedral current* in $\mathbb{R}^{n+k}$. Polyhedral currents in $\Omega$ are obtained by restriction.

By the constancy theorem, every $h$-dimensional current *without boundary* supported on a union $M$ of $h$-dimensional simplices with pairwise disjoint interiors can be represented as in (2.3) with a multiplicity function which is constant on each simplex, and therefore is a real polyhedral current (cf. [20, Section 4.1.32]).

**2.4. Pull-back of forms and push-forward of currents.** Given a linear map $L : \mathbb{R}^{n+k} \to \mathbb{R}^m$ and $\beta \in \bigwedge^h(\mathbb{R}^m)$, then $L^\sharp \beta$ is the $h$-covector on $\mathbb{R}^{n+k}$ defined by the identity $(L^\sharp \beta) \cdot (v_1 \wedge \cdots \wedge v_h) = \beta \cdot (Lv_1 \wedge \cdots \wedge Lv_h)$ for every simple vector $v_1 \wedge \cdots \wedge v_h$. Given an open set $\Omega'$ in $\mathbb{R}^m$ and a map $f : \Omega \to \Omega'$, the *pull-back* of an $h$-form $\omega$ on $\Omega'$ according to $f$ is the $h$-form $f^\sharp \omega$ on $\Omega$ defined by $f^\sharp \omega(x) := (Df(x))^\sharp \omega(f(x))$ for every $x$. The *push-forward* of an $h$-current $T$ on $\Omega$ is the $h$-current $f_\sharp T$ on $\Omega'$ defined by the obvious duality $f_\sharp T[\omega] = T[f^\sharp \omega]$. If $T$ has compact support in $\Omega$, then $\partial(f_\sharp T) = f_\sharp(\partial T)$.

The push-forward has a clear geometric meaning: if $T$ is the current associated $(M, \tau_M, \sigma_M)$ as in (2.3), then $f_\sharp T$ is the current associated to $(M', \tau_{M'}, \sigma_{M'})$ where $M'$ is the rectifiable set $f(M)$, $\tau_{M'}$ is any orientation of $M'$, and $\sigma_{M'}(y)$ is the sum of the the multiplicities $\sigma_M(x)$ over all $x \in f^{-1}(y)$, computed taking the orientation into account.

**2.5. Flat norm.** Throughout this paper we mostly deal with currents that are boundaries, or differ by a boundary. Therefore it is convenient to define the following flat norm of a current $T \in \mathscr{D}_h(\Omega)$:

$$\mathbf{F}_\Omega(T) := \inf \left\{ \|S\|_\Omega : \ S \in \mathscr{D}_{h+1}(\Omega) \text{ and } T = \partial S \right\}, \qquad (2.4)$$

where the infimum is taken to be $+\infty$ if $T$ is not a boundary. We say that $T_i$ converge to $T$ in the flat norm $\mathbf{F}_\Omega$ if $\mathbf{F}_\Omega(T_i - T) \to 0$. This implies in particular that $T_i$ and $T$ differ by a boundary (for $i$ sufficiently large).

Of course, $\mathbf{F}_\Omega$ is related to, but not exactly the same as the usual flat seminorms for integral currents (see [20, Section 4.1.24], or [39, Section 31]). Clearly, $\mathbf{F}_\Omega(T) \geq T[\omega]$ for every $\omega \in \mathscr{D}^h(\Omega)$ such that $\|d\omega\|_\infty \leq 1$, and therefore convergence in the flat norm $\mathbf{F}_\Omega$ implies convergence in the dual of the space of $h$-forms on $\Omega$ of class $C_0^1$.

The following is a version of the closure theorem for integral currents.

**Proposition 2.6** (see Section 8 for the proof). *Given a sequence of $h$-dimensional integral boundaries $(M_i)$ in $\Omega$ which are uniformly bounded in mass, and an open set $U \Subset \Omega$, we can extract a subsequence which converges in the flat norm $\mathbf{F}_U$ to an integral boundary in $U$.*

**2.7. Coarea formula for Sobolev maps.** Following [3, Section 7.5], we extend the coarea formula to maps $u$ in $W^{1,k}(\Omega, \mathbb{R}^k)$. Let $E$ be the set of points of approximate differentiability of $u$ (more precisely, of the representative of $u$ under consideration) and set $M_y = M_y(u) := u^{-1}(y) \cap E$ for every $y \in \mathbb{R}^k$. Then the sets $M_y$ are $n$-rectifiable and have finite $\mathscr{H}^n$-measure for a.e. $y$. Moreover, they can be oriented in such a way that the resulting rectifiable $n$-currents on $\mathbb{R}^{n+k}$—still denoted by $M_y$—satisfy, for every bounded continuos function $\rho : \mathbb{R}^k \to \mathbb{R}$,

$$\|\rho(u) \star Ju\| = \int_{\mathbb{R}^k} \|M_y\| \, |\rho(y)| \, dy \ , \tag{2.5}$$

$$\rho(u) \star Ju = \int_{\mathbb{R}^k} M_y \, \rho(y) \, dy \ . \tag{2.6}$$

The integral in (2.6) is understood in the weak* sense; in other words, identity (2.6) means that

$$(\rho(u) \star Ju)[\omega] = \int_{\mathbb{R}^k} M_y[\omega] \cdot \rho(y) \, dy$$

for every $n$-form $\omega$ on $\mathbb{R}^{n+k}$ of class $C_0^0$. Note that if $u$ is a Lipschitz map then $\|M_y\| = \mathscr{H}^n(u^{-1}(y))$ for a.e. $y$, and setting $\rho = 1$ in (2.5) we recover the usual coarea formula.

The map $y \mapsto M_y$ is well-defined and approximately continuous in the weak* sense at a.e. $y \in \mathbb{R}^k$, meaning that $y \mapsto \|M_y\|$ and $y \mapsto M_y[\omega]$ are approximately continuous at $y$ for all $n$-forms $\omega$ on $\mathbb{R}^{n+k}$ of class $C_0^0$. We call such $y$ regular values, and the corresponding $M_y$ regular level sets. In the following we write $M_y$ only for regular level sets.

**Remark 2.8.** (i) If $u$ is a map of class $C^1$, every value $y$ which is regular in the classical sense is also regular in our sense, and the corresponding $M_y$ is an $n$-surface of class $C^1$ oriented by the $n$-vector $\star Ju$. By Sard theorem the set of regular values in the classical sense has full measure if $u$ is of class $C^{k+1}$, but it may be even empty for maps of class $C^k$.

(ii) For maps in $W^{1,k-1}(\Omega, S^{k-1})$, the notion of regular value and regular set is the obvious generalization of the one for maps in $W^{1,k-1}(\Omega, \mathbb{R}^{k-1})$. In particular, $M_y$ is a rectifiable current of dimension $(n+1)$ for $\mathscr{H}^{k-1}$-a.e. $y \in S^{k-1}$ (see [3, Section 7.5] for more details).

**2.9. Jacobians of Sobolev maps.** We recall here some basic facts about Jacobians of Sobolev maps; see [3, 28] for further details and additional references. Let $y = (y_1, \ldots, y_k)$ denote the variable in $\mathbb{R}^k$, and $\{dy_1, \ldots, dy_k\}$ the standard basis of $\bigwedge^1(\mathbb{R}^k)$. Then the wedge product $dy := dy_1 \wedge \cdots \wedge dy_k$ is the standard volume-form on $\mathbb{R}^k$, and $k \cdot dy$ is the differential of the $(k-1)$-form

$$\omega_0(y) := \sum_i (-1)^{i-1} y_i \, \widehat{dy_i} \ , \tag{2.7}$$

where $\widehat{dy_i}$ stands for the wedge product of all $dy_j$ with $j \neq i$. The $k$-dimensional *Jacobian* of a map $u \in W^{1,k}(\Omega, \mathbb{R}^k)$ is the pull-back of $dy$ according to $u$, namely the $k$-form

$$Ju := du_1 \wedge \cdots \wedge du_k = u^\sharp(dy) = \frac{1}{k} d(u^\sharp \omega_0) , \qquad (2.8)$$

where $du_i = \sum_j D_j u_i \, dx_j$ is the differential of the $i$-th component of $u$. $J$ is a continuous operator from $W^{1,k}(\Omega, \mathbb{R}^k)$ into $L^1(\Omega, \bigwedge^k(\mathbb{R}^{n+k}))$.

**2.10. Jacobians as currents.** Let $\{e_1, \ldots, e_{n+k}\}$ be the standard basis of $\mathbb{R}^{n+k}$. We identify vector and covectors on $\mathbb{R}^{n+k}$ using the operator $\star$ defined in [3, Section 2.7]: given a covector $\alpha \in \bigwedge^h(\mathbb{R}^{n+k})$, the vector $\star\alpha \in \bigwedge_{n+k-h}(\mathbb{R}^{n+k})$ is uniquely determined by the identity

$$\alpha' \cdot (\star\alpha) = (\alpha' \wedge \alpha) \cdot (e_1 \wedge \cdots \wedge e_{n+k}) \quad \text{for every } \alpha' \in \bigwedge^{n+k-h}(\mathbb{R}^{n+k}).$$

Thus, given an $h$-form $\omega$ in $L^1_{\text{loc}}(\Omega)$, $\star\omega$ is the $(n+k-h)$-current on $\Omega$ defined by

$$\star\omega[\omega'] := \int_\Omega (\omega' \wedge \omega) \cdot \tau_\Omega \quad \text{for every } \omega' \in \mathscr{D}^{n+k-h}(\Omega), \qquad (2.9)$$

where $\tau_\Omega := e_1 \wedge \cdots \wedge e_{n+k}$ is the standard orientation of $\Omega$. There holds

$$\partial(\star\omega) = (-1)^{n+k-h} \star(d\omega) . \qquad (2.10)$$

Give a map $u \in W^{1,k}(\Omega, \mathbb{R}^k)$, $\star Ju$ is an $n$-current in $L^1(\Omega)$, and since $Ju$ is a differential (cf. (2.8)), then $\star Ju$ is a boundary. For $n = 0$, the 0-current $\star Ju$ is the real function $\det(Du)$; for $k = 2$ and $n = 1$, the 1-current $\star Ju$ is the vector product $Du_1 \times Du_2$.

**Remark 2.11.** Unlike the wedge product $du_1 \wedge \cdots \wedge du_k$, the last term in line (2.8) makes sense, at least as a distribution, also for maps $u$ in $L^\infty \cap W^{1,k-1}(\Omega, \mathbb{R}^k)$, and is therefore taken as a definition of Jacobian in this case (cf. [3, 28]). In this case $\star Ju$ may have infinite mass, but still $\mathbf{F}_\Omega(\star Ju) \leq \|u\|_\infty \|Du\|_{k-1}^{k-1}$. Hence, for every closed bounded set $E \subset \mathbb{R}^k$, $\star J$ is a continuous operator from $W^{1,k-1}(\Omega, E)$ into the space of $n$-dimensional boundaries in $\Omega$ with finite flat norm $\mathbf{F}_\Omega$.

## 3. PROOF OF STATEMENT (i) OF THEOREM 1.1

The main step is the following proposition:

**Proposition 3.1.** *Assume a (countable) sequence of smooth maps $(u_\varepsilon)$ on $\Omega$ such that $F_\varepsilon(u_\varepsilon) = O(|\log \varepsilon|)$. Then, for every open set $U \Subset \Omega$, we can extract a subsequence (not relabelled) such that $\mathbf{F}_U(\star Ju_\varepsilon - \alpha_k M) \to 0$, where $M$ is an integral boundary with finite mass in $U$. Moreover, for every simple $n$-covector $\eta$ such that $|\eta| = 1$ there holds*

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} F_\varepsilon(u_\varepsilon) \geq \beta_k |\eta \cdot M|(U) \ . \tag{3.1}$$

*Here $M$ is viewed as a vector measure on $U$ and $\eta \cdot M$ is the real measure defined by $(\eta \cdot M)(B) := \eta \cdot (M(B))$ for every set $B \subset U$.*

The proof of this statement is based on a variant of the polyhedral deformation theorem. To this purpose, we need some additional notation.

**3.2. Grids.** Given $\ell > 0$ and $a \in \mathbb{R}^{n+k}$, we call grid of size $\ell$ and center $a$ the collection $\mathscr{G} = \mathscr{G}(\ell, a)$ of all closed cubes of the form $Q = a + \ell z + [0, \ell]^{n+k}$ such that $z$ has integral coordinates; $\mathscr{G}' = \mathscr{G}'(\ell, a)$ is the dual grid associated to $\mathscr{G}$, and precisely $\mathscr{G}' := \mathscr{G}(\ell, a')$ with $a' := a + (\ell/2, \ldots, \ell/2)$. Given $h = 0, 1, \ldots, n+k$, we call $h$-cells (of the grid) the $h$-dimensional faces of the cubes in the grid. For every $h$-cell $Q$ we denote by $Q'$ the unique $(n+k-h)$-cell in the dual grid which intersects $Q$. Hence $Q$ and $Q'$ lie on orthogonal affine spaces which intersect at the center of $Q$ (see Figure 3.1 below).
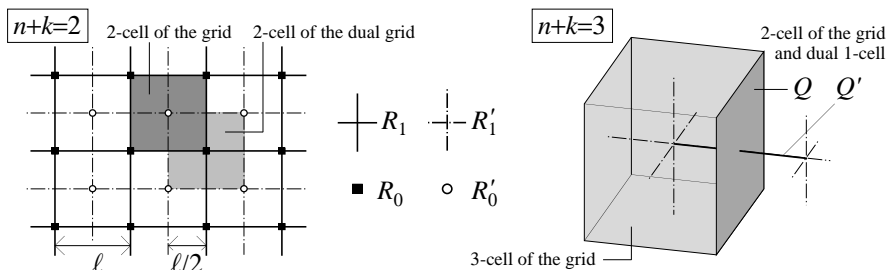


Figure 3.1.

We denote by $R_h = R_h(\ell, a)$ the $h$-skeleton of $\mathscr{G}$, that is, the union of all $h$-cells in the grid $\mathscr{G}$, and by $R'_h = R'_h(\ell, a)$ the $h$-skeleton of the dual grid. In the following, we consider sequences of grids $\mathscr{G}_\varepsilon = \mathscr{G}(\ell(\varepsilon), a(\varepsilon))$. In this case $\mathscr{G}'_\varepsilon$, $R_{\varepsilon,h}$, $R'_{\varepsilon,h}$, etc., denote the obvious objects.

**3.3. Structure of the proof of Proposition 3.1.** We first notice that if the grids $\mathscr{G}_\varepsilon$ satisfy certain assumptions, then $|u_\varepsilon| \geq 1/2$ on the $(k-1)$-skeleton $R_{\varepsilon,k-1}$ for all $\varepsilon$ sufficiently small. We then introduce a modified Jacobian operator $J_\psi$ with the following properties: the sequences $(\star Ju_\varepsilon)$ and $(\star J_\psi u_\varepsilon)$ are

asymptotically equivalent in the flat norm $\mathbf{F}_\Omega$, and $J_\psi u_\varepsilon$ is supported away from $R_{\varepsilon,k-1}$.

In the next step, we project $\star J_\psi u_\varepsilon$ on the dual $n$-skeleton $R'_{\varepsilon,n}$ using a suitable retraction of the complement of $R_{\varepsilon,k-1}$ onto $R'_{\varepsilon,n}$, and show that the multiplicity of the projected current on each dual $n$-cell $Q'$ is equal to $\alpha_k d_Q$, where $d_Q$ is the degree of the restriction of $u_\varepsilon/|u_\varepsilon|$ to the boundary of the $k$-cell $Q$. In particular the projected currents can be written as $\alpha_k M_\varepsilon$ for suitable integral currents $M_\varepsilon$. Moreover, if certain additional assumptions are satisfied, then the sequences $(\star J_\psi u_\varepsilon)$ and $(\alpha_k M_\varepsilon)$ are asymptotically equivalent in the flat norm $\mathbf{F}_U$, and the masses $\|M_\varepsilon\|_U$ can be efficiently estimated in terms of the energies $F_\varepsilon(u_\varepsilon)$.

Thus the currents $M_\varepsilon$, and consequently the Jacobians $\star J u_\varepsilon$, admit a converging subsequence by the closure theorem for integral currents. We conclude the proof by showing that the grids $\mathscr{G}_\varepsilon$ can be chosen so that all assumptions made in the previous steps are satisfied.

We first gather some preliminary results (Lemmas 3.4 to 3.11) and then prove Proposition 3.1.

**Lemma 3.4.** *Let $\mathscr{G}_\varepsilon$ be a (countable) sequence of grids with size $\ell = \ell(\varepsilon)$, and denote by $R_{\varepsilon,k-1}(\Omega)$ the union of all $(k-1)$-cells in $R_{\varepsilon,k-1}$ which are contained in $\Omega$. Assume smooth maps $u_\varepsilon : \Omega \to \mathbb{R}^k$ such that, for some $b < 2/k$,*

$$\ell^{n+1} \int_{R_{\varepsilon,k-1}(\Omega)} e_\varepsilon(u_\varepsilon) \, d\mathscr{H}^{k-1} = O(\varepsilon^{-b}) \ . \tag{3.2}$$

*If $\ell$ tends to $0$ sufficiently slow, and more precisely if $\varepsilon^c = O(\ell(\varepsilon))$ for some positive $c < (2/k - b)/(n+1)$, then the functions $|u_\varepsilon|$ converge to $1$ uniformly on $R_{\varepsilon,k-1}(\Omega)$ as $\varepsilon \to 0$. In particular, $|u_\varepsilon| \geq 1/2$ on $R_{\varepsilon,k-1}(\Omega)$ for $\varepsilon$ sufficiently small.*

**Remark.** It can be easily shown (cf. Lemma 3.11) that condition (3.2) is verified if $F_\varepsilon(u_\varepsilon, \Omega) = O(\varepsilon^{-b})$ and the centers of the grids $\mathscr{G}_\varepsilon$ are properly chosen.

*Proof.* Let $v_\varepsilon := |u_\varepsilon|$ and $R_\varepsilon := R_{\varepsilon,k-1}(\Omega)$. To prove the claim we will show that the oscillations of $v_\varepsilon$ over each $(k-1)$-cell of $R_\varepsilon$ tend uniformly to $0$ as $\varepsilon \to 0$, while the averages of $v_\varepsilon$ tend uniformly to $1$.

Fix for the time being $\lambda \in (0,1)$. Take a positive continuos function $\phi$ on $[0, +\infty)$ which vanishes only in $1$ and satisfies $\phi(|y|) \leq W(y)$ for every $y \in \mathbb{R}^k$, and denote by $\Phi$ a primitive of $\phi^{\lambda/((1-\lambda)k)}$.

To begin with, we denote by $g(\varepsilon)$ the left-hand side of (3.2). Then[12]

$$g(\varepsilon) \geq \ell^{n+1} \int_{R_\varepsilon} \frac{1}{k}|Dv_\varepsilon|^k + \frac{1}{\varepsilon^2}\phi(v_\varepsilon) \, d\mathscr{H}^{k-1}$$

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[12] In the first inequality we use that $|Du_\varepsilon| \geq |Dv_\varepsilon|$ and $W(u_\varepsilon) \geq \phi(v_\varepsilon)$, while the second one follows from Young's inequality $a + b \geq K a^{1-\lambda} b^\lambda$ applied with $a := |Dv_\varepsilon|^k$ and $b := \varepsilon^{-2}\phi(v_\varepsilon)$.

$$\geq K\ell^{n+1} \int_{R_\varepsilon} |Dv_\varepsilon|^{(1-\lambda)k} \varepsilon^{-2\lambda} \phi^\lambda(v_\varepsilon) \, d\mathscr{H}^{k-1}$$

$$\geq K\ell^{n+1} \varepsilon^{-2\lambda} \int_{R_\varepsilon} \left| D(\Phi(v_\varepsilon)) \right|^{(1-\lambda)k} d\mathscr{H}^{k-1} \, . \tag{3.3}$$

Now we use (3.3) and the Sobolev-Morrey embedding theorem to estimate the oscillation of $\Phi(v_\varepsilon)$ on each $(k-1)$-cell $Q$ of $R_\varepsilon$. To this end we need that $(1-\lambda)k > k-1$, that is, $\lambda < 1/k$:

$$\left[ \mathrm{osc}(\Phi(v_\varepsilon), Q) \right]^{(1-\lambda)k} \leq K\ell^{1-\lambda k} \int_Q \left| D(\Phi(v_\varepsilon)) \right|^{(1-\lambda)k} d\mathscr{H}^{k-1}$$

$$\leq K\ell^{-(n+\lambda k)} \varepsilon^{2\lambda} g(\varepsilon) = O(\varepsilon^{-(n+\lambda k)c + 2\lambda - b}) \, .$$

Hence these oscillations converge uniformly to 0 if $-(n+\lambda k)c + 2\lambda - b > 0$, i.e.,

$$c < \frac{2\lambda - b}{n + \lambda k} \, ,$$

and because of the assumption on $c$, it is clearly possible to choose $\lambda < 1/k$ so that this inequality holds. Moreover, since $\Phi$ is continuous and strictly increasing, also the oscillations of $v_\varepsilon$ on each $(k-1)$-cell $Q$ of $R_\varepsilon$ converge uniformly to 0.

Now we use (3.2) to estimate the average of $\phi(v_\varepsilon)$ over each cell

$$\frac{1}{\ell^{k-1}} \int_Q \phi(v_\varepsilon) \, d\mathscr{H}^{k-1} \leq \frac{1}{\ell^{k-1}} \int_{R_\varepsilon} W(u_\varepsilon) \, d\mathscr{H}^{k-1} \leq O(\varepsilon^{-(n+k)c + 2 - b}) \, .$$

As the assumption on $c$ implies $-(n+k)c + 2 - b > 0$, these averages converge uniformly to 0. Moreover, since $\phi$ is continuous, vanishes in 1 and is strictly positive elsewhere, then the averages of $v_\varepsilon$ converge uniformly to 1. $\qquad\square$

**3.5. Modified Jacobians.** Let a bounded continuous $k$-form $\psi$ on $\mathbb{R}^k$ be given. For every function $u \in W^{1,k}(\Omega, \mathbb{R}^k)$, we denote by $J_\psi u$ the pull-back of $\psi$ according to $u$. In other words, writing $\psi = \rho \, dy$ with $\rho$ a bounded continuous function on $\mathbb{R}^k$,

$$J_\psi u := u^\sharp \psi = u^\sharp(\rho \, dy) = \rho(u) \, Ju \, . \tag{3.4}$$

We show below (Lemma 3.7) that, when the integral of $\rho$ over the unit ball $B_1^k$ is $\alpha_k$, the sequences $\star Ju_\varepsilon$ and $\star J_\psi u_\varepsilon$ are asymptotically equivalent in the flat norm $\mathbf{F}_\Omega$ provided that $F_\varepsilon(u_\varepsilon) = o(\varepsilon^{-2/k})$. This will turn quite useful when proving the compactness part of statement (i) of Theorem 1.1.

**Lemma 3.6.** *Let $\rho : \mathbb{R}^k \to \mathbb{R}$ be a bounded function of class $C^1$ such that*

$$\int_{B_1^k} \rho(y) \, dy = \alpha_k \, . \tag{3.5}$$

*Then there exists a $(k-1)$-form $\omega$ on $\mathbb{R}^k$ of class $C^1$ such that $d\omega = (1-\rho) \, dy$, $\omega(y) = 0$ for every $y \in S^{k-1}$, and $\omega$ has growth at most linear at infinity.*

*Proof.* If we write $\omega = \sum(-1)^{i-1}a_i\,\widehat{dy_i}$ (cf. Section 2.9), then the problem reduces to finding a bounded $C^1$ vectorfield $a = (a_1, \ldots, a_k)$ on $\mathbb{R}^k$ which is null on $S^{k-1}$ and solves $\operatorname{div} a = f$ on $\mathbb{R}^k$, where $f := 1 - \rho$.

By assumption (3.5), the integral of $f$ on $B_1^k$ is 0. Using this fact we can decompose $f$ as $f = f_1 + f_2$ where $f_1$ satisfies

$$\int_0^1 f_1(ty)\,t^{k-1}dt = 0 \quad \text{for every } y \in S^{k-1}, \tag{3.6}$$

and $f_2$ is of the form $f_2(y) = h_1(|y|) \cdot h_2(y/|y|)$ with $h_1$ a non-negative $C^1$ function on $\mathbb{R}$ with support in $(0,1)$ and integral equal to 1, and $h_2$ a $C^1$ function on $S^{k-1}$ with integral equal to 0. To obtain such a decomposition, one can choose $h_1$ arbitrarily, set

$$h_2(y) := \int_0^1 f(ty)\,t^{k-1}dt \; ,$$

and then take $f_1 := f - f_2$.

We first look for a solution of $\operatorname{div} a_1 = f_1$ of the form $a_1 = g(y)\,y$ with $g$ a real function on $\mathbb{R}^k$. Then the equation reduces to $y \cdot Dg + k\,g = f_1$, and the theory of characteristics provides the explicit solution

$$g(y) := \int_0^1 f_1(ty)\,t^{k-1}dt \; .$$

The function $g$ is of class $C^1$ and bounded because so is $f_1$, and (3.6) implies $g(y) = 0$ for every $y \in S^{k-1}$. Hence $a_1$ is of class $C^1$, grows at most linearly at infinity, and is null on $S^{k-1}$.

Since the integral of the function $h_2$ on $S^{k-1}$ must be 0, and $S^{k-1}$ is a connected manifold, there exists a $C^1$ vectorfield $b$ on $S^{k-1}$ whose divergence is $h_2$ (one can take the gradient of the solution of the Laplace equation $\Delta u = h_2$, or exploit the fact that the $(k-1)$-form $h_2\omega_0$, with $\omega_0$ the standard volume form on $S^{k-1}$, has integral 0 on $S^{k-1}$ and therefore is a differential—cf. [12, Corollary 5.8]). Now it is easy to check that the vectorfield $a_2(y) := |y|\,h_1(|y|)\,b(y/|y|)$ solves $\operatorname{div} a_2 = f_2$, is of class $C^1$, and $\operatorname{supp}(a_2) \subset B_1^k$.

Finally, we take $a := a_1 + a_2$. $\qquad\square$

**Lemma 3.7.** *Let $\psi := \rho\,dy$ with $\rho : \mathbb{R}^k \to \mathbb{R}$ a bounded function of class $C^1$ such that (3.5) holds. For every $u \in W^{1,k}(\Omega, \mathbb{R}^k)$ and $\varepsilon > 0$ there holds*

$$\mathbf{F}_\Omega(\star Ju - \star J_\psi u) \le C\varepsilon^{2/k}F_\varepsilon(u) \; , \tag{3.7}$$

*where $C$ is a constant that depends only on the choice of $\rho$ .*

*Proof.* Take $\omega$ as in Lemma 3.6. Taking formula (2.10) into account we obtain
$$\star Ju - \star J_\psi u = \star u^\sharp ((1-\rho)\,dy) = \star u^\sharp(d\omega) = (-1)^n \partial(\star u^\sharp \omega) \;,$$
and by the definition of flat norm,
$$\mathbf{F}_\Omega(\star Ju - \star J_\psi u) \le \|\star u^\sharp \omega\| \le \int_\Omega |\omega(u)| \cdot |Du|^{k-1} \;. \tag{3.8}$$

The form $\omega$ grows at most linearly at infinity, and since it is of class $C^1$ and vanishes on $S^{k-1}$, then it grows at most linearly around $S^{k-1}$, too. On the other hand, the potential $W$ has growth of order at least $k$ at infinity and around $S^{k-1}$, and therefore there exists a constant $C$, depending on $\rho$ only, such that
$$|\omega(y)| \le C\, W^{1/k}(y) \;. \tag{3.9}$$
To conclude the proof, it is enough to combine estimates (3.8) and (3.9):
$$\mathbf{F}_\Omega(\star Ju - \star J_\psi u) \le C \int_\Omega W^{1/k}(u) \cdot |Du|^{k-1} \le C\varepsilon^{2/k} \int_\Omega |Du|^k + \frac{1}{\varepsilon^2} W(u) \;,$$
where the last inequality follows by applying Young's inequality $ab \le a^k + b^{k/(k-1)}$ with $a := \varepsilon^{-2/k} W^{1/k}(u)$ and $b := |Du|^{k-1}$. $\qquad\square$

The next lemma contains a version (in fact, a simplified one) of Federer-Fleming deformation theorem. Since none of the statements is exactly included in the standard formulations of this theorem (see for instance [39, Section 29], [20, Section 4.2]) we have included a detailed sketch of the proof.

**Lemma 3.8.** *Let a grid $\mathcal{G} = \mathcal{G}(\ell, a)$ and a bounded open set $V$ in $\mathbb{R}^{n+k}$ be given. Denote by $d := 2\sqrt{n+k}\,\ell$ the double of the diameter of the cubes in $\mathcal{G}$, and by $V_d$ the closed $d$-neighbourhood of $V$. The following statements hold.*

(i) *There exists a map $\Phi : \mathbb{R}^{n+k} \setminus R_{k-1} \to R'_n$ which is locally Lipschitz, maps every cube in the grid $\mathcal{G}$ into itself, and satisfies*
$$|D\Phi(x)| \le K\ell/\mathrm{dist}(x, R_{k-1}) \quad \text{for every } x \in \mathbb{R}^{n+k} \setminus R_{k-1}. \tag{3.10}$$

(ii) *Let $T$ be an $n$-dimensional current in $\mathbb{R}^{n+k}$ with finite mass and compact support such that $\mathrm{supp}(T) \cap R_{k-1} = \varnothing$. The push-forward $\Phi_\sharp T$ is a well-defined current with finite mass in $\mathbb{R}^{n+k}$ supported on $R'_n$. If in addition $\mathrm{supp}(\partial T) \cap V_d = \varnothing$, then the restriction of $\Phi_\sharp T$ to $V$ is a real polyhedral current without boundary, and*
$$\mathbf{F}_V(T - \Phi_\sharp T) \le K\ell^{n+1} \int_{V_d} \frac{d|T|(x)}{\big[\mathrm{dist}(x, R_{k-1})\big]^n} \;. \tag{3.11}$$

(iii) *Let $M$ be a compact smooth $n$-surface in $\mathbb{R}^{n+k}$ such that $\mathrm{supp}(M) \cap R_{k-1} = \varnothing$ and $\mathrm{supp}(\partial M) \cap V_d = \varnothing$, and let $Q$ be a $k$-cell in the grid whose dual $n$-cell $Q'$ intersects $V$. If $M$ is transversal to $Q$, then the multiplicity of $\Phi_\sharp M$ on $Q'$ is constant and agrees, up to a sign, with the intersection number of $M$ and $Q$ (namely, the sum over all $x \in M \cap Q$ of $+1$ if $\tau_M(x) \wedge \tau_Q(x)$ agrees with the standard orientation of $\mathbb{R}^{n+k}$, and $-1$ otherwise).*

*Proof.* First we construct the map $\Phi$ on the convex hull $H_Q$ of $Q \cup Q'$, where $Q$ is a $k$-cell in the grid and $Q'$ the associated dual cell. We write $x \in \mathbb{R}^{n+k}$ as $(s', s) \in \mathbb{R}^n \times \mathbb{R}^k$. We can safely assume that $Q$ is the set of all $(s', s)$ such that $s' = 0$ and $|s|_\infty \le \ell/2$, where $|s|_\infty := \max_i |s_i|$, and $Q'$ is the set of all $(s', s)$ such that $s = 0$ and $|s'|_\infty \le \ell/2$ (see Figure 3.2). Then $H_Q$ is given by $|s'|_\infty + |s|_\infty \le \ell/2$. We define $\Phi_Q : H_Q \setminus \partial Q \to Q'$ by

$$\Phi_Q(s', s) := \left( \frac{\ell \, s'}{\ell - 2|s|_\infty}, 0 \right) .$$

One verifies that $\Phi_Q$ is singular on $\partial Q$, and satisfies $|D\Phi_Q(x)| \le K\ell/\mathrm{dist}(x, \partial Q)$.

The hulls $H_Q$, with $Q$ ranging among all $k$-cells in the grid, cover $\mathbb{R}^{n+k}$ and have pairwise disjoint interiors, and the associated maps $\Phi_Q$ agree on their common faces. Therefore, to conclude the proof of statement (i), it suffices to take $\Phi$ equal to $\Phi_Q$ on each $H_Q$.
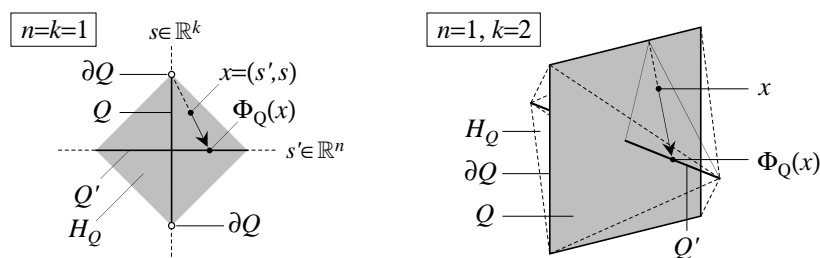


FIGURE 3.2.

The proof of statement (ii) is standard. The push-forward $\Phi_\sharp T$ is a well-defined current because $T$ is supported away from the singular set of $\Phi$. Denote by $V'$ the interior of the union of all closed cubes in the grid $\mathscr{G}$ that are contained in $V_d$. Thus $V_d \supset V' \supset V$. We set $\Psi(t, x) := (1 - t)x + t\Phi(x)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^{n+k} \setminus R_{k-1}$, and define

$$S := \Psi_\sharp(I \times T) ,$$

where $I$ is the 1-current associated to the oriented segment $[0, 1]$. Since $\Psi(0, x) = x$ and $\Psi(1, x) = \Phi(x)$, if we assume that $\mathrm{supp}(\partial T) \cap V_d = \varnothing$, then[13]

$$\partial S = \Phi_\sharp T - T \quad \text{in } V'. \tag{3.12}$$

---

[13] As $I \times T$ is supported away from the singular set of $\Psi$, the push-forward $S$ is well-defined and $\partial S = \Psi_\sharp(\partial I \times T - I \times \partial T)$. Since $\Phi$ maps each cube $Q$ in the grid into itself, then $\Psi$ maps $[0, 1] \times Q$ into $Q$, and the assumption $\mathrm{supp}(\partial T) \cap V_d = \varnothing$ implies $\Psi$ maps $[0, 1] \times \mathrm{supp}(\partial T)$ in the complement of $V'$. Therefore, inside $V'$ there holds $\partial S = \Psi_\sharp(\partial(I \times T)) = \Psi_\sharp(\delta_1 \times T - \delta_0 \times T) = \Phi_\sharp T - T$.

In particular, since $T$ has no boundary in $V'$, the same holds for $\Phi_\sharp T$. Hence the multiplicity of $\Phi_\sharp T$ on each $n$-cell of $R'_n$ contained in $V'$ is constant (and therefore $\Phi_\sharp T$ agrees in $V'$ with a real polyhedral current).

Given a unit vector $e_0$ in $\mathbb{R} \times \{0\}$ and a unit $n$-vector $\tau$ in $\{0\} \times \mathbb{R}^{n+k}$, there holds

$$
\begin{aligned}
\left| [D\Psi(t,x)]_\sharp (e_0 \wedge \tau) \right| &\leq |D_t \Psi(t,x)| \cdot |D_x \Psi(t,x)|^n \\
&\leq |x - \Phi(x)| \cdot |D\Phi(x)|^n \leq K\ell^{n+1} \cdot [\mathrm{dist}(x, R_{k-1})]^{-n}
\end{aligned}
$$

and integrating this inequality with respect to the mass of the current $I \times T$—namely the measure $\mathscr{L}^1 \llcorner [0,1] \times |T|$—we obtain

$$
\|S\|_{V'} = \|\Psi_\sharp(I \times T)\|_{V'} \leq K\ell^{n+1} \int_{\overline{V}'} \frac{d|T|(x)}{\left[ \mathrm{dist}(x, R_{k-1}) \right]^n} \ ,
$$

which, in view of (3.12) and of the inclusions $V \subset V'$ and $\overline{V}' \subset V_d$, implies (3.11).

It remains to prove statement (iii). Since $Q'$ intersects $V$, then $Q$ and $Q'$ are contained in $V'$, and, as pointed out before, the multiplicity of $\Phi_\sharp M$ on $Q'$ is constant. Moreover, if $x$ is the intersection point of $Q$ and $Q'$, then $\Phi^{-1}(x) = Q$ (cf. Figure 3.2), and therefore the multiplicity of $\Phi_\sharp M$ at $x$ is the intersection number of $M$ and $Q$ (up to a sign which depends on the orientations of $Q$ and $Q'$, which have not been fixed). $\qquad\square$

The next statement is an almost straightforward application of the previous lemma to Jacobians. $\mathscr{G}$, $\Phi$, $V$, $d$, and $V_d$ are taken as in Lemma 3.8, while $\psi = \rho\,dy$ and $J_\psi u$ are taken as in Section 3.5.

**Lemma 3.9.** *Take $0 < t < 1$, and assume that $\psi = \rho\,dy$ is supported in the ball $B_t^k$ and has integral $\alpha_k$. Let $V'$ be a neighbourhood of $V_d$ and $u : V' \to \mathbb{R}^k$ a map, smooth up to the boundary, such that $|u| \geq t$ on $R_{k-1}$. Let $Q$ be a $k$-cell in the grid $\mathscr{G}$ whose dual $n$-cell $Q'$ intersects $V$. The following statements hold.*

(i) *If $|y| < t$ and $M_y$ is a regular level set of $u$, the push-forward $\Phi_\sharp M_y$ defines an integral polyhedral current in $V$ whose multiplicity on $Q'$ agrees, up to a sign, with $d_Q := \deg(u/|u|, \partial Q, S^{k-1})$. In particular $\Phi_\sharp M_y$ does not depend on the choice of $y$. Moreover*

$$
|d_Q| \leq K \int_Q |Du|^k \, d\mathscr{H}^k \ . \tag{3.13}
$$

(ii) *The push-forward $\Phi_\sharp(\star J_\psi u)$ is well-defined and agrees with $\alpha_k \Phi_\sharp M_y$ for every regular value $y$ with $|y| < t$. Thus $\Phi_\sharp(\star J_\psi u)$ is a real polyhedral boundary in $V$ with multiplicity $\pm\alpha_k d_Q$ on $Q'$. Moreover*

$$
\mathbf{F}_V(\star J_\psi u - \Phi_\sharp(\star J_\psi u)) \leq K\|\psi\|_\infty \ell^{n+1} \int_{V_d} \frac{|Du(x)|^k \, dx}{\left[ \mathrm{dist}(x, R_{k-1}) \right]^n} \ . \tag{3.14}
$$

*Proof.* We prove statement (i) first. By a density argument we can assume that $y$ is a regular value in the classical sense and $M_y$ is transversal to $Q$. Since $|y| < t$ and $|u| \geq t$ on $R_{k-1}$, the level set $M_y$ is supported away from $R_{k-1}$, and therefore satisfies the assumption of statement (iii) of Lemma 3.8. Hence $\Phi_\sharp M_y$ agrees with a real polyhedral current on $V$, and its multiplicity on $Q'$ is the intersection number of $M_y$ and $Q$, which in turn agrees—almost by definition— with $\deg(u, Q, \mathbb{R}^k, y)$. We claim that

$$\deg(u, Q, \mathbb{R}^k, y) = d_Q \ . \tag{3.15}$$

By integrating identity (3.15) over all $y \in B_t^k$ we obtain

$$\alpha_k \, t^k \, |d_Q| \leq \int_{B_t^k} |\deg(u, Q, \mathbb{R}^k, y)| \, dy \leq \int_Q |\det Du| \leq \int_Q |Du|^k \ ,$$

which gives (3.13). It remains to prove (3.15). Consider the truncated map

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } |u(x)| \leq t, \\ t \, u(x)/|u(x)| & \text{otherwise.} \end{cases}$$

Since $|u| > t$ on $R_{k-1}$, which contains $\partial Q$, and $|y| < t$, then $\deg(u, Q, \mathbb{R}^k, y) = \deg(\tilde{u}, Q, \mathbb{R}^k, y)$. Since $\tilde{u}$ maps $\partial Q$ in the sphere of radius $t$, then $\deg(\tilde{u}, Q, \mathbb{R}^k, y)$ is equal to $\deg(\tilde{u}, \partial Q, S_t^{k-1})$ (see, e.g., [3, Section 2.10]), which is clearly the same as $d_Q$.

To prove statement (ii), we denote by $T$ the the current obtained by extending $\star J_\psi u$ to $0$ outside $V'$. Then $T$ satisfies the assumptions of statement (ii) of Lemma 3.8, and (3.14) follows from (3.11) and the estimate $|J_\psi u| \leq \|\psi\|_\infty |Du|^k$. Note that $\Phi_\sharp(\star J_\psi u)$ is a boundary in $V$ because it has finite flat distance from $\star J_\psi u$, which is a boundary in $V$ (see Sections 2.5 and 2.10).

By the coarea formula (see Section 2.7), $\star J_\psi u = \rho(u) \star Ju$ is given by the integral of the currents $M_y$ with respect to the measure $\rho(y) \, d\mathscr{L}^k(y)$. Then, by the linearity of the push-forward operator, $\Phi_\sharp(\star J_\psi u)$ is the integral of the currents $\Phi_\sharp M_y$. But these currents are all the same for $|y| < t$, and to conclude the proof it suffices to recall that $\rho$ is supported in $B_t^k$ and its integral is $\alpha_k$. $\quad\square$

The following key estimate is a direct application of the results in [27].

**Lemma 3.10.** *There exist a universal finite constant $K$ and, for every $r > 0$, finite constants $C_0$ and $C_1$ (depending on $r$, $k$ and the potential $W$) so that, given a cube $Q_\ell := [-\ell/2, \ell/2]^k$ with $\ell > 0$, and a smooth map $u : Q_\ell \to \mathbb{R}^k$ such that $|u| > 1/2$ on $\partial Q_\ell$ and $d := \deg(u/|u|, \partial Q_\ell, S^{k-1}) \neq 0$, then*

$$\beta_k |d| \cdot \big[ |\log(\varepsilon/\ell^{k/2})| - C_1(1 + \log |d|) \big]$$
$$\leq \int_{Q_\ell} e_\varepsilon(u) \, d\mathscr{L}^k + K r \ell \int_{\partial Q_\ell} e_\varepsilon(u) \, d\mathscr{H}^{k-1} \tag{3.16}$$

*for every $\varepsilon$ such that $\varepsilon/\ell^{k/2} < 1$ and $(\varepsilon/\ell^{k/2}) \cdot |\log(\varepsilon/\ell^{k/2})| \leq C_0/|d|$.*

*Proof.* Because of the growth assumptions on the potential $W$ (see Section 2.1), there exists a constant $K$ such that $(1 - |y|^2)^2 \leq K\,W(y)$. Then it suffices to prove the statement for $W(y) = K^{-1}(1 - |y|^2)^2$, and by rescaling we can also assume that $K = 1$.

It is proved in [27] that for every bounded Lipschitz domain $U \subset \mathbb{R}^k$ and $r > 0$, there exist constants $C_0 = C_0(U, r)$ and $C_1 = C_1(U, r)$ so that

$$\beta_k |d| \cdot \big[ |\log \varepsilon| - C_1(1 + \log |d|) \big] \leq \int_U e_\varepsilon(u)\, d\mathscr{L}^k$$

for every map $u : U \to \mathbb{R}^k$ such that $|u(x)| \geq 1/2$ for $\mathrm{dist}(x, \partial U) \leq r$ and every $\varepsilon$ such that $\varepsilon < 1$ and $\varepsilon |\log \varepsilon| \leq C_0/|d|$.[14]

To prove Lemma 3.10 for $\ell = 1$, it suffices to apply this result with $U := Q_{1+r}$ and $u$ extended to $Q_{1+r} \setminus Q_1$ by the formula $u(x) := u(x/|x|_\infty)$, where $|x|_\infty := \max\{|x_i|\}$. Indeed a simple computation gives

$$\int_{Q_{1+r} \setminus Q_1} e_\varepsilon(u)\, d\mathscr{L}^k \leq Kr \int_{\partial Q_1} e_\varepsilon(u)\, d\mathscr{H}^{k-1}\ .$$

The statement for general $\ell$ is obtained by scaling. $\qquad\square$

We finally show that the grid can always be chosen so to satisfy certain additional energy estimates that will be used in the proof of Proposition 3.1.

**Lemma 3.11.** *Assume $u : \Omega \to \mathbb{R}^k$ smooth, and positive numbers $\delta$, $\varepsilon$ and $\ell$. Then there exists $a \in \mathbb{R}^{n+k}$ such that the grid $\mathscr{G} = \mathscr{G}(\ell, a)$ satisfies*

$$\ell^n \int_{\widetilde{R}_k \cap \Omega} e_\varepsilon(u)\, d\mathscr{H}^k \leq (1 + \delta)\, F_\varepsilon(u) \tag{3.17}$$

$$\ell^n \int_{R_k \cap \Omega} e_\varepsilon(u)\, d\mathscr{H}^k \leq \frac{K}{\delta}\, F_\varepsilon(u) \tag{3.18}$$

$$\ell^{n+1} \int_{R_{k-1} \cap \Omega} e_\varepsilon(u)\, d\mathscr{H}^{k-1} \leq \frac{K}{\delta}\, F_\varepsilon(u) \tag{3.19}$$

$$\ell^n \int_\Omega \frac{e_\varepsilon(u)\, dx}{\big[ \mathrm{dist}(x, R_{k-1}) \big]^n} \leq \frac{K}{\delta}\, F_\varepsilon(u)\ , \tag{3.20}$$

*where $\widetilde{R}_k$ is the union of all $k$-cells in $R_k$ that are parallel to the $k$-plane spanned by $\{e_{n+1}, \ldots, e_{n+k}\}$.*

---

[14] We have used estimate $\int_U e_\varepsilon(u) \geq \beta_k |d| \cdot \log(\sigma/\varepsilon) - C$ in Theorem 1.2 of [27]. However, the constants $C$ and $\sigma$ given in that statement depend on the domain $U$ and the trace of $u$ on the boundary of $U$. To show that their contribution can be estimated by $C_1(1 + \log |d|)|d|$ independently of the trace of $u$ and of the degree $d$, one has to go through the proof given in Section 4 of that paper. More precisely, if one rewrites the constant $\sigma$ in Proposition 4.1 as $\sigma_0/|d|$, then $\sigma_0$ can be arbitrarily chosen in the interval $(0, r/4)$, independently of $d$, and looking through the proof one finds that the desired inequality holds provided that $\sigma_0/|d| \geq s(\varepsilon)$, where $s(\varepsilon)$ is a universal function of order $\varepsilon |\log \varepsilon|$.

*Proof.* Denote the left-hand sides of (3.17), (3.18), (3.19), and (3.20) by $f_0(a)$, $f_1(a)$, $f_2(a)$, and $f_3(a)$, respectively. We claim that the average of $f_0(a)$ over all $a$ in the cube $Q_\ell := [0, \ell]^{n+k}$ is $F_\varepsilon(u)$, while the average of $f_i(a)$ agrees with $F_\varepsilon(u)$ times some universal constant for $i = 1, 2, 3$. Were this claim true, the rest of the proof would follow by Lemma 8.4 (with $X := Q_\ell$ and $\mu := \mathscr{L}^{n+k} \llcorner Q_\ell$).

We first compute the average of $f_0$. Notice that the set $\widetilde{R}_k = \widetilde{R}_k(\ell, a)$ depends only on the first $n$-coordinates of $a$ (see Figure 3.3), and then, denoting by $Q_\ell^n$ the set of all $a \in Q_\ell$ whose last $k$-coordinates are null, the average of $f_0$ on $Q_\ell$ is the same as the average on $Q_\ell^n$. Hence

$$\fint_{Q_\ell} f_0 = \int_{Q_\ell^n} \left[ \int_{\widetilde{R}_k(\ell,a)\cap\Omega} e_\varepsilon(u)\, d\mathscr{H}^k \right] d\mathscr{L}^n(a) = \int_\Omega e_\varepsilon(u)\, d\mathscr{L}^{n+k} = F_\varepsilon(u) \ ,$$

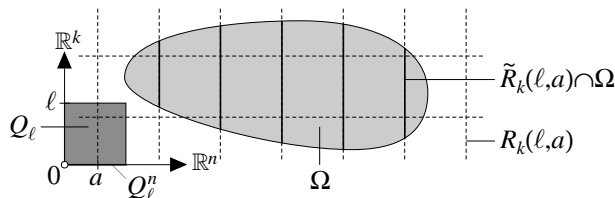where the second identity follows by Fubini's theorem (see Figure 3.3).



FIGURE 3.3.

We compute now the average of $f_1$. For every $k$-plane $P$ spanned by subsets of $\{e_1, \ldots, e_{n+k}\}$ we take the union $R_k(P)$ of all $k$-cell in the grid that are parallel to $P$, then we proceed as above with $\widetilde{R}_k$ replaced by $R_k(P)$ and take the sum over all $P$. We thus obtain that the average of $f_1$ over $Q_\ell$ is $\binom{n+k}{k}$ times $F_\varepsilon(u)$.

In a similar way one proves that the average of $f_2$ is $\binom{n+k}{k-1}$ times $F_\varepsilon(u)$.

Finally, we compute the average of $f_3$. Let $R$ be the $(k-1)$-skeleton of the grid with size 1 and center 0. Since $R_{k-1} = a + \ell R$, we have

$$\operatorname{dist}(x, R_{k-1}) = \ell \cdot \operatorname{dist}(\ell^{-1}(x - a), R) \ ,$$

and therefore[15]

$$\fint_{Q_\ell} f_3 = \fint_{Q_\ell} \left[ \ell^n \int_\Omega \frac{e_\varepsilon(u)\, dx}{\left[ \ell \operatorname{dist}(\ell^{-1}(x-a), R) \right]^n} \right] da$$

$$= \int_\Omega \left[ \fint_{Q_\ell} \frac{da}{\left[ \operatorname{dist}(\ell^{-1}(x-a), R) \right]^n} \right] e_\varepsilon(u)\, dx$$

$$= \int_\Omega \left[ \fint_{Q_1} \frac{dy}{\left[ \operatorname{dist}(y, R) \right]^n} \right] e_\varepsilon(u)\, dx \ . \tag{3.21}$$

---

[15] The last identity follows from the change of variable $y = \ell^{-1}(x - a)$ and the fact that $Q_\ell$ and $Q_1$ are periodicity cells for $a \mapsto \operatorname{dist}(\ell^{-1}(x-a), R)$ and $y \mapsto \operatorname{dist}(y, R)$, respectively.

Since the integral between square brackets in the last line of formula (3.21) is finite (Lemma 8.3), the average of $f_3$ is $F_\varepsilon(u)$ times some universal constant $K$. □

**Proof of Proposition 3.1/compactness.** Let $u_\varepsilon$ be as in the statement of Proposition 3.1, and set $e_\varepsilon := e_\varepsilon(u_\varepsilon)$. Passing to a subsequence, we can assume that the liminf at the left-hand side of (3.1) is a limit (and therefore will not be affected by further extractions of subsequences).

We fix a $k$-form $\psi = \rho\,dy$ on $\mathbb{R}^k$ with support contained in $B^k_{1/2}$ and integral equal to $\alpha_k$, and set

$$\ell = \ell(\varepsilon) := |\log\varepsilon|^{-2} \quad \text{for every } \varepsilon. \qquad (3.22)$$

We take the grids $\mathscr{G}_\varepsilon = \mathscr{G}(a(\varepsilon), \ell(\varepsilon))$ where the centers $a = a(\varepsilon)$ are chosen according to Lemma 3.11, so that (3.17)–(3.20) hold for some fixed positive number $\delta$, with $u$ and $\mathscr{G}$ replaced by $u_\varepsilon$ and $\mathscr{G}_\varepsilon$. Recall that

$$F_\varepsilon(u_\varepsilon) = O(|\log\varepsilon|) . \qquad (3.23)$$

We choose two open sets $V$ and $V'$ so that $U \Subset V \Subset V' \Subset \Omega$. First of all, we notice that Lemma 3.7 and (3.23) yield

$$\mathbf{F}_\Omega(\star Ju_\varepsilon - \star J_\psi u_\varepsilon) \to 0 . \qquad (3.24)$$

Lemma 3.4 and estimates (3.19) and (3.23) imply $|u_\varepsilon| \geq 1/2$ on $R_{\varepsilon,k-1}(\Omega)$ for $\varepsilon$ sufficiently small.

We want to apply Lemma 3.9 with $u = u_\varepsilon$, $\mathscr{G} = \mathscr{G}_\varepsilon$, and $t = 1/2$. Since the quantity $d$ which appears in that lemma tends to 0 as $\varepsilon \to 0$, then $R_{\varepsilon,k-1}(\Omega) \supset R_{\varepsilon,k-1} \cap V_d$ and $V' \supset V_d$ for $\varepsilon$ sufficiently small, and the assumptions of the lemma are satisfied. Hence the projected current $\Phi_{\varepsilon\sharp}(\star J_\psi u_\varepsilon)$ can be represented in $V$ as $\alpha_k M_\varepsilon$ where $M_\varepsilon$ is an integral boundary in $V$ supported on $R'_{\varepsilon,n}$. Combining estimates (3.14) and $|Du_\varepsilon|^k \leq ke_\varepsilon$, and recalling (3.20), (3.22), (3.23), we obtain

$$\mathbf{F}_V(\star J_\psi u_\varepsilon - \alpha_k M_\varepsilon) = \ell \cdot O(|\log\varepsilon|) \to 0 . \qquad (3.25)$$

Now we estimate the mass of $M_\varepsilon$ in $V$. Given a dual $n$-cell $Q' \in R'_{\varepsilon,k}$ which intersects $V$, the corresponding $k$-cell $Q$ is contained in $\Omega$, and by Lemma 3.9 the multiplicity of $M_\varepsilon$ on $Q'$ agrees with $d_Q := \deg(u_\varepsilon/|u_\varepsilon|, \partial Q, S^{k-1})$. Hence

$$\|M_\varepsilon\|_V \leq \ell^n \sum_{Q \subset \Omega} |d_Q| . \qquad (3.26)$$

We fix $r > 0$, and use Lemma 3.10 to estimate $|d_Q|$. Since $Q \subset V_d \subset \Omega$ for $\varepsilon$ small enough, estimates (3.13) and (3.18) yield

$$|d_Q| \leq K \int_Q |Du|^k \, d\mathscr{H}^k \leq K \int_{R_{\varepsilon,k}(\Omega)} e_\varepsilon \, d\mathscr{H}^k = O(|\log\varepsilon|^{2n+1}) . \qquad (3.27)$$

Hence $(\varepsilon/\ell^{k/2})\,|\log(\varepsilon/\ell^{k/2})| \leq K\varepsilon|\log\varepsilon|^{k+1} = o(1/d_Q)$, which implies that, for $\varepsilon$ sufficiently small, estimate (3.16) holds for all $Q$. Moreover, by (3.27), the term between square brackets in the first line of (3.16) is asymptotically equivalent to $|\log\varepsilon|$, and therefore (3.16) can be rewritten as

$$(1 - c_r(\varepsilon))\,\beta_k|d_Q| \leq \frac{1}{|\log\varepsilon|}\int_Q e_\varepsilon d\mathscr{H}^k + \frac{Kr\ell}{|\log\varepsilon|}\int_{\partial Q} e_\varepsilon d\mathscr{H}^{k-1} \qquad (3.28)$$

where $c_r(\varepsilon)$ tends to 0 as $\varepsilon \to 0$ (and does not depend on $Q$). For $\varepsilon$ sufficiently small $c_r(\varepsilon) \leq 1/2$, and the previous inequality and (3.26) becomes

$$\|M_\varepsilon\|_V \leq \frac{K\ell^n}{|\log\varepsilon|}\int_{R_{\varepsilon,k}(\Omega)} e_\varepsilon d\mathscr{H}^k + \frac{K\ell^{n+1}}{|\log\varepsilon|}\int_{R_{\varepsilon,k-1}(\Omega)} e_\varepsilon d\mathscr{H}^{k-1}\,. \qquad (3.29)$$

Estimates (3.29), (3.18), (3.19), and (3.23) imply that the integral boundaries $M_\varepsilon$ are uniformly bounded in mass in $V$, and therefore converge in the flat norm $\mathbf{F}_U$ to some integral boundary $M$ in $U$ (Proposition 2.6). Finally, (3.25) and (3.24) imply that the currents $\star Ju_\varepsilon$ and $\star J_\psi u_\varepsilon$ converge to $\alpha_k M$ in the flat norm $\mathbf{F}_U$. $\qquad\square$

**Proof of Proposition 3.1/inequality (3.1).** We follow the notation of the previous section. With no loss in generality we may assume $\eta = dx_1 \wedge \cdots \wedge dx_n$. Then the proof of (3.1) just follows from a refinement of estimate (3.29).

Since the current $M_\varepsilon$ is supported on the dual $n$-skeleton $R'_{\varepsilon,n}$, the positive measure $|\eta \cdot M_\varepsilon|$ is just the part of $|M_\varepsilon|$ supported on the union of the dual $n$-cells $Q'$ parallel to the $n$-plane spanned by $\{e_1, \ldots, e_n\}$, that is, the duals of the $k$-cells $Q$ contained in $\widetilde{R}_{\varepsilon,k}$ (see Lemma 3.11). Moreover, if $Q'$ intersects $V$, then $Q$ is contained in $\Omega$. Hence, taking into account (3.28), (3.17), and (3.19)

$$\begin{aligned}
(1 - c_r(\varepsilon))\,&\beta_k\,|\eta \cdot M_\varepsilon|(V) \\
&\leq (1 - c_r(\varepsilon))\,\beta_k\,\ell^n \sum\nolimits_{Q \subset \widetilde{R}_{\varepsilon,k}\cap\Omega} |d_Q| \\
&\leq \frac{\ell^n}{|\log\varepsilon|}\int_{\widetilde{R}_{\varepsilon,k}\cap\Omega} e_\varepsilon d\mathscr{H}^k + \frac{Kr\ell^{n+1}}{|\log\varepsilon|}\int_{R_{\varepsilon,k-1}\cap\Omega} e_\varepsilon d\mathscr{H}^{k-1} \\
&\leq \frac{1+\delta}{|\log\varepsilon|}F_\varepsilon(u_\varepsilon) + rO(1)\,,
\end{aligned}$$

and passing to the limit as $\varepsilon \to 0$,

$$\beta_k|\eta \cdot M|(V) \leq \liminf_{\varepsilon\to 0}\frac{1+\delta}{|\log\varepsilon|}F_\varepsilon(u_\varepsilon) + C\,r\,.$$

Finally, (3.1) follows because $\delta$ and $r$ can be taken arbitrarily small. $\qquad\square$

**Proof of Theorem 1.1(i).** We use the reflection map $\Phi$ described in Remark 8.2 to extend $u_\varepsilon$ to $\Omega' := \Omega \cup U$ so that the energy of $u_\varepsilon$ on $\Omega'$ remains of order $|\log \varepsilon|$. By a density argument, we can also assume that $u_\varepsilon$ is smooth in $\Omega'$, and then the compactness statement follows from Proposition 3.1 (with $\Omega$ and $U$ replaced by $\Omega'$ and $\Omega$, respectively).

We prove the lower-bound inequality (1.3) using a standard localization argument. Given an open set $A \Subset \Omega$ and a simple unit $n$-covector $\eta$, we apply Proposition 3.1 with $\Omega$ replaced by $A$, and taking the supremum in (3.1) over all $U \Subset A$ we obtain

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon, A)}{|\log \varepsilon|} \geq \beta_k |\eta \cdot M|(A) \ .$$

Now we apply the previous inequality to finitely many pairwise disjoint open sets $A_i \Subset \Omega$ and simple unit $n$-covectors $\eta_i$, and summing over all $i$ we get

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \geq \beta_k \sum_i |\eta_i \cdot M|(A_i) \ .$$

Finally, taking the supremum over all choices of $A_i$ and $\eta_i$ we obtain (1.3).    $\square$

## 4. Proof of Statement (ii) of Theorem 1.1

The upper bound inequality in Theorem 1.1 is a direct consequence of Theorem 5.10 in [3] and of the construction described in Lemma 4.2 below. We begin with some additional notation.

**4.1. Additional Notation.** Given a polyhedral current $M = \sum_i \sigma_i F_i$, we tacitly assume that the intersection of any two simplexes $F_i$ and $F_j$ is either empty or a common face of $F_i$ and $F_j$. With a slight abuse of notation, we use the letter $M$ also to denote the supporting polyhedral set $\cup_i F_i$.

Given an $n$-dimensional simplex $F$ in $\mathbb{R}^{n+k}$ and $\delta, \gamma > 0$, let $U = U(F, \delta, \gamma)$ be the set of all $x \in \mathbb{R}^{n+k}$ such that

$$\text{dist}(x, F) \leq \delta \wedge \frac{\gamma}{\sqrt{1+\gamma^2}} \text{dist}(x, \partial F) \ . \tag{4.1}$$

If we identify the $n$-plane spanned by $F$ with $\mathbb{R}^n$ and write $x \in \mathbb{R}^{n+k}$ as $x = (x', x'') \in \mathbb{R}^n \times \mathbb{R}^k$, then $U$ is the set of all $x$ such that (see Figure 4.1 below, or Figure 9.1 in Appendix B)

$$x' \in F \text{ and } |x''| \leq g(x'), \text{ where } g(x') := \delta \wedge \gamma \, \text{dist}(x', \partial F). \tag{4.2}$$

Jacobians of maps of class $W^{1,k-1}$ valued in the sphere $S^{k-1}$ are intended in the distributional sense (see Remark 2.11).

The next proposition contains the basic construction for the approximating maps $u_\varepsilon$. In view of the proof of the upper bound inequality in Theorem 5.5, we have chosen a more general statement than needed for the proof of Theorem 1.1(ii).

**Lemma 4.2.** *Let $N$ be an integral polyhedral current in $\mathbb{R}^{n+k}$ with dimension $n+1$ such that $|\partial N|(\partial\Omega) = 0$ and $\partial N$ has multiplicity 1, and denote by $M$ the restriction of $\partial N$ to $\Omega$. Let these be given: positive numbers $\delta, \gamma$, a map $\tilde{u} : \overline{\Omega} \to S^{k-1}$, and a finite union $S$ of $(n-1)$-dimensional simplexes in $\mathbb{R}^{n+k}$ which contains all $(n-1)$-dimensional faces of $N$, so that the following holds:*

(i) $\tilde{u} \in W^{1,k-1}(\Omega, S^{k-1})$ and $\star J\tilde{u} = \alpha_k M$;

(ii) $\tilde{u}$ *is locally Lipschitz in* $\overline{\Omega} \setminus (M \cup S)$ *and there exists* $p < 1 + \frac{1}{k}$ *such that*

$$|D\tilde{u}(x)| = O\big(1/\mathrm{dist}(x,M)\big) + O\big(1/(\mathrm{dist}(x,S))^p\big) \ .$$

(iii) *for every $n$-dimensional face $F$ of $\partial N$,*

$$\tilde{u}(x) = x''/|x''| \quad \text{for } x \in U(F,\delta,\gamma) \cap \Omega.$$

*Then the approximating maps $u_\varepsilon : \overline{\Omega} \to \mathbb{R}^k$ defined by*

$$u_\varepsilon(x) := g_\varepsilon(x)\,\tilde{u}(x) \quad \text{where} \quad g_\varepsilon(x) := 1 \wedge \frac{\mathrm{dist}(x, M \cup S)}{\varepsilon} \tag{4.3}$$

*are locally Lipschitz in the complement of $S$, belong to $W^{1,q}(\Omega, \mathbb{R}^k)$ for every $q < \frac{k+1}{p-1}$, and converge strongly to $\tilde{u}$ in $W^{1,q}(\Omega)$ for every $q < k \wedge \frac{k+1}{p}$, and in particular for $q = k-1$. The Jacobians $\star J u_\varepsilon$ converge to $\star J\tilde{u} = \alpha_k M$ in the flat norm $\mathbf{F}_\Omega$, and*

$$\limsup_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \le \beta_k \, \|M\|_\Omega \ . \tag{4.4}$$

*Proof.* Using Lemma 8.3, the regularity of $u_\varepsilon$ and the convergence of $u_\varepsilon$ to $\tilde{u}$ are matter of simple computations. The strong convergence in $W^{1,q}$ for $q = k-1$ follows by the assmption on $p$, and implies the convergence of Jacobians (see Remark 2.11). It remains to prove (4.4).
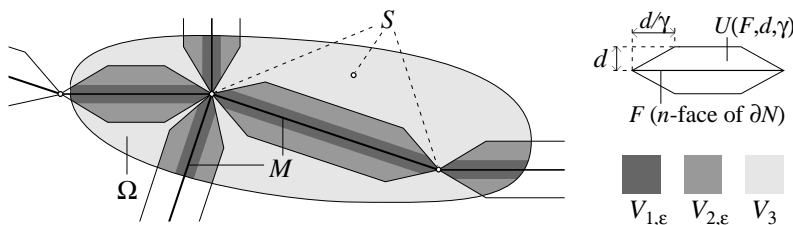


FIGURE 4.1.

Let $d \in (0, \delta \wedge 1]$ be fixed for the time being, and denote by $U$ the union of the set $U(F, d, \gamma)$ over all $n$-dimensional faces $F$ of $\partial N$. For every $\varepsilon \in (0, d]$ we decompose $\Omega$ as union of the sets $V_{1,\varepsilon}$, $V_{2,\varepsilon}$, $V_3$ defined as follows (see Figure 4.1):

$$V_{1,\varepsilon} := \big\{ x \in \Omega \cap U : \ \mathrm{dist}(x, M \cup S) \leq \varepsilon \big\}$$
$$V_{2,\varepsilon} := (\Omega \cap U) \setminus V_{1,\varepsilon}$$
$$V_3 := \Omega \setminus U \ .$$

For the rest of this proof, the letter $C$ denotes any positive finite constant, possibly different at each occurrence, that depends only on the choice of $\Omega$, $N$, $\tilde{u}$, $\delta$ and $\gamma$. Since $V_{1,\varepsilon} \subset U$, assumption (iii) implies $|D\tilde{u}(x)| \leq C/\mathrm{dist}(x, M)$, and therefore formula (4.3) yields

$$|Du_\varepsilon(x)| \leq g_\varepsilon(x) \cdot |D\tilde{u}(x)| + |Dg_\varepsilon(x)| \cdot |\tilde{u}(x)|$$
$$\leq \frac{C \cdot \mathrm{dist}(x, M \cup S)}{\varepsilon \cdot \mathrm{dist}(x, M)} + \frac{1}{\varepsilon} \leq \frac{C}{\varepsilon} \ .$$

Using that $W(u_\varepsilon) \leq C$ and $\mathscr{L}^{n+k}(V_{1,\varepsilon}) \leq C\varepsilon^k$ (cf. Lemma 8.3), we obtain

$$F_\varepsilon(u_\varepsilon, V_{1,\varepsilon}) \leq C \ . \tag{4.5}$$

In $V_{2,\varepsilon}$, $u_\varepsilon$ agrees with $\tilde{u}$, and since $\tilde{u}$ takes values in $S^{k-1}$, the energy density $e_\varepsilon(u_\varepsilon)$ reduces to $\frac{1}{k}|D\tilde{u}|^k$. For every $n$-dimensional face $F$ of $\partial N$ and every $x \in U(F, d, \gamma)$, assumption (iii) implies

$$|D\tilde{u}(x)| = \left| D\frac{x''}{|x''|} \right| = \left| \frac{I}{|x''|} - \frac{x'' \otimes x''}{|x''|^3} \right| = \frac{(k-1)^{1/2}}{|x''|} \ . \tag{4.6}$$

Since the projection of $V_{2,\varepsilon} \cap U(F, d, \gamma)$ on the affine plane spanned by $F$ is contained in $F \cap \Omega_d$, where $\Omega_d$ is the closed $d$-neighbourhood of $\Omega$, from (4.6) we obtain

$$F_\varepsilon(u_\varepsilon, V_{2,\varepsilon} \cap U(F, d, \gamma)) \leq \mathscr{H}^n(F \cap \Omega_d) \frac{(k-1)^{k/2}}{k} \int_{B_d^k \setminus B_\varepsilon^k} \frac{dx''}{|x''|^k}$$
$$\leq \mathscr{H}^n(F \cap \Omega_d) \frac{(k-1)^{k/2}}{k} \int_\varepsilon^1 \frac{k\alpha_k \rho^{k-1} d\rho}{\rho^k}$$
$$= \beta_k |\log \varepsilon| \cdot \mathscr{H}^n(F \cap \Omega_d) \ .$$

Since the sets $V_{2,\varepsilon} \cap U(F, d, \gamma)$ cover $V_{2,\varepsilon}$ (and have negligible intersections), summing the previous estimate over all $F$ yields

$$F_\varepsilon(u_\varepsilon, V_{2,\varepsilon}) \leq \beta_k |\log \varepsilon| \cdot \mathscr{H}^n(M \cap \Omega_d) \ . \tag{4.7}$$

It remains to estimate the energy in $V_3 := \Omega \setminus U$. Using (4.1) it is easy to check that $\text{dist}(x, S) \leq C \, \text{dist}(x, M \cup S)$ in the complement of $U$. Thus assumption (ii) becomes $|D\tilde{u}| \leq C/(\text{dist}(x, S))^p$, and by formula (4.3)

$$|Du_\varepsilon(x)| \leq \frac{C}{\text{dist}(x, S)^p} \ .$$

Taking into account that $W(u_\varepsilon) = 0$ in the complement of the $\varepsilon$-neighbourhood of $M \cup S$, and using Lemma 8.3 we obtain

$$F_\varepsilon(u_\varepsilon, V_{3,\varepsilon}) \leq \int_\Omega \frac{C \, dx}{\text{dist}(x, S)^{kp}} + \frac{C}{\varepsilon^2} \mathscr{L}^{n+k}((M \cup S)_\varepsilon) \leq C \ . \qquad (4.8)$$

Putting together (4.5), (4.7), and (4.8) we obtain

$$\limsup_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \leq \beta_k \mathscr{H}^n(M \cap \Omega_d) = \beta_k |M|(\Omega_d) \ .$$

Finally, the assumption $|M|(\partial\Omega) = 0$ implies that the infimum of $|M|(\Omega_d)$ over all $d \in (0, 1 \wedge \delta]$ is $|M|(\overline{\Omega}) = |M|(\Omega) = \|M\|_\Omega$, and (4.4) is proved. $\qquad\square$

**Remark 4.3.** Let $\Sigma$ be a Lipschitz $m$-surface contained in $\overline{\Omega}$ and transversal to $M$ and $S$.[16] Then the proof of Lemma 4.2 gives

$$\limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_\Sigma e_\varepsilon(u_\varepsilon) \, d\mathscr{H}^m \leq C \mathscr{H}^{m-k}(M \cap \Sigma) \ ,$$

where $C$ is a constant that depends only on the choice of $M$ and $\Sigma$.

**Proof of Theorem 1.1(ii).** Let $M$ be taken as in Lemma 4.2: by Theorem 5.10 in [3] (see also Theorem 9.6 in Appendix B), there exist a map $\tilde{u}$ which satisfies assumptions (i)–(iii) of Lemma 4.2 with $S$ the union of all $(n-1)$-dimensional faces of $N$; in fact, $\tilde{u}$ is defined on $\mathbb{R}^{n+k}$, and satisfies

$$|D\tilde{u}(x)| = O\left(\frac{1}{\text{dist}(x, \partial N \cup S)}\right) \ .$$

Then the maps $u_\varepsilon$ defined in (4.3) satisfy statement (ii) of Theorem 1.1—note that (4.4) and (1.3) imply (1.4). The proof for general $M$ follows by the polyhedral approximation given in Proposition 8.6 and a suitable diagonal argument. $\qquad\square$

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[16] If $N_1$ and $N_2$ are finite union of Lipschitz surfaces in $\mathbb{R}^d$ of dimension $d_1$ and $d_2$, respectively, we say that they are transversal when either $d_1 + d_2 < d$ and $N_1 \cap N_2$ is empty, or $d_1 + d_2 \geq d$ and $N_1 \cap N_2$ is contained in a finite union of Lipschitz surfaces with dimension $d_1 + d_2 - d$, and there exists a finite $C$ such that $\text{dist}(x, N_1 \cap N_2) \leq C \, \text{dist}(x, N_1)$ for all $x \in N_2$, and viceversa. In case of polyhedral sets, this notion is equivalent to $\dim(N_1 \cap N_2) \leq d_1 + d_2 - d$.

## 5. Boundary Value Problems

In the rest of this paper we deal with Dirichlet boundary conditions. In this section we describe some related notions and then state the Γ-convergence and compactness result in this setting (Theorem 5.5) and the concentration result for minimizing sequences (Corollary 5.6). Proofs will be given in Sections 6 and 7.

Here and in the rest of this paper we assume that $n \geq 1$ and $\Omega$ is a bounded Lipschitz domain. Unless otherwise stated, we only consider currents on the entire space $\mathbb{R}^{n+k}$, and convergence will be intended in the sense of the flat norm $\mathbf{F}_{\mathbb{R}^{n+k}}$. In particular, given $u \in W^{1,k}(\Omega, \mathbb{R}^k)$, $\star Ju$ denotes now the current in $\mathbb{R}^{n+k}$ which is obtained by extending the Jacobian of $u$ to 0 outside $\Omega$.

**5.1. Cobordant currents.** We say that the rectifiable $n$-currents $M$ and $M'$ are *cobordant* in the compact set $E$, and write $M \sim_E M'$, if $M - M' = \partial N$ where $N$ is an integral $(n+1)$-current supported in $E$.

If $M \sim_E M'$ and $E$ is the closure of a bounded Lipschitz domain, by the isoperimetric theorem we can find $N$ supported in $E$ such that $M - M' = \partial N$ and $\|N\| \leq C\|M - M'\|^{1+1/n}$, where the constant $C$ depends only on $E$ (apply Theorem 4.4.2(2) in [20] with $A$ equal to $E$, $U$ a neighbourhood of $E$ that admits a Lipschitz retraction on $E$, $B$ and $W$ empty). This fact and the closure theorem for integral currents imply the following closure property of the relation $\sim_E$: given sequences of rectifiable $n$-currents $(M_i)$ and $(M'_i)$ which are uniformly bounded in mass, if $M_i \sim_E M'_i$ for all $i$ and $M_i \to M$, $M'_i \to M'$, then $M \sim_E M'$.

Clearly $M \sim_E M'$ implies that $\mathrm{supp}(M - M') \subset E$ and $\partial M = \partial M'$, but the converse may not hold. However, if $E$ is the closure of a bounded Lipschitz domain and the $n$-th integral homology group $H_n(E, \mathbb{Z})$ is trivial, then a rectifiable current supported in $E$ is the boundary of a rectifiable current supported in $E$ if and only if it has no boundary, and therefore $M \sim_E M'$ if and only if $\mathrm{supp}(M - M') \subset E$ and $\partial M = \partial M'$.

**5.2. Jacobians of traces of Sobolev maps.** If $v : \partial\Omega \to \mathbb{R}^k$ is a smooth map, then $Jv$ and $\star Jv$ can be defined as in Sections 2.9 and 2.10, and in particular $\star Jv$ is an $(n-1)$-current in $\mathbb{R}^{n+k}$ supported on $\partial\Omega$. Moreover, if $u$ is a smooth extension of $v$ to $\Omega$, then $\star Jv$ is the boundary of $\star Ju$ (cf. Lemma 6.1). This identity motivates the following general definition due to F.-B. Hang and F.-H. Lin [23] (inspired by similar notions introduced in [13], see also [36]): the $k$-dimensional Jacobian of a map $v$ in the trace space $W^{1-1/k,k}(\partial\Omega, \mathbb{R}^k)$ is the current in $\mathscr{D}_{n-1}(\mathbb{R}^{n+k})$ defined by

$$\star Jv := \partial(\star Ju) \ , \tag{5.1}$$

where $u$ is any map in $W^{1,k}(\Omega, \mathbb{R}^k)$ with trace $v$ on $\partial\Omega$, and $\star Ju$ is the current on $\mathbb{R}^{n+k}$ obtained by extending the Jacobian to 0 outside $\Omega$, as already agreed.

As shown by identity (5.2) below, $\star Jv$ does not depend on the choice of the extension $u$, and since $\star Ju$ has no boundary in $\Omega$, $\star Jv$ is always supported on $\partial\Omega$, and clearly $\mathbf{F}_{\mathbb{R}^{n+k}}(\star Jv) \le \|\star Ju\| \le \|Du\|_k^k$ (but $\star Jv$ may have infinite mass). Moreover $\star J$ is a continuous operator from $W^{1-1/k,k}(\partial\Omega,\mathbb{R}^k)$ into the space of $(n-1)$-dimensional boundaries in $\mathbb{R}^{n+k}$ with finite $\mathbf{F}_{\mathbb{R}^{n+k}}$ norm.

**Proposition 5.3.** *Let $\Omega$ be a bounded Lipschitz domain and $v$ a map in $W^{1-1/k,k}(\partial\Omega,\mathbb{R}^k)$. Take maps $u, u' \in W^{1,k}(\Omega,\mathbb{R}^k)$ which extend $v$, and regular level sets $M_y(u)$, $M_{y'}(u)$, $M_y(u')$ in the sense of Section 2.7. Then*

$$M_y(u) \sim_{\overline{\Omega}} M_y(u') \ . \tag{5.2}$$

*If in addition $v$ takes values in $S^{k-1}$, then*

(i) *$M_y(u)$ is cobordant to $0$ in $\overline{\Omega}$ when $|y| > 1$;*

(ii) *$M_y(u)$ and $M_{y'}(u)$ are cobordant in $\overline{\Omega}$ when $|y|, |y'| < 1$;*

(iii) *$M_y(u)$ is cobordant in $\overline{\Omega}$ to $(-1)^n M_{y'}(v)$ when $|y| < 1$, $v$ is of class $W^{1,k-1}$, and $y'$ is a regular value of $v$ in the sense of Remark 2.8(ii);*

(iv) *$\partial M_y(u) = \alpha_k^{-1}\star Jv$ when $|y| < 1$; then $\alpha_k^{-1}\star Jv$ is the boundary of a rectifiable current, and is itself rectifiable when it has finite mass.*

**Remark 5.4.** (i) The proof of statement (iii) above (see Section 6) gives a slightly more explicit result: for a.e. $y$ and $y'$, $M_y(u) - (-1)^n M_{y'}(v)$ is the boundary of $M_e(u_y)$ where $u_y := (u - y)/|u - y|$ and $e := (y' - y)/|y' - y|$ (by Proposition 6.4(ii), the map $u_y$ belongs to $W^{1,k-1}(\Omega, S^{k-1})$ for a.e. $y$). In particular $M_y(u)$ agrees in $\Omega$ with $\partial M_e(u_y)$.

(ii) The rectifiability of $\alpha_k^{-1}\star Jv$ (see statement (iv)) was first proved in [23]. We give here a simplified proof.

(iii) If $v$ is a map of class $W^{1,k}$, then $Jv$ agrees with the usual Jacobian, namely the pull-back of the standard volume form on $\mathbb{R}^k$. Therefore $Jv = 0$ when $v$ takes values in $S^{k-1}$ because $\mathscr{L}^k(S^{k-1}) = 0$. This is not true in general for maps of class $W^{1-1/k,k}$, and the typical example is $v(x) := x'/|x'|$, with $x = (x', x'')$ in $\mathbb{R}^k \times \mathbb{R}^{n-1}$, whose Jacobian $\star Jv$ is the current associated to the plane $\{0\} \times \mathbb{R}^{n-1}$ and constant multiplicity $\alpha_k$.

(iv) If $v \in W^{1-1/k,k}(\partial\Omega, S^{k-1})$ is smooth in the complement of an $(n-1)$-dimensional surface $M \subset \partial\Omega$ without boundary, then $\star Jv$ is supported on $M$, and more precisely is of the form $\alpha_k\, \sigma\, \tau_M \cdot \mathscr{H}^{n-1} \llcorner M$, where $\sigma$ is an integral multiplicity that can be explicitly computed in terms of the degree of the restriction of $u$ to certain $k$-surfaces (cf. [28] and [3, Section 3.7]).

We can now state the main results of this section.

**Theorem 5.5.** *Let $\Omega$ be a bounded Lipschitz domain and let $v$ be a map in $W^{1-1/k,k}(\partial\Omega, S^{k-1})$. Choose $u \in W^{1,k}(\Omega, \mathbb{R}^k)$ with trace $v$ on $\partial\Omega$ and a regular level set $M_y = M_y(u)$ with $|y| < 1$. The following statements hold.*

(i) *Assume a (countable) sequence of maps $u_\varepsilon \in W^{1,k}(\Omega.\mathbb{R}^k)$ with trace $v$ on $\partial\Omega$ such that $F_\varepsilon(u_\varepsilon) = O(|\log \varepsilon|)$. Then we can extract a subsequence such that the Jacobians $\star Ju_\varepsilon$ converge in the flat norm $\mathbf{F}_{\mathbb{R}^{n+k}}$ to $\alpha_k M$, where $M$ is a rectifiable $n$-current supported in $\overline{\Omega}$ such that $M \sim_{\overline{\Omega}} M_y$. The following localized version of the lower bound inequality holds: for every open set $A \subset \mathbb{R}^{n+k}$*

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon, A \cap \Omega)}{|\log \varepsilon|} \geq \beta_k \|M\|_A \ . \tag{5.3}$$

(ii) *Given a rectifiable $n$-current $M$ supported in $\overline{\Omega}$ such that $M \sim_{\overline{\Omega}} M_y$, for every $\varepsilon > 0$ we can find $u_\varepsilon$ such that $u_\varepsilon = v$ on $\partial\Omega$, $\mathbf{F}_{\mathbb{R}^{n+k}}(\star Ju_\varepsilon - \alpha_k M) \to 0$, and*

$$\lim_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon, \Omega)}{|\log \varepsilon|} = \beta_k \|M\|_{\mathbb{R}^{n+k}} \ . \tag{5.4}$$

**Remarks.** (i) By Proposition 5.3, the cobordancy condition $M \sim_{\overline{\Omega}} M_y(u)$ in Theorem 5.5 (and in Corollary 5.6 below) is independent of the choice of the extension $u$ and of the value $y$. If $v$ is of class $W^{1,k-1}$, by Proposition 5.3(iii) we can replace $M_y(u)$ by any regular level set of $v$ in the sense of Remark 2.8(ii).

(ii) Statement (i) of Theorem 5.5 is stronger than the corresponding statement in Theorem 1.1 under many regards. First of all, the rectifiable current $M$ is the limit of the Jacobians of $u_\varepsilon$ in $\mathbb{R}^{n+k}$, and not just in $\Omega$: it may happen that part of $M$ is supported on the boundary of $\partial\Omega$, and in that case the rectifiable current given by statement (ii) of Theorem 1.1 is the restriction of $M$ to $\Omega$. Accordingly, the right-hand side of (5.3) takes also into account the part of the mass of $M$ which is located on the boundary of $\Omega$ and therefore (5.3) cannot be deduced from (1.3) except when $A \cap \partial\Omega = \varnothing$.

(iii) As already pointed out in Remark (i) after Theorem 1.1, the right-hand side of (5.3) does not depend on the choice of the potential $W$. Hence, replacing $W$ with $\sigma W$ and then taking the limit in (5.3) as $\sigma \to 0$, we obtain the stronger estimate

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_{\Omega \cap A} \frac{1}{k} |Du_\varepsilon|^k \geq \beta_k \|M\|_A \ .$$

In particular, if the sequence $(u_\varepsilon)$ satisfies the upper bound inequality (5.4), then the potential part of the energy is asymptotically negligible, in the sense that (1.5) holds.

**Corollary 5.6.** *Take $\Omega$, $v$, $u$, $M_y$ as in Theorem 5.5, and denote by $m_\varepsilon$ the minimum of $F_\varepsilon(u, \Omega)$ over all maps $u \in W^{1,n}(\Omega, \mathbb{R}^k)$ with trace $v$ on $\partial\Omega$. Then*

$$m_\varepsilon = O(|\log \varepsilon|) \ . \tag{5.5}$$

*Let $(u_\varepsilon)$ be any (countable) sequence of maps with trace $v$ on $\partial\Omega$ such that*

$$F_\varepsilon(u_\varepsilon, \Omega) = m_\varepsilon + o(|\log \varepsilon|) \ . \tag{5.6}$$

*Then, up to a subsequence, the Jacobians $\star Ju_\varepsilon$ converge in the flat norm $\mathbf{F}_{\mathbb{R}^{n+k}}$ to $\alpha_k M$, where $M$ minimizes the mass among all rectifiable $n$-current supported in $\overline{\Omega}$ which are cobordant to $M_y$ in $\overline{\Omega}$. Moreover the potential part of the energy is asymptotically negligible, in the sense that (1.5) holds, and the energy densities $e_\varepsilon(u_\varepsilon)$ and $\frac{1}{k}|Du_\varepsilon|^k$, rescaled by $1/|\log \varepsilon|$, converge to $\beta_k |M|$ in the sense of measures on $\mathbb{R}^{n+k}$.*

**Remarks.** (i) By Proposition 5.3(iv), the cobordancy condition $M \sim_{\overline{\Omega}} M_y$ implies

$$\partial M = \alpha_k^{-1} \star Jv \ .$$

When the $n$-th homology group $H_n(\overline{\Omega}, \mathbb{Z})$ is trivial, $M \sim_{\overline{\Omega}} M_y$ is equivalent to $\partial M = \alpha_k^{-1} \star Jv$ (see Section 5.1). Thus Corollary 1.2 is a particular case of Corollary 5.6.

(ii) If $\star Jv$ vanishes—which happens, e.g., when $v$ is of class $W^{1,k}$—then the current $M$ in Corollary 5.6 minimizes the area among all cycles (rectifiable currents without boundary) in the integral homology class of $M_y$. If in addition $H_n(\overline{\Omega}, \mathbb{Z}) = 0$, then $M$ is trivial.

## 6. PROOF OF PROPOSITION 5.3 AND THEOREM 5.5(i)

We prove first Proposition 5.3: even though all claims are obvious (or empty) when $u$ and $u'$ are smooth up to the boundary, the proof in the general case is rather delicate. We follow the notation of Section 5.

**Lemma 6.1.** *Let $u$ be a map in $W^{1,k}(\Omega, \mathbb{R}^k)$ with trace $v$ in $W^{1,k}(\partial\Omega, \mathbb{R}^k)$, and let $\rho$ be a bounded continuous function on $\mathbb{R}^k$. Then*

$$\partial(\rho(u) \star Ju) = \rho(v) \star Jv \quad in \ \mathbb{R}^{n+k}. \tag{6.1}$$

*Proof.* By a density argument, it suffices to prove (6.1) when $u$ is smooth up to the boundary. Denoting by $\tau_\Omega = e_1 \wedge \cdots \wedge e_{n+k}$ the standard orientation of $\Omega$, for every $(n-1)$-form $\omega \in \mathscr{D}^{n-1}(\mathbb{R}^{n+k})$ there holds

$$
\begin{aligned}
(\rho(u) \star Ju)[d\omega] &= \int_\Omega (d\omega \wedge \rho(u)\, Ju) \cdot \tau_\Omega \\
&= \int_\Omega d(\omega \wedge \rho(u)\, Ju) \cdot \tau_\Omega \\
&= \int_{\partial\Omega} (\omega \wedge \rho(u)\, Jv) \cdot \tau_{\partial\Omega} = (\rho(u) \star Jv)[\omega] \ ,
\end{aligned}
$$

where the first and fourth identities follow from (2.9), the third one is Stokes theorem, and the second one is a particular case of the general identity $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^m \omega \wedge d\omega'$, where $m$ is the order of the form $\omega$ (in this case $m := n-1$ and $\omega' := \rho(u)\, Ju = u^\sharp(\rho\, dy)$, thus $d\omega' = u^\sharp(d(\rho\, dy)) = u^\sharp(0) = 0$).     $\square$

**Lemma 6.2.** *Assume a bounded continuous function $\rho$ on $\mathbb{R}^k$, and maps $u$, $u'$ in $W^{1,k}(\Omega, \mathbb{R}^k)$. We denote by $\pi$ the projection of $\mathbb{R} \times \mathbb{R}^{n+k}$ on $\mathbb{R}^{n+k}$, and set*

$$
U(t,x) := t\, u(x) + (1-t)\, u'(x) \quad \text{for every } (t,x) \in (0,1) \times \Omega,
$$

$$
\bar{\rho}(y) := \int_0^1 \rho(ty)\, t^{k-1} dt \quad \text{for every } y \in \mathbb{R}^k.
$$

*Thus $\pi_\sharp \partial\big(\rho(U) \star JU\big)$ is an $n$-current on $\mathbb{R}^{n+k}$ of class $L^1$. If $u$ and $u'$ have the same trace on $\partial\Omega$ then*

$$
\pi_\sharp \partial\big(\rho(U) \star JU\big) = \rho(u) \star Ju - \rho(u') \star Ju' \quad \text{in } \mathbb{R}^{n+k}, \tag{6.2}
$$

*whereas if $u$ has trace $v \in W^{1,k-1}(\partial\Omega, \mathbb{R}^k)$ and $u' = 0$ then*[17]

$$
\pi_\sharp \partial\big(\rho(U) \star JU\big) = \rho(u) \star Ju - (-1)^n \star v^\sharp(\bar{\rho}\, \omega_0) \quad \text{in } \mathbb{R}^{n+k}. \tag{6.3}
$$

*Proof.* By a density argument, we can assume that the traces of $u$ and $u'$ belongs to $W^{1,k}(\partial\Omega, \mathbb{R}^k)$. If we denote by $V$ the trace of $U$ on the boundary of $A := (0,1) \times \Omega$, Lemma 6.1 yields

$$
\pi_\sharp \partial(\rho(U) \star JU) = \pi_\sharp(\rho(V) \star JV) \ . \tag{6.4}
$$

---

[17] Here $\omega_0$ is the $(k-1)$-form defined in (2.7). Thus $v^\sharp(\bar{\rho}\, \omega)$ is a $(k-1)$-form on $\partial\Omega$ and $\star v^\sharp(\bar{\rho}\, \omega)$ is an $n$-current supported on $\partial\Omega$.

The boundary of $A$ is the union of $\{1\} \times \Omega$, $\{0\} \times \Omega$, and $(0,1) \times \partial\Omega$, and we denote the restrictions of $\rho(V) \star JV$ to these three sets by $T_1$, $T_2$ and $T_3$. We claim that

$$\pi_\sharp T_1 = \rho(u) \star Ju \quad \text{and} \quad \pi_\sharp T_2 = -\rho(u') \star Ju' . \tag{6.5}$$

Indeed, since $V = u$ and $\tau_{\partial A} = \tau_\Omega$ on $\{1\} \times \Omega$ (cf. Note 11), for every smooth $n$-form $\omega$ on $\Omega$ there holds

$$\pi_\sharp T_1[\omega] = T_1[\pi^\sharp \omega] = \int_{\{1\} \times \Omega} (\pi^\sharp \omega \wedge \rho(V) \, JV) \cdot \tau_{\partial A}$$

$$= \int_\Omega (\omega \wedge \rho(u) \, Ju) \cdot \tau_\Omega = (\rho(u) \star Ju) [\omega] .$$

A similar argument gives the second identity in (6.5).

    We compute now $T_3$. Let $e$ be the unit vector that orients $\mathbb{R} \times \{0\}$. Then $\tau_{\partial A} = -e \wedge \tau_{\partial\Omega}$ on $(0,1) \times \partial\Omega$. If we assume that $u$ and $u'$ have the same trace $v$ on $\partial\Omega$, then $V(t,x) = v(x)$ and $JV(t,x) = Jv(x)$ on $(0,1) \times \partial\Omega$, and therefore[18]

$$\pi_\sharp T_3[\omega] = \int_{(0,1) \times \partial\Omega} \rho(v) \, (\omega \wedge Jv) \cdot (-e \wedge \tau_{\partial\Omega}) = 0 .$$

Recalling (6.5), (6.4) and identity $\rho(V) \star JV = T_1 + T_2 + T_3$ we obtain (6.2).

    On the other hand, if we assume $u' = 0$, then $V(t,x) = t \, v(x)$ on $(0,1) \times \partial\Omega$ and an easy computation yields

$$JV = \bigwedge_i (t \, dv_i + v_i \, dt) = t^k Jv + t^{k-1} dt \wedge v^\sharp \omega_0 .$$

Proceeding as before,

$$\pi_\sharp T_3[\omega] = \int_{(0,1) \times \partial\Omega} \rho(tv) \, (t^k \, \omega \wedge Jv + t^{k-1} \omega \wedge dt \wedge v^\sharp \omega_0) \cdot (-e \wedge \tau_{\partial\Omega})$$

$$= \int_{(0,1) \times \partial\Omega} (-1)^{n+1} \rho(tv) \, t^{k-1} (dt \wedge \omega \wedge v^\sharp \omega_0) \cdot (e \wedge \tau_{\partial\Omega})$$

$$= (-1)^{n+1} \int_{\partial\Omega} \bar\rho(v) \, (\omega \wedge v^\sharp \omega_0) \cdot \tau_{\partial\Omega}$$

$$= (-1)^{n+1} \star v^\sharp (\bar\rho \, \omega_0)[\omega] ,$$

and (6.3) is proved.                                              $\square$

---

[18] Since no component of $\omega \wedge Jv$ contains $dt$, $(\omega \wedge Jv) \cdot (e \wedge \tau) = 0$ for every $(n-1)$-vector $\tau$.

**Proof of Proposition 5.3.** We first prove (5.2). Let $y$ be a regular value of $u$, $u'$ and of the map $U$ defined in Lemma 6.2. Take a smooth non-negative function $\rho : \mathbb{R}^k \to \mathbb{R}$ with compact support and integral 1, and set $\rho_\varepsilon(z) := \varepsilon^{-k}\rho((z-y)/\varepsilon)$. Using the coarea formula (2.6), and replacing $\rho$ with $\rho_\varepsilon$, identity (6.2) can be rewritten as

$$\int_{\mathbb{R}^k} \rho_\varepsilon(z)\, \partial(\pi_\sharp M_z(U))\, dz = \int_{\mathbb{R}^k} \rho_\varepsilon(z)\, \big(M_z(u) - M_z(u')\big)\, dz \ ,$$

and passing to the limit as $\varepsilon \to 0$ we obtain (5.2). A density argument and the closure property of the cobordism relation give (5.2) when $y$ is not a regular value of $U$.

Statement (i) follows from (5.2) by taking $u' := \Phi(u)$, where $\Phi$ agrees with the identity on $S^{k-1}$ (so that $u = u'$ on $\partial\Omega$), and projects $\mathbb{R}^k$ onto the closed unit ball (so that $M_y(u') = 0$).

Statement (ii) follows from (5.2) by taking $u' := \Phi(u)$, where $\Phi$ is a smooth diffeomorphism that agrees with the identity on $S^{k-1}$ and takes $y'$ into $y$ (so that $M_y(u') = M_{y'}(u)$).

The coarea formula (2.6) yields

$$\star Jv = \partial(\star Ju) = \int_{\mathbb{R}^k} \partial M_y(u)\, dy \ ,$$

and since the currents $M_y(u)$ have all the same boundary for $|y| < 1$, and no boundary for $|y| > 1$, we obtain that $\star Jv = \alpha_k\, \partial M_y(u)$. The rest of statement (iv) follows by the boundary rectifiability theorem.

It remains to prove statement (iii). Let $y$ be a regular value of $u$ and $U$, and assume that $y' := y/|y|$ is a regular value of $v$. Using the coarea formula (2.6) and the coarea formula for maps in $W^{1,k-1}(\partial\Omega, S^{k-1})$ (see [3, Section 7.5]), and replacing $\rho$ with the function $\rho_\varepsilon$ defined above, identity (6.3) can be rewritten as

$$\int_{\mathbb{R}^k} \rho_\varepsilon(z)\, \partial(\pi_\sharp M_z(U))\, dz$$
$$= \int_{\mathbb{R}^k} \rho_\varepsilon(z)\, M_z(u)\, dz - (-1)^n \int_{S^{k-1}} \bar\rho_\varepsilon(z')\, M_{z'}(u)\, d\mathscr{H}^{k-1}(z') \ ,$$

and passing to the limit as $\varepsilon \to 0$ we obtain $\partial(\pi_\sharp M_y(U)) = M_y(u) - M_{y'}(v)$, which implies $M_y(u) \sim_{\overline{\Omega}} M_{y'}(v)$. We prove statement (iii) in full generality using statement (ii) and a density argument. $\qquad\square$

**6.3. Construction of maps with good energy bounds.** For every $a \in \mathbb{R}^k$ and $\varepsilon \geq 0$ such that $\varepsilon + |a| < 1$ we choose a map $\Phi_{a,\varepsilon} : \mathbb{R}^k \to \mathbb{R}^k$ that maps the open ball $B_{a,\varepsilon} := (1-\varepsilon)a + B_\varepsilon^k$ onto the unit open ball $B_1^k$ homothetically, maps the rest of $\mathbb{R}^k$ in the sphere $S^{k-1}$, and agrees with the identity on $S^{k-1}$.

Here is a more precise definition: we set $\Phi_{a,\varepsilon}(a) := a$, then we write every $y \in \mathbb{R}^k$ different from $a$ as $a + tv$ with $t > 0$ and $v$ such that $a + v \in S^{k-1}$, and set

$$\Phi_{a,\varepsilon}(y) := \begin{cases} a + \frac{1}{\varepsilon}tv & \text{for } t < \varepsilon, \\ a + v & \text{for } t \geq \varepsilon. \end{cases}$$

In particular $\Phi_{a,\varepsilon}(y) = a + \frac{1}{\varepsilon}(y - a)$ for every $y \in B_{a,\varepsilon}$. Moreover $\Phi_{a,\varepsilon}$ is Lipschitz for $\varepsilon > 0$, and locally Lipschitz on $\mathbb{R}^k \setminus \{a\}$ for $\varepsilon = 0$. A lengthy but straightforward computation yields

$$|D\Phi_{a,\varepsilon}(y)| \leq \frac{K}{|y - a| \vee \varepsilon} \quad \text{for a.e. } y. \tag{6.6}$$

For every $u : \Omega \to \mathbb{R}^k$ we set

$$u_a = u_{a,0} := \Phi_{a,0}(u) \quad \text{and} \quad u_{a,\varepsilon} := \Phi_{a,\varepsilon}(u) . \tag{6.7}$$

The following lemma was inspired by the proof of Proposition 4 in [8], based in turn on computations from [24].

**Proposition 6.4.** *Take $u$, $a$, $\varepsilon$, $u_a$, $u_{a,\varepsilon}$, $B_{a,\varepsilon}$ as above.*

(i) *The map $u_{a,\varepsilon}$ belongs to $W^{1,k}(\Omega, \mathbb{R}^k)$ for all $a$ and all $\varepsilon > 0$, and*

$$\star J u_{a,\varepsilon} = \frac{1}{\varepsilon^k} \int_{B_{a,\varepsilon}} M_y(u) \, dy . \tag{6.8}$$

(ii) *The map $u_a$ belongs to $W^{1,k-1}(\Omega, S^{k-1})$ for a.e. $a \in B_1^k$, and $\star J u_a = \alpha_k M_a(u)$.*

(iii) *There exists a universal constant $K$ such that, for every $\varepsilon > 0$,*

$$\int_{B_{1-\varepsilon}^k} F_\varepsilon(u_{a,\varepsilon}) \, da \leq K \left[ |\log \varepsilon| \cdot \|Du\|_k^k + \varepsilon^{k-2} \mathscr{L}^{n+k}(\Omega) \right] \tag{6.9}$$

$$\int_{B_{1-\varepsilon}^k} \|Du_{a,\varepsilon} - Du_a\|_{k-1}^{k-1} \, da \leq K\varepsilon \|Du\|_{k-1}^{k-1} . \tag{6.10}$$

(iv) *for every $\delta > 0$ there exists $E_\delta \subset B_1^k$ with $\mathscr{L}^k(E_\delta) \leq \delta$ such that*

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon(u_{a,\varepsilon})}{|\log \varepsilon|} \leq \frac{K}{\delta} \|Du\|_k^k \quad \text{for every } a \in B_1^k \setminus E_\delta. \tag{6.11}$$

(v) *For every $a \in B_1^k$, $u_{a,\varepsilon} \to u_a$ pointwise as $\varepsilon \to 0$, and for a.e. $a \in B_1^k$*

$$\liminf_{\varepsilon \to 0} \|u_{a,\varepsilon} - u_a\|_{W^{1,k-1}} = 0 , \tag{6.12}$$

$$\liminf_{\varepsilon \to 0} \mathbf{F}_\Omega(\star J u_{a,\varepsilon} - \star J u_a) = 0 . \tag{6.13}$$

(vi) *If the trace of $u$ on $\partial\Omega$ takes values in $S^{k-1}$, for a.e. $a \in B_1^k$ there holds*

$$\liminf_{\varepsilon \to 0} \mathbf{F}_{\mathbb{R}^{n+k}}(\star J u_{a,\varepsilon} - \alpha_k M_a(u)) = 0 . \tag{6.14}$$

**Remarks.** (i) As shown in [3, Theorem 3.8], if $u_a$ is of class $W^{1,k-1}$ then $\star Ju_a = (-1)^{n+1}\partial M_{\bar{y}}(u_a)$ for every $\bar{y} \in S^{k-1}$ which is a regular value of $u_a$ in the sense of Remark 2.8(ii).

(ii) Statements (iv)–(vi) hold even when applied to a subsequence of $(\varepsilon)$. However, the set of all $a$ for which one among (6.11)–(6.14) fails may depend on the choice of the subsequence, and therefore we cannot infer that the liminf in (6.11)–(6.14) can be replaced by a limsup. As far as we know, these stronger versions of statements (iv)–(vi) could be true. If so, some of the proofs below could be simplified.

*Proof.* The first part of statement (i) follows by the fact that $\Phi_{a,\varepsilon}$ is Lipschitz for $\varepsilon > 0$, while (6.8) follows by the coarea formula (2.6) and the fact that $\det(D\Phi_{a,\varepsilon})$ is equal to $\varepsilon^{-k}$ on the ball $B_{a,\varepsilon}$, and vanishes elsewhere.

In order to prove (6.9), we define the following auxiliary function:

$$\sigma_\varepsilon(a, x) := \begin{cases} 1 & \text{if } |u(x) - a| \le 2\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Thus (6.6) implies $|Du_{a,\varepsilon}| \le K\varepsilon^{-1}|Du|$ for a.e. $x$ such that $\sigma_\varepsilon(a, x) = 1$, while $|Du_{a,\varepsilon}| \le K|u - a|^{-1}|Du|$ for a.e. $x$ such that $\sigma_\varepsilon(a, x) = 0$. Moreover, in the latter case we also have $u_{a,\varepsilon} \in S^{k-1}$, because $\Phi_{a,\varepsilon}$ maps the complement of $B_{a,\varepsilon}$ into $S^{k-1}$. Hence

$$F_\varepsilon(u_{a,\varepsilon}) \le K \int_\Omega \sigma_\varepsilon(a, x)\left(\frac{|Du(x)|^k}{\varepsilon^k} + \frac{1}{\varepsilon^2}\right)$$
$$+ (1 - \sigma_\varepsilon(a, x))\frac{|Du(x)|^k}{|u(x) - a|^k}\, dx\,. \qquad (6.15)$$

Now we integrate both sides of (6.15) over all $a$ such that $|a| + \varepsilon < 1$, and change the order of integration in the right-hand side:

$$\int_{B_{1-\varepsilon}^k} F_\varepsilon(u_{a,\varepsilon})\, da \le K \int_\Omega \left[\left(\frac{|Du(x)|^k}{\varepsilon^k} + \frac{1}{\varepsilon^2}\right)\int_{B_{1-\varepsilon}^k} \sigma_\varepsilon(a, x)\, da\right.$$
$$\left. + |Du(x)|^k \int_{B_{1-\varepsilon}^k} \frac{1 - \sigma_\varepsilon(a, x)}{|u(x) - a|^k}\, da\right] dx\,.$$

Thus (6.9) follows by the estimates

$$\int_{B_{1-\varepsilon}^k} \sigma_\varepsilon(x, a)\, da \le \mathscr{L}^k(B_{2\varepsilon}^k) = 2^k \alpha_k\, \varepsilon^k$$

and

$$\int_{B_{1-\varepsilon}^k} \frac{1 - \sigma_\varepsilon(x, a)}{|u(x) - a|^k}\, da \le \int_{B_{3/2}^k \setminus B_\varepsilon^k} \frac{dy}{|y|^k} \le K \int_\varepsilon^{3/2} \frac{d\rho}{\rho} \le K|\log \varepsilon|\,.$$

To prove (6.10), we use that $|Du_{a,\varepsilon}| \leq K\varepsilon^{-1}|Du|$ and $|Du_a| \leq K|u-a|^{-1}|Du|$ for a.e. $x$ such that $\sigma_\varepsilon(a,x) = 1$, while $Du_{a,\varepsilon} = Du_a$ for a.e. $x$ such that $\sigma_\varepsilon(a,x) = 0$ because $u_{a,\varepsilon} = u_a$ in the complement of $B_{a,\varepsilon}$. Hence

$$\int_\Omega |Du_{a,\varepsilon} - Du_a|^{k-1} \leq K \int_\Omega \sigma_\varepsilon(a,x) \left( \frac{1}{\varepsilon^{k-1}} + \frac{1}{|u(x) - a|^{k-1}} \right) |Du(x)|^{k-1} \, dx \ .$$

The rest of the proof of (6.10) follows that of (6.9). Estimate (6.9) and Fatou's lemma imply

$$\int_{B_1^k} \liminf_{\varepsilon \to 0} \frac{F_\varepsilon(u_{a,\varepsilon})}{|\log \varepsilon|} \, da \leq \liminf_{\varepsilon \to 0} \int_{B_1^k} \frac{F_\varepsilon(u_{a,\varepsilon})}{|\log \varepsilon|} \, da \leq K\|Du\|_k^k \ , \qquad (6.16)$$

which yields (6.11). In a similar way, we use (6.10) to obtain (6.12), which implies that, up to a subsequence, the maps $u_{a,\varepsilon}$ converge strongly to $u_a$ in $W^{1,k-1}$ (and in particular $u_a$ belongs to $W^{1,k-1}$). Now (6.13) follows by the fact that strong convergence in $W^{1,k-1}$, implies convergence of Jacobians in the flat norm $\mathbf{F}_\Omega$ (see Remark 2.11).

The convergence of $\star Ju_{a,\varepsilon}$ to $\star Ju_a$ (up to a subsequence) and formula (6.8) imply $\star Ju_a = \alpha_k M_a(u)$ for every regular value $a$ of $u$, and the proof of (ii) is completed.

It remains to prove (vi). We take a regular value $a$ of $u$ such that $\star Ju_a = \alpha_k M_a(u)$ (cf. statement (ii)) and $u_{a,\varepsilon}$ converge to $u_a$ in $W^{1,k-1}$ *up to a subsequence*, not relabelled in the following (cf. (6.12)). Now we choose a map $\tilde{u}$ in $W^{1,k}(\mathbb{R}^{n+k} \setminus \overline{\Omega}, \mathbb{R}^k)$ with the same trace as $u$ on $\partial\Omega$ and for every $\varepsilon > 0$ we set

$$\tilde{u}_{a,\varepsilon}(x) := \begin{cases} u_{a,\varepsilon}(x) & \text{if } x \in \Omega \\ \tilde{u}(x) & \text{if } x \in \mathbb{R}^{n+k} \setminus \overline{\Omega}, \end{cases}$$

and define $\tilde{u}_a$ in the same way. Since trace of $u_{a,\varepsilon}$ and $u_a$ on $\partial\Omega$ agrees with that of $u$, and therefore with that of $\tilde{u}$, the maps $\tilde{u}_{a,\varepsilon}$ belong to $W^{1,k}(\mathbb{R}^{n+k}, \mathbb{R}^k)$ and converge in $W^{1,k-1}(\mathbb{R}^{n+k}, \mathbb{R}^k)$ to $\tilde{u}_a$. Hence

$$\mathbf{F}_{\mathbb{R}^{n+k}}(\star J\tilde{u}_{a,\varepsilon} - \star J\tilde{u}_a) \to 0 \ . \qquad (6.17)$$

On the other hand, $\star J\tilde{u}_{a,\varepsilon}$ can be decomposed as $\star J\tilde{u} + \star Ju_{a,\varepsilon}$ on $\mathbb{R}^{n+k}$ (we assume that $\star J\tilde{u}$ is extended to 0 in $\overline{\Omega}$, while $\star Ju_{a,\varepsilon}$ is extended to 0 outside $\Omega$). Now, the first addendum agrees with the restriction of $\star J\tilde{u}_a$ to $\mathbb{R}^{n+k} \setminus \overline{\Omega}$, while formula (6.8) and the fact that $a$ is a regular value of $u$ imply that the second addendum converges to $\alpha_k M_a(u)$ in the weak topology of currents. In particular $\alpha_k M_a(u)$ must agree with the restriction of $\star J\tilde{u}_a$ to $\overline{\Omega}$, and then (6.17) yields (6.14) (recall that (6.17) holds for a subsequence only). $\qquad\square$

**Proof of statement (i) of Theorem 5.5.** Let a (countable) sequence $(u_\varepsilon)$ be given such that $u_\varepsilon = v$ on $\partial\Omega$ and $F_\varepsilon(u_\varepsilon) = O(|\log\varepsilon|)$, and an open set $A \subset \mathbb{R}^{n+k}$. As in the proof of statement (i) of Theorem 1.1, we can assume that the liminf at the left-hand side of (5.3) is a limit.

*First step: compactness and lower bound inequality.*

The idea is to apply statement (i) of Theorem 1.1 with $\Omega$ replaced by a larger set (and $u_\varepsilon$ suitably extended outside of $\Omega$). Let $\delta > 0$ be fixed for the time being. First we take $u \in W^{1,k}(\mathbb{R}^{n+k}, \mathbb{R}^k)$ with trace $v$ on $\partial\Omega$. Then we choose an open set $\Omega'$ so that $\Omega \Subset \Omega'$ and

$$\int_{\Omega'\setminus\Omega} |Du|^k \le \delta \ .$$

We set $U := \Omega' \setminus \overline{\Omega}$. Now, let $u_{a,\varepsilon}$ be defined as in (6.7). By Proposition 6.4 we can choose $a \in B_1^k$ so that, *passing to a subsequence,*

$$F_\varepsilon(u_{a,\varepsilon}, \Omega') = O(|\log\varepsilon|) \tag{6.18}$$

$$F_\varepsilon(u_{a,\varepsilon}, U) \le K|\log\varepsilon| \int_U |Du|^k \le K\delta|\log\varepsilon| \tag{6.19}$$

$$\mathbf{F}_{\Omega'}\big(\star Ju_{a,\varepsilon} - \alpha_k M_a(u)\big) \to 0 \tag{6.20}$$

$$\mathbf{F}_{\mathbb{R}^{n+k}}\big(\star Ju_{a,\varepsilon} \llcorner U - \alpha_k M_a(u) \llcorner U\big) \to 0 \ . \tag{6.21}$$

Finally we set

$$u'_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & \text{if } x \in \Omega, \\ u_{a,\varepsilon}(x) & \text{if } x \in \Omega' \setminus \Omega. \end{cases}$$

The maps $u'_\varepsilon$ belong to $W^{1,k}(\Omega', \mathbb{R}^k)$, and $F(u'_\varepsilon, \Omega') = O(|\log\varepsilon|)$ by (6.19). By Theorem 1.1(i) implies that, up to a subsequence, $\star Ju'_\varepsilon$ converge to $\alpha_k M'$ in the flat norm $\mathbf{F}_{\Omega'}$, where $M'$ is a rectifiable current supported in $\Omega'$. Then (6.21) implies that $\star Ju_\varepsilon = \star Ju'_\varepsilon - \star Ju_{a,\varepsilon} \llcorner U$ converge in the flat norm $\mathbf{F}_{\Omega'}$ to $\alpha_k M$, where

$$M := M' - M_a(u) \llcorner U = M' \llcorner \overline{\Omega} \ . \tag{6.22}$$

Since $\star Ju_\varepsilon$ and $M$ are supported in $\overline{\Omega}$, we have convergence in the flat norm $\mathbf{F}_{\mathbb{R}^{n+k}}$.[19] Moreover the lower bound inequality (1.3) applied to the maps $u'_\varepsilon$ and to the set $\Omega' \cap A$ and estimate (6.19) yield

$$\begin{aligned}
\beta_k |M|(A) &= \beta_k |M'|(A \cap \overline{\Omega}) \\
&\le \beta_k |M'|(A \cap \Omega') \\
&\le \liminf_{\varepsilon\to 0} \frac{F_\varepsilon(u'_\varepsilon, A \cap \Omega')}{|\log\varepsilon|} \le \liminf_{\varepsilon\to 0} \frac{F_\varepsilon(u_\varepsilon, A \cap \Omega)}{|\log\varepsilon|} + K\delta \ .
\end{aligned}$$

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[19]$\mathbf{F}_{\Omega'}(T_i - T) \to 0$ means that there exist currents $S_i$ on $\Omega'$ such that $\partial S_i = T_i - T$ in $\Omega'$ and $\|S_i\| \to 0$. Clearly, if $S_i$ is regarded as a current on $\mathbb{R}^{n+k}$, $\partial S_i$ may have an additional part supported on $\partial\Omega'$. However, if all current $T_i$ are supported on $\Omega$, a careful use of truncation and of the cone construction allows to modify each $S_i$ so that $\partial S_i = T_i - T$ in $\mathbb{R}^{n+k}$ (and $\|S_i\|$ still converge to 0).

Since $\delta$ can be taken arbitrarily small, the lower bound inequality (5.3) is proved.

*Second step: $M$ is cobordant to $M_a(u) \llcorner \Omega$ in $\overline{\Omega}$.*

While it is easy to prove that these two currents differ by the boundary of a real current with finite mass supported in $\overline{\Omega}$, it is more difficult to show that they differ by a rectifiable one.

Choose an open set $V$ such that $\Omega \Subset V \Subset \Omega'$ and whose closure admits a Lipschitz retraction onto $\overline{\Omega}$ (use Proposition 8.1). We claim that $M' \sim_{\overline{V}} M_a(u)$. Were this proved, (6.22) and the fact that $M_a(u) \llcorner \partial\Omega = 0$ (recall that the trace of $u$ on $\partial\Omega$ takes values in $S^{k-1}$) would imply $M \sim_{\overline{V}} M_a(u) \llcorner \Omega$, and we would conclude the proof using the retraction of $\overline{V}$ to $\Omega$. We prove the claim by showing that $M'$ and $M_a(u)$ can be obtained as limits—with respect to the weak convergence of currents—of rectifiable currents which are cobordant in $\overline{V}$ (cf. Section 5.1).

By the choice of $a$, the current $\alpha_k M_a(u)$ is the limit of $\star Ju_{a,\varepsilon}$ in the flat norm $\mathbf{F}_{\Omega'}$, and $F_\varepsilon(u_{a,\varepsilon}, \Omega') = O(|\log \varepsilon|)$. Following the proof of the compactness statement in Proposition 3.1 (see Section 3), we see that the currents $\star Ju_{a,\varepsilon}$ are asymptotically equivalent in the flat norm $\mathbf{F}_{\Omega'}$ to $\star J_\psi u_{a,\varepsilon}$, where $\psi = \rho\,dy$ is a $k$-form with support contained in $B^k_{1/2}$ and integral $\alpha_k$, and $J_\psi$ is defined in Section 3.5. In turn, $\star J_\psi u_{a,\varepsilon}$ are asymptotically equivalent in the flat norm $\mathbf{F}_V$ to the polyhedral deformations $\Phi_{\varepsilon\sharp}(\star J_\psi u_{a,\varepsilon})$, which are uniformly bounded in mass ($\Phi_\varepsilon$ are the retractions associated to the grids $\mathscr{G}_\varepsilon$ as in Lemmas 3.8 and 3.9).

By Lemma 3.9(ii), $\Phi_{\varepsilon\sharp}(\star J_\psi u_{a,\varepsilon}) = \alpha_k \Phi_{\varepsilon\sharp}(M_y(u_{a,\varepsilon}))$ for any regular value $y = y(\varepsilon)$ of $u_{a,\varepsilon}$ with $|y| < 1/2$. Thus $M_a(u)$ is the limit in the flat norm $\mathbf{F}_V$ of the integral polyhedral currents $\Phi_{\varepsilon\sharp}(M_y(u_{a,\varepsilon}))$, which are uniformly bounded in mass. Similarly, $M'$ is the limit of in the flat norm $\mathbf{F}_V$ of the polyhedral currents $\Phi_{\varepsilon\sharp}(M_y(u'_\varepsilon))$, which are also uniformly bounded in mass (we take the same $y = y(\varepsilon)$ as above).

It remains to show that $\Phi_{\varepsilon\sharp}(M_y(u'_\varepsilon))$ and $\Phi_{\varepsilon\sharp}(M_y(u_{a,\varepsilon}))$ are cobordant in $\overline{V}$. Since $u_{a,\varepsilon}$ and $u'_\varepsilon$ agree outside $\Omega$, $M_y(u'_\varepsilon) - M_y(u_{a,\varepsilon})$ is supported in $\overline{\Omega}$, and by (5.2) is cobordant to 0 in $\Omega$. Hence $\Phi_{\varepsilon\sharp}(M_y(u'_\varepsilon) - M_y(u_{a,\varepsilon}))$ is cobordant to 0 in $\overline{V}$ for $\varepsilon$ small enough by a known property of polyhedral deformation,[20] and this concludes the proof.

Let us point out a couple of technical issues in this second step of the proof. Firstly, the argument above, as most lemmas in Section 3, requires that $u'_\varepsilon$ and $u_{a,\varepsilon}$ are smooth in $\Omega'$. This can be fixed by regularizing these maps (e.g., by convolution). Indeed we never used that they have trace $v$ on $\partial\Omega$, but only that they agree outside $\overline{\Omega}$ (in fact, it is enough that they agree outside $\overline{V}$). Secondly, the argument works provided that the size of the grid $\mathscr{G}_\varepsilon$ is taken as in (3.22),

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[20] More precisely, the deformation of the boundary of a current supported in $\overline{\Omega}$ is the boundary of a polyhedral current supported on a $d$-neighbourhood of $\Omega$, where $d$ is, say, twice the diameter of a cube in the grid (cf. [20, Theorem 4.2.9]).

and the center is chosen so that estimates (3.18)–(3.20)—but not (3.17)—hold for $u'_\varepsilon$ and $u_{a,\varepsilon}$ at the same time. This can be achieved by a suitable modification of Lemma 3.11. □

## 7. Proof of Theorem 5.5(ii) and Corollary 5.6

In this section we assume that the potential $W$ in the definition of $F_\varepsilon$ takes a constant value inside the unit ball of $\mathbb{R}^k$: it is clear that proving statement (ii) of Theorem 5.5 for this particular choice of $W$ is sufficient.[21] This assumption is used in the proof of Lemma 7.1 only.

For the rest, we follow the notation of Section 6. In particular, $n \geq 1$ and $\Omega$ is a bounded Lipschitz domain.

The proof of statement (ii) of Theorem 5.5 is a modification of the proof of statement (ii) of Theorem 1.1 given in Section 4. Hence we need an $S^{k-1}$-valued map $\tilde{u}$ with the same properties as the one used in Section 4, which satisfies the additional condition $\tilde{u} = v$ on $\partial\Omega$. This refinement of Theorem 5.10 of [3] is proved in Theorem 9.6 for special boundary data $v$. In order to handle a general $v$, we paste together the maps $u_\varepsilon$ defined in (4.3) and the maps $u_{a,\varepsilon}$ defined in (6.7). This last step is based on the following lemma:

**Lemma 7.1.** *Let $\Omega_1$ be a Lipschitz domain relatively compact in $\Omega$, and set $\Omega_2 := \Omega \setminus \overline{\Omega}_1$. Let maps $u_i \in W^{1,k}(\Omega_i, \mathbb{R}^k)$ for $i = 1, 2$ be given so that $|u_1|, |u_2| \leq 1$ everywhere and their traces on $\Sigma := \partial\Omega_1$ (the interface between $\Omega_1$ and $\Omega_2$) satisfy the following compatibility condition:*

$$|u_1(x)| = |u_2(x)| = 1 \Rightarrow u_1(x) = u_2(x) \quad \text{for } \mathscr{H}^{n+k-1}\text{-a.e. } x \in \Sigma. \qquad (7.1)$$

*Then there exist positive finite constants $\eta_0$ and $C$ (depending on $\Omega$ and $\Omega_1$) so that, for every $0 < \eta \leq \eta_0$ there exists $u \in W^{1,k}(\Omega, \mathbb{R}^k)$ which agrees with $u_1$ in $\Omega_1$ and with $u_2$ in $\partial\Omega$, and for every $\varepsilon > 0$ satisfies*

$$F_\varepsilon(u, \Omega_2) \leq 2\, F_\varepsilon(u_2, \Omega_2) + C\eta \int_\Sigma \frac{1}{\eta^k} + e_\varepsilon(u_1) + e_\varepsilon(u_2) \ . \qquad (7.2)$$

*Proof.* By Proposition 8.1, we can find a neighbourhood $U$ of $\partial\Omega_1 = \Sigma$ which is relatively compact in $\Omega$ and homeomorphic to $(-1, 1) \times \Sigma$ via some bi-Lipschitz map $\Psi$. For every $0 < \eta < 1$, we set $u := u_1$ in $\Omega_1$, $u := u_2$ in $\Omega_2 \setminus U$, and $u := u' \circ \Psi^{-1}$ in $U \setminus \Omega_1 = \Psi([0, 1) \times \Sigma)$, where

$$u'(t, x') := \begin{cases} (1 - t/\eta)\, u_1(x') + (t/\eta)\, u_2(x') & \text{for } 0 \leq t < \eta, \\ u_2\big(\Psi\big((t - \eta)/(1 - \eta), x'\big)\big) & \text{for } \eta \leq t < 1. \end{cases} \qquad (7.3)$$

---

[21] $W$ is not continuous, but this is not needed in the construction that follows.

Hence

$$F_\varepsilon(u, \Omega_2 \setminus U) = F_\varepsilon(u_2, \Omega_2 \setminus U) . \tag{7.4}$$

It remains to estimate the energy of $u$ in $U \cap \Omega_2$. To this end, we decompose $U \cap \Omega_2$ as $U_\eta \cup U'_\eta$ with $U_\eta := \Psi([0,\eta) \times \Sigma)$ and $U'_\eta := \Psi([\eta, 1) \times \Sigma)$. For almost every $(t, x') \in [0, \eta) \times \Sigma$ there holds

$$|Du'(t, x')|^k \leq K \left[ |Du_1(x')|^k + |Du_2(x')|^k + \frac{1}{\eta^k} \right] ,$$

while condition (7.1) and the fact that $W(y)$ is constant for $|y| < 1$ yield

$$W(u'(t, x')) \leq W(u_1(x')) + W(u_2(x')) .$$

Hence

$$F_\varepsilon(u', [0, \eta) \times \Sigma) \leq K\eta \int_\Sigma \frac{1}{\eta^k} + e_\varepsilon(u_1) + e_\varepsilon(u_2) .$$

The change of variable formula gives the estimate

$$F_\varepsilon(u, \Psi(A)) \leq C_1 F_\varepsilon(u \circ \Psi, A) , \tag{7.5}$$

where $C_1 := (1 \vee \mathrm{Lip}(\Psi^{-1}))^k \cdot \mathrm{Lip}(\Psi)^{n+k}$, and therefore

$$\begin{aligned} F_\varepsilon(u, U_\eta) &\leq C_1 F_\varepsilon(u', (0, \eta) \times \Sigma) \\ &\leq K C_1 \eta \int_\Sigma \frac{1}{\eta^k} + e_\varepsilon(u_1) + e_\varepsilon(u_2) . \end{aligned} \tag{7.6}$$

By (8.2) and (7.3), for every $x \in U'_\eta$ there holds $u(x) = u_2(R_\eta(x))$ where

$$R_\eta(x) := x - \eta \frac{1-t}{1-\eta} v(x') ,$$

and $t, x'$ satisfy $x = x' + t\, v(x) = \Psi(t, x')$. Hence $\mathrm{Lip}(R_\eta)$ and $\mathrm{Lip}(R_\eta^{-1})$ are both of order $1 + O(\eta)$, and formula (7.5) yields

$$F_\varepsilon(u, U'_\eta) \leq (1 + O(\eta)) \cdot F_\varepsilon(u_2, U \setminus \Omega_1) . \tag{7.7}$$

For $\eta$ small enough, estimates (7.4), (7.6), (7.7) yield (7.2). $\qquad\square$

**Proof of statement (ii) of Theorem 5.5.** Let $u$ be a map in $W^{1,k}(\Omega, \mathbb{R}^k)$ with trace $v$ on $\partial\Omega$ such that $|u| \leq 1$ and $u$ is locally piecewise affine in $\Omega$, that is, agrees with some piecewise affine map on every compact subset of $\Omega$ (such a map can be obtained from any $u$ with trace $v$ by truncation and a local regularization argument *à la* Meyers-Serrin).

*First reduction.* A simple diagonal argument shows that the statement we want to prove is equivalent to the following: for every $\delta > 0$, every admissible current $M$ and every countable subsequence of $(\varepsilon)$ there exists a subsequence (not relabelled) for which we can construct maps $u_\varepsilon$ such that $\star Ju_\varepsilon$ converge to $\alpha_k M$ and

$$\limsup \frac{F_\varepsilon(u_\varepsilon, \Omega)}{|\log\varepsilon|} \leq \beta_k \|M\| + \delta \ . \tag{7.8}$$

This remark allows us to pass freely to subsequences in the construction below.

Let $\delta$ and the countable subsequence of $(\varepsilon)$ be fixed. We take now $d > 0$ so that $\int_{U_d} |Du|^k \leq \delta^2/(2K)$, where $U_d$ the set of all $x \in \Omega$ such that $\mathrm{dist}(x, \partial\Omega) < d$ and $K$ is the universal constant in (6.11). Then, by Proposition 6.4(iv) we can find $y \in B_1^k$ so that, *passing to a subsequence*, the maps $u_{y,\varepsilon}$ given in (6.7) satisfy

$$\lim_{\varepsilon \to 0} \frac{F_\varepsilon(u_{y,\varepsilon}, U_d)}{|\log\varepsilon|} \leq \frac{K}{\delta} \int_{U_d} |Du|^k \leq \frac{\delta}{2} \ . \tag{7.9}$$

*Second reduction.* A simple diagonal argument and Proposition 8.7 show that it suffices to prove the claim above when the current $M$ has multiplicity 1 and can be written as $M = M_y(u) + \partial N$, where $N$ is a polyhedral current of dimension $n+1$ with compact support in $\Omega$.

Let $M$ and $N$ be fixed, and for every $\varepsilon > 0$ let a positive $\eta = \eta(\varepsilon)$ be given—the correct choice of $\eta$ will be specified below. Now choose a polyhedral domain $\Omega_d$ so that $N \Subset \Omega_d \Subset \Omega$, $\Omega_d \cup U_d = \Omega$ (see Figure 7.1 below) and

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log\varepsilon|} \int_{\partial\Omega_d} \frac{1}{\eta^k} + e_\varepsilon(u_{y,\varepsilon}) \leq \frac{\delta}{2d} + \frac{|\Omega|}{\eta^k |\log\varepsilon|} \ . \tag{7.10}$$

To find such $\Omega_d$, apply Lemma 8.5 with $g : U_d \to [0, d/2]$ a piecewise affine 1-Lipschitz function such that $g = 0$ on $\partial\Omega$ and $g = d/2$ on $\partial U_d \setminus \Omega$—such a function exists because the distance between these two sets is $d$—and then use estimate (7.9). After *passing to a subsequence*, we may assume that the liminf at the left-hand side of (7.10) is a limit.
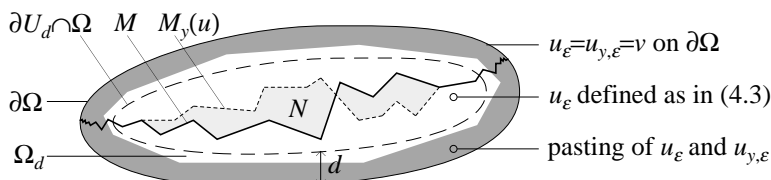


FIGURE 7.1.

By Theorem 9.6, there exists a map $\tilde{u} : \overline{\Omega}_d \to S^{k-1}$ which satisfies assumptions (i), (ii), (iii) of Lemma 4.2 with $\Omega$ replaced by $\Omega_d$, and $\tilde{u} = u_y$ on $\partial\Omega_d$, where $u_y$ is defined as in (6.7). Take $u_\varepsilon : \overline{\Omega}_d \to \mathbb{R}^k$ as in (4.3): then the Jacobians $\star J u_\varepsilon$ converge in the sense of currents to $\alpha_k M$ on $\Omega_d$, and

$$\limsup_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon, \Omega_d)}{|\log \varepsilon|} \le \beta_k \|M\|_{\Omega_d} \le \beta_k \|M\|_\Omega \ . \tag{7.11}$$

Moreover, as pointed out in Remark 4.3,

$$\int_{\partial\Omega_d} e_\varepsilon(u_\varepsilon) = O(|\log \varepsilon|) \ . \tag{7.12}$$

It is easy to check that $u_\varepsilon$ and $u_{y,\varepsilon}$ satisfy the compatibility condition (7.1) on $\partial\Omega_d$, and therefore we can apply Lemma 7.1 to extend the maps $u_\varepsilon$ to $\Omega$ so that they agree with $u_{y,\varepsilon} = v$ on $\partial\Omega$. Moreover estimate (7.2) yields

$$F_\varepsilon(u_\varepsilon, \Omega) \le F_\varepsilon(u_\varepsilon, \Omega_d) + 2\, F_\varepsilon(u_{y,\varepsilon}, \Omega \setminus \overline{\Omega}_d) +$$
$$+ C\eta \int_{\partial\Omega_d} \frac{1}{\eta^k} + e_\varepsilon(u_\varepsilon) + e_\varepsilon(u_{y,\varepsilon}) \ . \tag{7.13}$$

Now we divide both sides of (7.13) by $|\log \varepsilon|$ and take the limsup as $\varepsilon \to 0$. If we have choosen $\eta = \eta(\varepsilon)$ so that it tends to 0 and $1/\eta = o(|\log \varepsilon|^{1/(k-1)})$, then estimates (7.10) and (7.12) imply that the term in the second line of formula (7.13) is negligible with respect to $|\log \varepsilon|$, and recalling estimates (7.9) and (7.11) we obtain (7.8).

It remains to prove that the Jacobians $\star J u_\varepsilon$ converge to $\alpha_k M$ in the flat norm $\mathbf{F}_{\mathbb{R}^{n+k}}$. On the one hand, we already know that $\star J u_\varepsilon$ converge in the sense of currents to $\alpha_k M$ on every $\Omega_d$, and therefore also on $\Omega$. On the other hand, by estimate (7.8) and statement (i) of Theorem 5.5, $\star J u_\varepsilon$ converge, up to a subsequence and in the flat norm $\mathbf{F}_{\mathbb{R}^{n+k}}$, to some $\alpha_k M'$, and

$$\beta_k \|M'\|_{\mathbb{R}^{n+k}} \le \liminf_{\varepsilon \to 0} \frac{F_\varepsilon(u_\varepsilon, \Omega)}{|\log \varepsilon|} \ . \tag{7.14}$$

It remains to prove that $M' = M$. Note that $M$ is supported in $\Omega$ and agrees with $M'$ in $\Omega$, that is, $M = M' \llcorner \Omega$. Combining (7.8) and (7.14) we obtain $\|M'\| \le \|M\|$, and therefore $M'$ must be equal to $M$. $\qquad\square$

**Proof of Corollary 5.6.** Estimate (5.5) follows by applying statement (ii) of Theorem 5.5 to any admissible $M$ with finite mass—for instance any regular level set $M_y(u)$ of any map $u \in W^{1,k}(\Omega, \mathbb{R}^k)$ with trace $v$ on $\partial\Omega$.

Let $(u_\varepsilon)$ be a sequence of maps that satisfy (5.6). Then $F_\varepsilon(u_\varepsilon) = O(|\log \varepsilon|)$, and Theorem 5.5(i) implies that $\star J u_\varepsilon$ converge, up to a subsequence and in the

flat norm $\mathbf{F}_{\mathbb{R}^{n+k}}$, to $\alpha_k M$, where $M$ is a rectifiable current cobordant to $M_y$ in $\overline{\Omega}$, and $\beta_k \|M\| \le \liminf |\log \varepsilon|^{-1} F_\varepsilon(u_\varepsilon)$.

On the other hand, by Theorem 5.5(ii), every $M'$ such that $M' \sim_{\overline{\Omega}} M_y$ can be realized as a limit of Jacobians $\star J u'_\varepsilon$ so that $\lim |\log \varepsilon|^{-1} F_\varepsilon(u'_\varepsilon) = \beta_k \|M'\|$. Hence the minimizing property (5.6) implies $\|M\| \le \|M'\|$ (and therefore $M$ is mass-minimizing) and $\lim |\log \varepsilon|^{-1} F_\varepsilon(u_\varepsilon) = \beta_k \|M\|$. As pointed out in Remark (iii) after Theorem 5.5, the latter condition implies (1.5).

Let $\lambda$ be any limit point (in the sense of measures on $\mathbb{R}^{n+k}$) of the rescaled energy densities $\lambda_\varepsilon := |\log \varepsilon|^{-1} e_\varepsilon(u_\varepsilon)$, extended to 0 outside $\Omega$. By Theorem 5.5(i), for every open set $A \subset \mathbb{R}^{n+k}$ there holds

$$\beta_k |M|(A) \le \liminf_{\varepsilon \to 0} \int_{A \cap \Omega} \lambda_\varepsilon \le \lambda(\overline{A}) \ .$$

Thus $\beta_k |M|(A) \le \lambda(A)$ when $A$ is an open set such that $|\lambda|(\partial A) = 0$, and by approximation also when $A$ is a Borel set. On the other hand, $\|\lambda\| \le \lim \|\lambda_\varepsilon\| = \beta_k \|M\|$, and therefore $\lambda = \beta_k |M|$. In other words, the rescaled energy densities $|\log \varepsilon|^{-1} e_\varepsilon(u_\varepsilon)$ converge to $\beta_k |M|$ in the sense of measures on $\mathbb{R}^{n+k}$. The same holds for $|\log \varepsilon|^{-1} \frac{1}{k} |Du_\varepsilon|^k$ because of (1.5).          $\square$

## 8. Appendix A: Technical Lemmas and Approximation Results

**Proposition 8.1.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Then there exists a bi-Lipschitz homeomorphism $\Psi$ which maps the product $(-1,1) \times \partial\Omega$ onto a neighbourhood $U$ of $\partial\Omega$. Moreover $\Psi$ takes $(-1,0) \times \partial\Omega$ onto $U \cap \Omega$ and $[0,1) \times \partial\Omega$ onto $U \setminus \Omega$, and $\Psi(0,x) = x$ for every $x \in \partial\Omega$.*

*Proof.* For every $x \in \partial\Omega$ and $\varepsilon > 0$, denote by $C(x)$ the set of all vectors $v \in \mathbb{R}^d$ such that $x + \lambda v \in \mathbb{R}^d \setminus \overline{\Omega}$ and $x - \lambda v \in \Omega$ for every $\lambda \in (0,1]$ (thus $v \ne 0$), and by $C_\varepsilon(x)$ the set of all $v$ such that $v + B_\varepsilon \subset C(x)$.

Assume $x \in \partial\Omega$, vectors $v, v' \in \mathbb{R}^d$ and $r > 0$ such that $|v|, |v'| \le r$ and $v, v' \in C(x')$ for every $x' \in (x + B_r) \cap \partial\Omega$. Then every convex combination of $v$ and $v'$ belongs to $C(x)$. Let indeed $T^+$ and $T^-$ be the triangles with vertices $x$, $x + v$, $x + v'$ and $x$, $x - v$ and $x - v'$, respectively. It is easy to check that $T^+$ and $T^-$ intersect $\partial\Omega$ only in the common vertex $x$, and therefore they are contained, except for $x$, in $\mathbb{R}^d \setminus \overline{\Omega}$ and $\Omega$, respectively.

From the previous remark we deduce the following property: given $\varepsilon, r > 0$ and a set $E \subset \mathbb{R}^d$, and denoting by $\text{Conv}(E)$ the convex hull of $E$,

$$E \subset C_\varepsilon(x') \cap \overline{B}_r \ \forall x' \in (x + B_r) \cap \partial\Omega \Rightarrow \text{Conv}(E) \subset C_\varepsilon(x) \ . \qquad (8.1)$$

The fact that $\Omega$ is a bounded Lipschitz domain means that there exist $\varepsilon > 0$, finitely many open sets $U_i$ which cover $\partial\Omega$, and vectors $v_i \in \overline{B}_1$ such that $v_i \in$

$C_\varepsilon(x_i)$ for every $x \in U_i$. We take non-negative smooth functions $\phi_i : \mathbb{R}^d \to [0,1]$ with compact support in $U_i$ such that $\sum \phi_i = 1$ on $\partial\Omega$, and denote by $r$ the minimum of $\mathrm{dist}(\mathrm{supp}(\phi_i), \partial U_i)$ over all $i$. Then the map $v(x) := r \sum \phi_i(x) \, v_i$ is smooth and compactly supported, and (8.1) implies $v(x) \in C_\varepsilon(x)$ for every $x \in \partial\Omega$, and in particular $v(x) \neq 0$. We set

$$\Psi(t,x) := x + t \, v(x) \quad \text{for every } t \in \mathbb{R}, \, x \in \mathbb{R}^d. \tag{8.2}$$

Thus $\Psi$ is smooth, $\Psi(0,x) = x$ for every $x \in \partial\Omega$, and the linear map $D\Psi(0,x)$ takes $(dt, dx) \in \mathbb{R} \times \mathbb{R}^d$ in $v(x) \, dt + dx$. Then the kernel of $D\Psi(0,x)$ is spanned by the vector $(1, -v(x))$.

We claim that there exists $\delta > 0$ such that the restriction of $\Psi$ to the set $(-\delta, \delta) \times \partial\Omega$ is bi-Lipschitz. This would also imply that $\Psi$ is open on this set, and conclude the proof. Were the claim not true, we could find sequences $(t_i, x_i)$ and $(t'_i, x'_i)$ in $\mathbb{R} \times \partial\Omega$, converging to $(0, x)$ and $(0, x')$, respectively, such that $(t_i, x_i) \neq (t'_i, x'_i)$ and

$$\frac{\left| \Psi(t_i, x_i) - \Psi(t'_i, x'_i) \right|}{\left| (t_i - t'_i, x_i - x'_i) \right|} \to 0 \; . \tag{8.3}$$

If $x \neq x'$, (8.3) implies $0 = \Psi(0,x) - \Psi(0,x') = x - x'$, which is a contradiction. If $x = x'$, (8.3) implies that every limit point of every renormalized sequence $\lambda_i(t_i - t'_i, x_i - x'_i)$ is in the kernel of $D\Psi(0,x)$, and can be written as $\lambda(1, -v(x))$ for some $\lambda \in \mathbb{R}$. In particular we can find $\lambda_i$ so that $\lambda_i(x_i - x'_i)$ converge to $v(x)$. On the other hand, $x_i$ and $x'_i$ belong to $\partial\Omega$, and since $v(x_i) \in C_{r\varepsilon}(x_i)$, the angle between $x'_i - x_i$ and $v(x_i)$ is at least $\arcsin(r\varepsilon/|v(x_i)|)$ for $i$ large enough. Since the map $v$ is continuous, the same is true, in the limit, for the angle between $x'_i - x_i$ and $v(x)$, and this contradicts the previous conclusion. $\qquad \square$

**Remark 8.2.** Let $\Psi$ be taken as in Proposition 8.1, and $R$ be the reflection map on $(-1,1) \times \partial\Omega$, namely $R(t,x) := (-t, x)$. Then $\Phi := \Psi \circ R \circ \Psi^{-1}$ is a bi-Lipschitz map of $U$ onto itself which agrees with the identity on $\partial\Omega$, and swaps $U \cap \overline{\Omega}$ with $U \setminus \Omega$, that is, a reflection with respect to $\partial\Omega$.

**Proof of Proposition 2.6.** Let $V$ be a smooth bounded domain such that $U \Subset V \subset \Omega$. By the isoperimetric theorem, for every rectifiable boundary $M$ in $V$ with finite mass there exists an integral current $N$ in $V$ such that $M = \partial N$ and $\|N\|_V \leq C\|M\|_V^{1+1/n}$, where the constant $C$ depends only on $V$ (apply Theorem 4.4.2(2) in [20] with $A := \overline{V}$, $B := \partial V$, and $U, W$ neighbourhoods of $A$ and $B$ that admit a Lipschitz retraction on $A$ and $B$, respectively).

Thus we can find integral currents $N_i$ in $V$ which are uniformly bounded in mass and satisfy $M_i = \partial N_i$, and by the closure theorem for integral currents (cf. [39, Theorem 27.3 and Remark 31.3]), we can extract a subsequence $N_j$ which converges to an integral current $N$ in the flat metric topology of $V$, as defined in

[39, Section 31]. This immediately implies the convergence of the corresponding boundaries $M_j$ to $M := \partial N$ in $\mathbf{F}_U$. $\qquad\square$

**Lemma 8.3.** *Let $S \subset \mathbb{R}^d$ be a bounded set contained in a finite union of Lipschitz surfaces of codimension $h$, and for every $t > 0$, denote by $S_t$ the $t$-neighbourhood of $S$. There exists a finite constant $C$ (depending on $S$) such that $\mathscr{L}^d(S_t) \le Ct^h$ for every $t \ge 0$ and*

$$\int_{S_t} \frac{dx}{[\operatorname{dist}(x,S)]^p} \le \frac{C}{h-p} t^{h-p} \quad \text{for every } p < h \text{ and } t \ge 0,$$

$$\int_{S_t \setminus S_{t'}} \frac{dx}{[\operatorname{dist}(x,S)]^h} \le C \log(t/t') \quad \text{for every } t \ge t' \ge 0.$$

*Proof.* The estimate on $\mathscr{L}^d(S_t) \le Ct^h$ follows from [20, Section 3.2.39]. For every $t > 0$ denote by $1_t$ the characteristic function of $S_t$. Hence

$$\frac{1}{[\operatorname{dist}(x,S)]^p} = t^{-p} + p \int_0^t \frac{1_\tau(x)}{\tau^{p+1}} \, d\tau \quad \text{for every } x \in S_t,$$

and

$$\int_{S_t} \frac{dx}{[\operatorname{dist}(x,S)]^p} = \mathscr{L}^d(S_t) \, t^{-p} + p \int_0^t \frac{\mathscr{L}^d(S_\tau)}{\tau^{p+1}} \, d\tau \le \frac{Ch}{h-p} t^{h-p} \ .$$

A similar computation yields the rest of the statement. $\qquad\square$

**Lemma 8.4.** *Let $X$ be a measurable space endowed with a finite measure $\mu$. Given positive measurable functions $f_0, \ldots, f_m$ on $X$ and $\delta > 0$, there exists $a \in X$ such that*

$$f_0(a) \le (1+\delta) \fint_X f_0 \, d\mu \ ,$$

$$f_i(a) \le (1+\delta) \frac{m}{\delta} \fint_X f_i \, d\mu \quad \text{for } i = 1, \ldots, m.$$

*Proof.* We can assume that $\mu$ is a probability measure. Let $E_i$ be the set of all $a$ where $f_i(a)$ fails to fulfil the required inequality. Then $\mu(E_0) < 1/(1+\delta)$, while $\mu(E_i) < \delta/(m(1+\delta))$ for $i = 1, \ldots, m$. Hence the measure of the set $E_0 \cup \cdots \cup E_m$ is less than 1, which implies that the complement is not empty. $\qquad\square$

**Lemma 8.5.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. Given a 1-Lipschitz function $g$ on $\Omega$, denote by $M$ and $m$ respectively the supremum and the infimum of $g$ on $\Omega$, and by $\Omega_t$ the $t$-sublevel set of $g$. Then, for every sequence of positive Borel functions $f_\varepsilon$ on $\Omega$ there exists $t$ such that $m < t < M$ and*

$$\liminf_{\varepsilon \to 0} \int_{\partial \Omega_t} f_\varepsilon \, d\mathcal{H}^{d-1} \le \frac{1}{M-m} \liminf_{\varepsilon \to 0} \int_\Omega f_\varepsilon \, d\mathcal{L}^d \ .$$

*Proof.* Since $|Dg| \le 1$, the coarea formula for Lipschitz functions yields, for every $\varepsilon > 0$,

$$\int_m^M \left[ \int_{\partial \Omega_t} f_\varepsilon \, d\mathcal{H}^{d-1} \right] dt = \int_\Omega f_\varepsilon \, |Dg| \, d\mathcal{L}^d \le \int_\Omega f_\varepsilon \, d\mathcal{L}^d \ ,$$

and by Fatou's lemma,

$$\int_m^M \left[ \liminf_{\varepsilon \to 0} \int_{\partial \Omega_t} f_\varepsilon \, d\mathcal{H}^{d-1} \right] dt \le \liminf_{\varepsilon \to 0} \int_\Omega f_\varepsilon \, d\mathcal{L}^d \ .$$

To conclude, it suffices to choose $t$ properly. $\qquad\square$

We conclude this appendix with two approximation results for integral currents.

**Proposition 8.6.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \ge 2$, and let $M$ be an integral boundary in $\Omega$ with dimension $h < d$ and finite mass. Then*

(i) *$M$ agrees (in $\Omega$) with the boundary of an integral $(h+1)$-current $N$ in $\mathbb{R}^d$ with compact support such that $|\partial N|(\partial \Omega) = 0$;*

(ii) *$M$ can be approximated in the flat norm $\mathbf{F}_\Omega$ by a sequence of polyhedral boundaries $M_i$ in $\mathbb{R}^d$ so that $\|M_i\|_\Omega \to \|M\|_\Omega$. Moreover we can require that $|M_i|(\partial \Omega) = 0$ and $M_i$ has multiplicity 1 for every $i$.*

*Proof.* Let $M = \partial N$, where $N$ is an integral current $N$ in $\Omega$.

We prove statement (i) first. We extend $N$ to a neighbourhood of $\overline{\Omega}$ using the reflection map $\Phi$ in Remark 8.2. More precisely, we set $\Omega' := \Omega \cup U$, where $U$ is the tubular neighbourhood in Proposition 8.1. Then the boundary of $N$ in $\Omega'$ is given by $M + T$, where $T$ is supported on $\partial \Omega$ ($\|T\|$ is not necessarily finite). If we set $N' := N - \Phi_\sharp N$, then $\partial N' = M - \Phi_\sharp M$ in $\Omega'$.[22] Hence $N'$ is an integral current in $\Omega'$ which agrees with $N$ in $\Omega$, and $|\partial N'|(\partial \Omega) = 0$. Finally, a suitable truncation makes $N'$ an integral current in $\mathbb{R}^d$.

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[22] This can be verified using the identification of $U$ with $(-1, 1) \times \partial \Omega$ provided by Proposition 8.1, and the explicit formula for the reflection map given in Remark 8.2.

We prove now statement (ii). By statement (i), we can assume that $M = \partial N$, where $N$ is an integral current in $\mathbb{R}^d$ with compact support, and $|\partial N|(\partial \Omega) = 0$. By standard approximation results (see, e.g., [20, Corollary 4.2.21]), we can find polyhedral currents $N_i$ which converge to $N$ in the usual flat norm for integral currents and $\|N_i\| + \|\partial N_i\| \to \|N\| + \|\partial N\|$ (where all masses are computed on the entire $\mathbb{R}^d$). It follows immediately that $\mathbf{F}_{\mathbb{R}^d}(\partial N_i - \partial N) \to 0$ and $\|\partial N_i\| \to \|\partial N\|$, and the latter implies $\|\partial N_i\|_U \to \|\partial N\|_U$ for every open set $U$ such that $|\partial N|(\partial U) = 0$, and in particular for $\|\partial N_i\|_\Omega \to \|\partial N\|_\Omega$.

We conclude the proof of statement (ii) with two simple remarks that allow us to choose $N_i$ so that $|\partial N_i|(\partial \Omega) = 0$ and $N_i$ has multiplicity 1. Firstly, if a polyhedral current $N$ does not satisfy $|\partial N|(\partial \Omega) = 0$, yet generic translations do. Indeed, since $\partial \Omega$ has null Lebesgue measure, for every $n$-plane $P$ in $\mathbb{R}^d$, the set of all vectors $v \in \mathbb{R}^d$ such that $\mathscr{H}^n((v + P) \cap \partial \Omega) > 0$ must be Lebesgue negligible by Fubini's theorem, and clearly the same holds if $P$ is a subset of a finite union of $n$-planes.

Secondly, every integral polyhedral boundary $M$ can be approximated in the flat norm by polyhedral boundaries with multiplicity 1. Write $M$ as a finite sum $M = \sum \sigma_i F_i$ where each $F_i$ is an oriented simplex with multiplicity 1 and $\sigma_i \geq 1$ an integer, and replace it by $M' := M + \sum_{ij} \partial G_{ij}$ where $j$ runs from 1 to $\sigma_i$ and each $G_{ij}$ is the cylinder which joins $F_i$ and a translation $F_i + v_{ij}$ (more precisely, the push-forward of the product current $[0,1] \times F_i$ in $\mathbb{R} \times \mathbb{R}^d$ according to the map $(t,x) \mapsto (x + tv_{ij})$). Thus $M'$ is polyhedral and cobordant to $M$, it has multiplicity 1 for almost every choice of the vectors $v_{ij}$, and the distance $\mathbf{F}_{\mathbb{R}^d}(M - M')$ is bounded by $\sum_{ij} \|G_{ij}\| \leq \sum_{ij} |v_{ij}| \cdot \|F_i\|$ and therefore can be taken arbitrarily small. $\qquad\square$

**Proposition 8.7.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Let $M'$ be a rectifiable current supported in $\overline{\Omega}$ with dimension $h < d$ such that $|M'|(\partial \Omega) = 0$, and let $M$ be a rectifiable current supported in $\overline{\Omega}$ of the form $M = M' + \partial N$, with $N$ an integral current supported in $\partial \Omega$.*

*The current $M$ can be approximated in the flat norm $\mathbf{F}_{\mathbb{R}^d}$ by a sequence of currents $M_\varepsilon$ of the form $M_\varepsilon = M' + \partial N_\varepsilon$ where $N_\varepsilon$ is an integral current with compact support in $\Omega$, and $\|M_\varepsilon\| \to \|M\|$. If in addition $M'$ has multiplicity 1 and agrees on every compact subset of $\Omega$ with a polyhedral current, then we can require that $N_\varepsilon$ is polyhedral, and $M_\varepsilon$ has multiplicity 1.*

*Proof.* Take $U$ and $\Psi : (-1,1) \times \partial \Omega \to U$ as in Proposition 8.1. For every $0 < \varepsilon < 1$ we set $U_\varepsilon := \Psi((-\varepsilon, \varepsilon) \times \partial \Omega)$. and take the following retraction of $\Omega \cup U_\varepsilon$ onto $\Omega \setminus U_\varepsilon$:

$$P_\varepsilon(x) := \begin{cases} x & \text{if } x \in \Omega \setminus U_\varepsilon, \\ \Psi(P'_\varepsilon(\Psi^{-1}(x))) & \text{if } x \in U_\varepsilon, \end{cases}$$

where $P'_\varepsilon(t, x') := (-\varepsilon, x')$ for every $(t, x') \in (-\varepsilon, \varepsilon) \times \partial\Omega$. We set

$$N_\varepsilon := P_{\varepsilon\sharp}(N) \quad \text{and} \quad M_\varepsilon := M' + \partial N_\varepsilon \ . \qquad (8.4)$$

Clearly, $\operatorname{supp}(N_\varepsilon) \subset \Omega$. Let us verify that $\mathbf{F}_{\mathbb{R}^d}(M_\varepsilon - M) \to 0$. To this end, we define the following homotopy between $P_\varepsilon$ and the identity map:

$$R_\varepsilon(\lambda, x) := \begin{cases} x & \text{if } \lambda \in [0, \varepsilon] \text{ and } x \in \Omega \setminus U_\varepsilon, \\ \Psi(R'_\varepsilon(\lambda, \Psi^{-1}(x))) & \text{if } \lambda \in [0, \varepsilon] \text{ and } x \in U_\varepsilon, \end{cases}$$

where $R'_\varepsilon(\lambda, t, x') := (-\lambda + (1 - \lambda/\varepsilon)t, x')$ for $(\lambda, t, x') \in [0, \varepsilon] \times (-\varepsilon, \varepsilon) \times \partial\Omega$. Thus $R_\varepsilon(0, x) = x$ and $R_\varepsilon(\varepsilon, x) = P_\varepsilon(x)$. We extend $R_\varepsilon$ to all $\lambda \notin [0, \varepsilon]$ by setting it equal to $x$ for $\lambda < 0$ and to $P_\varepsilon(x)$ for $\lambda > \varepsilon$. Using definition (8.4), and denoting by $I_\varepsilon$ the current associated to the oriented segment $[0, \varepsilon]$, we obtain

$$\begin{aligned} M_\varepsilon - M = P_{\varepsilon\sharp}(\partial N) - \partial N &= R_{\varepsilon\sharp}(\delta_\varepsilon \times \partial N - \delta_0 \times \partial N) \\ &= R_{\varepsilon\sharp}(\partial I_\varepsilon \times \partial N) \\ &= R_{\varepsilon\sharp}(\partial(I_\varepsilon \times \partial N)) = \partial R_{\varepsilon\sharp}(I_\varepsilon \times (M - M')) \ , \end{aligned}$$

and since $\operatorname{Lip}(R_\varepsilon)$ is bounded by a constant $C$ independent of $\varepsilon$,

$$\begin{aligned} \mathbf{F}_{\mathbb{R}^{n+k}}(M_\varepsilon - M) &\leq \|R_{\varepsilon\sharp}(I_\varepsilon \times (M - M'))\| \\ &\leq C^{h+1}\|I_\varepsilon \times (M - M')\| \leq C^{h+1}\varepsilon \|M - M'\| \ . \end{aligned}$$

We prove now that $\|M_\varepsilon\|$ converge to $\|M\|$. Using the identity $M = M' + \partial N$ and definition (8.4), and recalling that $P_\varepsilon$ agrees with the identity on $\Omega \setminus U_\varepsilon$, one easily verify that

$$\begin{aligned} M_\varepsilon &= M' - P_{\varepsilon\sharp}M' + \partial P_{\varepsilon\sharp}(M) \\ &= M' \llcorner U_\varepsilon - P_{\varepsilon\sharp}(M' \llcorner U_\varepsilon) + M \llcorner (\Omega \setminus U_\varepsilon) \\ &\quad + P_{\varepsilon\sharp}(M \llcorner (U_\varepsilon \setminus \partial\Omega)) + P_{\varepsilon\sharp}(M \llcorner \partial\Omega) \ . \end{aligned} \qquad (8.5)$$

The assumption $|M'|(\partial\Omega) = 0$ implies that $\|M' \llcorner U_\varepsilon\| = |M|(U_\varepsilon)$ tends to 0 as $\varepsilon \to 0$, and the same holds for $\|P_{\varepsilon\sharp}(M' \llcorner U_\varepsilon)\|$ because the Lipschitz constants of the maps $P_\varepsilon$ are bounded independently of $\varepsilon$. Similarly, $\|M' \llcorner (U_\varepsilon \setminus \partial\Omega)\| = |M|(U_\varepsilon \setminus \partial\Omega)$ tends to 0, and therefore the same holds for $\|P_{\varepsilon\sharp}(M' \llcorner (U_\varepsilon \setminus \partial\Omega))\|$.

Obviously $\|M \llcorner (\Omega \setminus U_\varepsilon)\| = |M|(\Omega \setminus U_\varepsilon)$ tends to $|M|(\Omega)$. The definition of $P_\varepsilon$ and formula (8.2) yields $P_\varepsilon(x) = x - \varepsilon v(x)$ for every $x \in \partial\Omega$, where $v$ is a smooth vectorfield. Hence the Lipschitz constant of the restriction of $P_\varepsilon$ to $\partial\Omega$ is of order $1 + O(\varepsilon)$ and therefore $\|P_{\varepsilon\sharp}(M \llcorner \partial\Omega)\|$ tends to $|M|(\partial\Omega)$.

Combining these remarks and (8.5) we finally obtain

$$\lim_{\varepsilon \to 0} \|M_\varepsilon\| = \lim_{\varepsilon \to 0} \left[ \|M \llcorner (\Omega \setminus U_\varepsilon)\| + \|P_{\varepsilon\sharp}(M \llcorner \partial\Omega)\| \right] = |M|(\overline{\Omega}) = \|M\| \ .$$

The rest of the statement follows by standard arguments like those used in the proof of Proposition 8.6, and we omit the details. $\qquad \square$

## 9. Appendix B: Maps with Prescribed Jacobians

The main result of this appendix is Theorem 9.6: it is a refinement of Theorem 5.10 of [3] and allows us to construct $S^{k-1}$-valued maps with prescribed Jacobian *and* prescribed boundary values. The proof requires some preliminary lemmas and additional notation.

We adopt the notation introduced in Section 4.1. In particular, given an $h$-dimensional polyhedral current, we denote by the same letter the supporting polyhedral set; given a simplex $F$ in $\mathbb{R}^{n+k}$ and $\delta, \gamma > 0$, the set $U(F, \delta, \gamma)$ is the set of all $x \in \mathbb{R}^{n+k}$ that satisfy (4.1) or equivalently (4.2). Jacobians of maps of class $W^{1,k-1}$ valued in $S^{k-1}$ are intended in the distributional sense (see Remark 2.11).

**9.1. Additional notation.** For every integer $\sigma$ we fix a smooth map $\phi_\sigma : S^{k-1} \to S^{k-1}$ so that $\phi_0$ is constant, $\phi_1$ is the identity, and each $\phi_\sigma$ has degree $\sigma$.

Let $F$ and $U = U(F, \delta, \gamma)$ be taken as above, and let $u : U \to S^{k-1}$ be a map which is continuous on $U \setminus F$. The *degree of the singularity* of $u$ at $F$ is $\sigma := \deg(u, S, S^{k-1})$, where $S$ is any $(k-1)$-dimensional sphere of the form $\{x'\} \times rS^{k-1}$ contained in $U$. Note that $\sigma$ does not depend on the choice of $S$.

Let $M$ and $S$ be finite unions of simplexes in $\mathbb{R}^{n+k}$ with dimension at most $n$ and $n-1$, respectively. We say that a map $u$ on $\overline{\Omega}$ has a *nice singularity at* $M, S$, if $u$ is locally Lipschitz on $\overline{\Omega} \setminus (M \cup S)$ and

$$|Du(x)| = O\left(\frac{1}{\mathrm{dist}(x, M)}\right) + O\left(\frac{1}{(\mathrm{dist}(x, S))^p}\right) \quad \text{for every } p > 1. \qquad (9.1)$$

We say that a map $u$ has a nice singularity at $M$ (resp., at $S$) if it is locally Lipschitz on $\overline{\Omega} \setminus M$ (resp., $\overline{\Omega} \setminus S$) and satisfies (9.1) when the second term (resp., the first term) at the right-hand side is dropped.

**Remark 9.2.** (i) Let $U'$ be the interior of $U = U(F, \delta, \gamma)$. The Jacobian of a map $u \in W^{1,k-1}(U', S^{k-1})$ which is continuous in $U' \setminus F$ is supported on the singularity $F$, and more precisely $\star Ju = \alpha_k \sigma F$, where $\sigma$ is the degree of the singularity—see [3, Section 3.7].[23]

(ii) If $u : \overline{\Omega} \to S^{k-1}$ has a nice singularity at $M, S$, then (9.1) and Lemma 8.3 imply $u \in W^{1,k-1}(\Omega \setminus (M \cup S))$, and therefore $u \in W^{1,k-1}(\Omega)$, because $M \cup S$ has codimension larger than $k-1$.

**Lemma 9.3.** *Let $F$ be an $n$-dimensional simplex in $\mathbb{R}^{n+k}$, take $U = U(F, \delta, \gamma)$ as above, and let $U'$ denote the interior of $U$. Given a map $u : U \to S^{k-1}$ with nice singularity at $F$ of degree $\sigma$, there exists $u' : U \to S^{k-1}$ such that*

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[23] We assume that the orientation of $F$ agrees with that of $\mathbb{R}^n$, and that the orientation of $\mathbb{R}^n \times \mathbb{R}^k$ agrees with that of $\mathbb{R}^{n+k}$.

(i)  $u' = u$  on  $\partial U$;

(ii)  $u'$  has a nice singularity at  $F, \partial F$  of degree  $\sigma$;

(iii)  $u' \in W^{1,k-1}(U', S^{k-1})$  and  $Ju' = Ju$;

(iv)  $u'(x) = \phi_\sigma(x''/|x''|)$  for all  $x \in U(F, \delta/4, \gamma/4)$.

*Proof.* Throughout this proof, the letter  $C$  denotes any positive finite constant, possibly different at each occurrence, that depends only on the choice of  $u, U$, and of the map  $\Psi$  below.

We define  $g$  as in (4.2), and then  $U$  can be written as  $U_1 \cup U_2$, where  $U_1$  is the set of all  $x$  such that  $x' \in F$  and  $g(x')/2 \leq |x''| \leq g(x')$, while  $U_2$  is the set of all  $x$  such that  $x' \in F$  and  $|x''| \leq g(x')/2$  (see Figure 9.1).
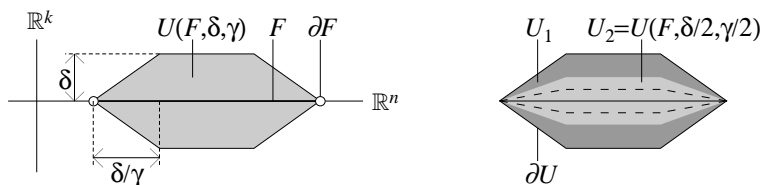


FIGURE 9.1.

Thus  $U_2 = U(F, \delta/2, \gamma/2)$,  $\partial U_1 = \partial U \cup \partial U_2$, and for every  $x \in U_1$

$$|x''| \leq g(x') \leq \gamma \operatorname{dist}(x', \partial F) \leq \gamma \operatorname{dist}(x, \partial F) \leq C|x''| . \qquad (9.2)$$

In other words, the ratio between any two of the quantities  $|x''|$, $\operatorname{dist}(x', \partial F)$, $\operatorname{dist}(x, \partial F)$, and  $g(x')$  is bounded over all  $x \in U_1$.

The proof is divided in two steps. First we define  $u'$  in  $U_1$  so that it agrees with  $u$  on  $\partial U$, and depends only on the variable  $x''/|x''|$  on  $\partial U_2$. Then we extend  $u'$  to the interior of  $U_2$  so that  $u'(x) = \phi_\sigma(x''/|x''|)$  for  $x \in U(F, \delta/4, \gamma/4)$. From the topological point of view, the first step is almost immediate, because  $U_1 \setminus \partial F$  is homeomorphic to the product  $(F \setminus \partial F) \times S^{k-1} \times [1/2, 1]$  and  $F \setminus \partial F$  is contractible; the required estimate on  $|Du'|$, however, is delicate. The second step is taken from Section 5.9 of [3], and will not be explained in full detail.

*Step 1.* Since  $F$  is convex, there exists a Lipschitz retraction  $\Psi : F \times [0,1] \to F$  such that  $\Psi(x', 0) = x'$  and  $\Psi(x', 1) = x'_0$  for every  $x' \in F$, where  $x'_0$  is an arbitrary point in the interior of  $F$; we can also assume that

$$\operatorname{dist}(\Psi(x', t), \partial F) \geq \operatorname{dist}(x', \partial F) + Ct \qquad (9.3)$$

for every  $x' \in F$,  $t \in [0, 1]$  (take for example  $\Psi(x', \tau) := (1 - \tau)x' + \tau x'_0$).

We choose a decreasing function  $\tau : [1/2, 1] \to [0, 1]$  of class  $C^1$  such that  $\tau(1) = 0$  and  $\tau(1/2) = 1$  (to be properly chosen later), and for every  $x \in U_1 \setminus \partial F$  we set

$$\Psi'(x) := \Psi\left(x', \tau\left(\frac{|x''|}{g(x')}\right)\right), \quad u'(x) := u\left(\Psi'(x), \frac{g(\Psi'(x))}{g(x')}x''\right) . \qquad (9.4)$$

One easily checks that $u'$ is locally Lipschitz on $U_1 \setminus \partial F$, it agrees with $u$ on $\partial U$—and therefore statement (i) holds—and for every $x \in \partial U_2$ it can be written as $u'(x) = \phi(x''/|x''|)$, with $\phi : S^{k-1} \to S^{k-1}$ Lipschitz.

Now we estimate $|Du'|$. Taking into account (9.2) and the fact that $\Psi$ and $g$ are Lipschitz, from (9.4) we obtain

$$|D\Psi'(x)| \leq C |D\Psi| \left[ 1 + |\dot\tau| \left( \frac{|Dx''|}{g(x')} + \frac{|Dg(x')| |x''|}{g^2(x')} \right) \right] \leq C \left[ 1 + \frac{|\dot\tau|}{|x''|} \right] , \quad (9.5)$$

where the argument of $\tau$ and $\dot\tau$ is $|x''|/g(x')$, and the argument of $\Psi$ is $(x', \tau)$. From (9.4) and (9.5) we obtain[24]

$$\begin{aligned}
|Du'(x)| &\leq C|Du| \left[ |D\Psi'| + \frac{|D\Psi'| \, |x''| + g(\Psi')}{g(x')} + \frac{g(\Psi') \, |x''|}{g^2(x')} \right] \\
&\leq C \frac{g(x')}{g(\Psi') \, |x''|} \left[ 1 + \frac{|\dot\tau|}{|x''|} + \frac{g(\Psi')}{g(x')} \right] \\
&\leq \frac{C}{|x''|} \left[ 1 + \frac{|\dot\tau|}{g(\Psi')} \right] \\
&\leq \frac{C}{\operatorname{dist}(x, \partial F)} \left[ 1 + \frac{|\dot\tau|}{\operatorname{dist}(x, \partial F) + \tau} \right] .
\end{aligned} \quad (9.6)$$

Let $f : [0, +\infty) \to [0, +\infty)$ be a continuous and strictly increasing function—to be properly chosen below—such that $f(0) = 0$, $f(t)/t$ is decreasing in $(0, t_0]$ for some $t_0 > 0$ and

$$\int_0^{t_0} \frac{dt}{f(t)} < \frac{1}{2} . \quad (9.7)$$

By (9.7) we can find a decreasing function $\tau : [1/2, 1] \to [0, 1]$ of class $C^1$ such that $\tau(1/2) = 1$, $\tau(1) = 0$, and $\dot\tau = -f(\tau)$ in the interval of all points where $\tau \leq t_0$. For every $d > 0$ and every point where $\tau \leq t_0$ there holds

$$\frac{|\dot\tau|}{d + \tau} = \frac{f(\tau)}{d + \tau} \leq \frac{f(d)}{d}$$

(if $d \leq \tau$ use that $f(t)/t$ is decreasing, and if $d \geq \tau$ use that $f(t)$ is increasing). Therefore estimate (9.6) becomes

$$|Du'(x)| \leq C \, \frac{f(\operatorname{dist}(x, \partial F))}{(\operatorname{dist}(x, \partial F))^2} \quad \text{for every } x \in U_1. \quad (9.8)$$

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[24] The argument of $Du$ in the first line is the same as that of $u$ in formula (9.4). The second inequality follows by estimates (9.2), (9.5) and $|Du(x)| \leq C/|x''|$—by assumption, $u$ has a nice singularity at $F$. For the third inequality we used the estimate $g(\Psi') = g(\Psi(x', \tau)) \geq g(x')$, while the last one follows from (9.2) and $g(\Psi') \geq C(\operatorname{dist}(x', \partial F) + \tau)$. Both estimates follow from (9.3).

*Step 2.* To extend $u'$ to $U_2$, we proceed as in Section 5.9 of [3]. Since both $\phi$ and $\phi_\sigma$ have degree $\sigma$, by the Hopf theorem they are homotopic, and we can construct a Lipschitz map $\Phi : S^{k-1} \times [0, 1/2]$ such that $\Phi(\cdot, 1/2) = \phi(\cdot)$ and $\Phi(\cdot, t) = \phi_\sigma(\cdot)$ for $t \leq 1/4$. For every $x$ in the interior of $U_2$ we set

$$u'(x) := \Phi\left(\frac{x''}{|x''|}, \frac{|x''|}{g(x')}\right) . \tag{9.9}$$

It is clear that $u'$ is locally Lipschitz on $U \setminus F$, satisfies statement (iv), and has a singularity of degree $\sigma$ at $F$. A straightforward computation gives $|Du'(x)| = O(1/|x''|)$ for $x \in U_2$. Together with (9.8), this implies

$$|Du'(x)| = O\left(\frac{1}{\text{dist}(x, F)}\right) + O\left(\frac{f(\text{dist}(x, \partial F))}{(\text{dist}(x, \partial F))^2}\right) . \tag{9.10}$$

In particular, taking $f(t) := t(\log t)^2$ (for $t \leq 1/e^2$), we obtain that $u'$ has a nice singularity at $F, \partial F$, and the proof of statement (ii) is complete. In turn, statement (ii) implies that $u'$ belongs to $W^{1,k-1}(U', S^{k-1})$ (cf. Remark 9.2(ii)), and $\star Ju = \star Ju'$ because the singularities of $u$ and $u'$ have both degree $\sigma$ (cf. Remark 9.2(i)). This concludes the proof of statement (iii). $\qquad\square$

**Remark 9.4.** (i) If $u$ has a singularity of degree $\sigma = 0$ at $F$, then definition (9.9) and the fact that $\phi_0$ is a constant map imply that the map $u'$ in Lemma 9.3 is constant in $U_2$. Thus the first term at the right-hand side of estimate (9.10) can be dropped, and $u'$ has a nice singularity at $\partial F$. In other words, Lemma 9.3 shows that $n$-dimensional singularities of degree 0 can be reduced to $(n-1)$-dimensional singularities.

(ii) The function $f(t)$ in the proof of Lemma 9.3 cannot be taken asymptotically equivalent to $t$ because of assumption (9.7), and therefore our construction does not provide a map $u'$ such that $|Du'(x)| = O(1/\text{dist}(x, \partial F))$.

In the following we assume that $\Omega$ is a bounded domain in $\mathbb{R}^{n+k}$ with polyhedral boundary, and $u : \overline{\Omega} \to \mathbb{R}^k$ a piecewise affine map. By Proposition 6.4(ii), we can choose a regular value $a \in B_1^k$ of $u$ such that the map $u_a$ defined in (6.7) belongs to $W^{1,k-1}(\Omega, S^{k-1})$, and $\star Ju_a = \alpha_k M_a(u)$. We also choose a regular value $y \in S^{k-1}$ of $u_a$ such that $M_a(u) = (-1)^{n-1} \partial M_y(u_a)$ in $\Omega$—see Proposition 5.3(iii) and Remark 5.4(i). Note that $M_a(u)$ and $M_y(u_a)$ are polyhedral current of dimension $n$ and $n+1$, supported on the sets $u^{-1}(a)$ and $u^{-1}(L_{a,y})$, where $L_{a,y}$ is the half-line $\{a + ty : t \geq 0\}$.

**Lemma 9.5.** *Let $\Omega$, $u$, $a$, and $y$ be taken as above. Let $\Omega_1$ be an open set relatively compact in $\Omega$ whose boundary is polyhedral and transversal to $M_a(u)$ and $M_y(u_a)$ (in the sense of Note 16) and set $\Omega_2 := \Omega \setminus \overline{\Omega}_1$. Let $N' := M_y(u_a) \llcorner \Omega_2$ be the restriction of the polyhedral current $M_y(u_a)$ to $\Omega_2$, and let $M'$ be*

the boundary of $N'$ in $\Omega$. Given $y' \in S^{k-1}$ with $y' \neq y$, there exists a map $u' : \overline{\Omega}_2 \to S^{k-1}$ such that

(i)  $u' = u_a$ on $\partial\Omega$ and $u' = y'$ on $\partial\Omega_1$;

(ii)  $u'$ has a nice singularity at $M'$;

(iii)  $u' \in W^{1,k-1}(\Omega_2, S^{k-1})$ and $M_y(u') = N'$.

*Proof.* Throughout this proof, the letter $C$ denotes any positive finite constant, possibly different at each occurrence, that depends only on the choice of $\Omega$, $\Omega_1$, $u$, $a$, $y$, and $y'$.

First of all, we choose a map $\Psi : S^{k-1} \times [0, +\infty) \to S^{k-1}$ such that

(i')  $\Psi(x, t) = x$ if $t \geq 1$;

(ii')  $\Psi(x, t) = y'$ if $|x - y| \geq Kt$;

(iii')  $\Psi$ is of class $C^1$ on the complement of $(y, 0)$, and $|D\Psi(x, t)| \leq C/t$.

(iv')  for $t > 0$, $\Psi(x, t) = y$ if and only if $x = y$, and the derivative of $\Psi(\cdot, t)$ at $y$ is orientation preserving.

(A way to construct the map $\Psi$ is the following: represent $S^{k-1}$ as a quotient of the closed unit ball $D$ in $\mathbb{R}^{k-1}$ so that $0$ corresponds to $y$ and $\partial D$ corresponds to $y'$, and take $\Psi(x, t) := x/(t \wedge 1)$ for $|x| \leq t$ and $\Psi(x, t) := x/|x| \simeq y'$ for $|x| \geq t$.)

We set $M'' := M_a(u) \cap \overline{\Omega}_2$, thus $M''$ is the closure of $M' \cap \Omega_2$ (or $\partial N' \cap \Omega_2$). Then, for every $x \in \overline{\Omega}_2$, we set

$$u'(x) := \Psi\big(u_a(x), \sigma(x)\, g_1(x)\big) \quad \text{where} \quad g_1(x) := \frac{\operatorname{dist}(x, M')}{\operatorname{dist}(x, M'') \wedge 1} \ , \qquad (9.11)$$

where $\sigma$ is a positive Lipschitz function on $\Omega_2$ bounded away from $0$ that will be made precise in the following. The verification of the properties of $u'$ is divided in several steps.

*Step 1.* Formula (9.11) and assumption (iii') imply that $u'$ is well-defined and locally Lipschitz in the set of all $x \in \Omega_2$ such that $(u_a(x), \sigma(x)\, g_1(x)) \neq (y, 0)$ and $\operatorname{dist}(x, M'') > 0$, that is, in the complement of the set $M'$.

*Step 2.* We estimate now $|Du'|$. The definition of $g_1$ yields

$$|Dg_1(x)| \leq \frac{1}{\operatorname{dist}(x, M'') \wedge 1} + \frac{\operatorname{dist}(x, M')}{(\operatorname{dist}(x, M'') \wedge 1)^2} \leq \frac{C}{\operatorname{dist}(x, M'')} \ . \qquad (9.12)$$

Since $u$ is piecewise affine, $u_a$ has a nice singularity at $M_a(u)$, that is, $|Du_a(x)| \leq C/\operatorname{dist}(x, M'')$ for every $x \in \Omega_2$. Hence, recalling the estimate on $|D\Psi|$ in assumption (iii') and (9.12) we obtain

$$\begin{aligned} |Du'(x)| &\leq |D\Psi| \cdot \big(|Du_a| + |D\sigma|\, g_1 + \sigma\, |Dg_1|\big) \\ &\leq \frac{C}{\sigma\, g_1} \cdot \frac{C}{\operatorname{dist}(x, M'')} \leq \frac{C}{\operatorname{dist}(x, M')} \ , \end{aligned}$$

Together with Step 1, this proves statement (ii).

*Step 3.* An easy computation shows that $g_1(x) \geq 1 \wedge \text{dist}(\partial\Omega, \partial\Omega_1)$ for every $x \in \partial\Omega$, and therefore assumption (i') and formula (9.11) imply $u'(x) = u_a(x)$ for every $x \in \partial\Omega$ provided that $\sigma \geq 1/(1 \wedge \text{dist}(\partial\Omega, \partial\Omega_1))$ on $\partial\Omega$.

*Step 4.* Using the definitions of $u_a$ and $N'$, and the fact that $u$ is piecewise affine, it is not difficult to check that for every $x \in \overline{\Omega}$ there holds[25]

$$|u_a(x) - y| \geq C \frac{\text{dist}(x, M_y(u_a))}{\text{dist}(x, M_a(u))} \vee C' \,\text{dist}(x, M_y(u_a)) \ .$$

Thus, for every $x \in \Omega_2$,

$$|u_a(x) - y| \geq C \, g_2(x) \ , \quad \text{where} \quad g_2(x) := \frac{\text{dist}(x, N')}{\text{dist}(x, M'') \wedge 1} \ . \tag{9.13}$$

By assumption (ii'), $u'(x) = y'$ provided that $|u_a(x) - y| \geq K \, \sigma(x) \, g_1(x)$. Taking into account (9.13) and the definitions of $g_1$ and $g_2$, we obtain the implication

$$\sigma(x) \, \text{dist}(x, M') \geq C \, \text{dist}(x, N') \ \Rightarrow \ u'(x) = y' \ . \tag{9.14}$$

On the other hand, $N'$ is transversal to $\partial\Omega_1$, and therefore

$$C \, \text{dist}(x, N') \geq \text{dist}(x, N' \cap \partial\Omega_1)$$

for every $x \in \partial\Omega_1$ (cf. Note 16). Since $N' \cap \partial\Omega_1 \subset M'$,

$$C \, \text{dist}(x, N') \geq \text{dist}(x, M') \quad \text{for every } x \in \partial\Omega_1. \tag{9.15}$$

Putting together (9.14) and (9.15) we finally obtain that $u'(x) = y'$ on $\partial\Omega_1$ provided that $\sigma$ is smaller than a given positive constant on $\partial\Omega_1$. Together with Step 3, this proves statement (i).

*Step 5.* $u'$ belongs to $W^{1,k-1}(\Omega, S^{k-1})$ because it has a nice singularity at $M'$ (cf. Remark 9.2(ii)). By assumption (iv'), $u'(x) = y$ if and only if $u_a(x) = y$, and since $\Psi(\cdot, t)$ is orientation-preserving at $y$, we deduce that $M_y(u') = M_y(u_a) \, \llcorner \, \Omega_2 = N'$, and the proof of statement (iii) is concluded. $\qquad\square$

We can now state and prove the main result of this section.

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[25] This estimate holds provided that $M_a(u)$ and $M_y(u_a)$ are not empty. It follows by the fact that the quantities in the left- and in the right-hand side vanish on the same set $M_y(u_a)$, and that $u$ is locally affine and non-degenerate in the directions orthogonal to $M_a(u)$.

**Theorem 9.6.** *Let $\Omega$, $u$, $a$, and $y$ be taken as above, and let $M$ be an $n$-dimensional polyhedral current of the form $M = M_a(u) + \partial N$, where $N$ is a polyhedral current with compact support in $\Omega$. Then there exists a map $\tilde{u} : \overline{\Omega} \to S^{k-1}$, an $(n-1)$-dimensional polyhedral set $S$, and $\delta, \gamma > 0$ such that*

  (i) *$\tilde{u} = u_a$ on $\partial\Omega$;*

  (ii) *$\tilde{u}$ has a nice singularity at $M, S$;*

  (iii) *$\tilde{u} \in W^{1,k-1}(\Omega, S^{k-1})$ and $\star J\tilde{u} = \alpha_k M$;*

  (iv) *for every $n$-dimensional face $F$ of $M$ there holds $u(x) = \phi_\sigma(x''/|x''|)$ on $U(F, \delta, \gamma)$, where $\sigma$ is the multiplicity of $M$ on $F$.*

*Proof.* We choose a point $y' \neq y$ in $S^{k-1}$ and a polyhedral open set $\Omega_1$ such that $\operatorname{supp}(N) \Subset \Omega_1 \Subset \Omega$ and $\partial\Omega_1$ is transversal to $M_a(u)$ and $M_y(u_a)$ (in the sense of Note 16). Then we take $u'$ as in Lemma 9.5.

We denote by $N''$ the restriction of $M_y(u_a) + (-1)^{n-1}N$ to $\Omega_1$, and by $M''$ the union of the $n$-dimensional faces of $N''$. We take $\delta, \gamma > 0$ so that $U(F, \delta, \gamma)$ is contained in the closure of $\Omega_1$ for every $(n+1)$-dimensional face $F$ of $N''$. By Proposition 5.8 in [3] we can find a map $u'' : \mathbb{R}^{n+k} \to S^{k-1}$ such that

  (i') *$u'' = y'$ in $\mathbb{R}^{n+k} \setminus \cup U(F, \delta, \gamma)$, where the union is taken over all $(n+1)$-dimensional faces $F$ of $N''$; in particular $u'' = y'$ in $\mathbb{R}^{n+k} \setminus \Omega_1$;*

  (ii') *$u''$ is locally Lipschitz in $\mathbb{R}^{n+k} \setminus M''$, and $|Du''(x)| = O(1/\operatorname{dist}(x, M''))$;*

  (iii') *$u'' \in W^{1,k-1}_{\mathrm{loc}}(\mathbb{R}^{n+k}, S^{k-1})$ and $\star Ju'' = \alpha_k \partial N''$.*

We set

$$\tilde{u}(x) := \begin{cases} u'(x) & \text{for } x \in \overline{\Omega} \setminus \Omega_1, \\ u''(x) & \text{for } x \in \Omega_1. \end{cases}$$

Taking into account the properties of $u''$ and $u'$ (see Lemma 9.5), and in particular that $u'' = u'$ on $\partial\Omega_1$, one easily verifies that $\tilde{u}$ agrees with $u_a$ on $\partial\Omega$, is locally Lipschitz in the complement of $M' \cup M''$, satisfies

$$|D\tilde{u}(x)| = O\left( \frac{1}{\operatorname{dist}(x, M' \cup M'')} \right) ,$$

and belongs to $W^{1,k-1}(\Omega, S^{k-1})$. Moreover $M_y(\tilde{u}) = M_y(u_a) + (-1)^{n-1}N$ by construction, and then

$$\star J\tilde{u} = (-1)^{n-1}\alpha_k \, \partial M_y(\tilde{u}) = \alpha_k M .$$

Thus $\tilde{u}$ satisfies statements (i) and (iii), and has a nice singularity at $M' \cup M''$.

Now we use Lemma 9.4 to modify $\tilde{u}$ in each $U(F, \delta, \gamma)$ so that it agrees with $\phi_\sigma$ in $U(F, \delta/4, \gamma/4)$, where $\sigma$ is the degree of the singularity of $\tilde{u}$ at $F$. Hence $\tilde{u}$ satisfies statement (iv).

By Remark 9.4(ii) we have that $\tilde{u}$ is no longer singular at $F$ if $\sigma = 0$, and since $\sigma$ agrees with the multiplicity of $M$ at $F$ (Remark 9.2(i)), $\sigma = 0$ if and only if $F$ does not belong to the support of $M$. Hence $\tilde{u}$ has a nice singularity at $M, S$, where $S$ is the union of the $(n-1)$-dimensional faces of $M' \cup M''$.    $\square$

**Remark 9.7.** By statement (i) of Theorem 9.6, the map $\tilde{u}$ satisfies estimate (9.1) for every $p > 1$. It seems possible to choose $\tilde{u}$ so that (9.1) holds with $p = 1$, that is, $|D\tilde{u}(x)| = O(1/\text{dist}(x, M \cup S))$. However, the only proof we could find is completely different from the current one and more complicated. Hence we opted for a slightly weaker statement, which is anyhow more than enough for our purposes.

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