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NON-OCCURRENCE OF GAP FOR ONE-DIMENSIONAL AUTONOMOUS FUNCTIONALS

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ABSTRACT

Let $F(u) = \int_0^1 f(u,u')\,dt$ be a weakly lower semicontinuous autonomous functional defined for all functions $u : [0,1] \to \mathbb{R}^n$ in the Sobolev space $W^{1,p}$. We show that under suitable hypotheses F agrees with the relaxation of the same functional restricted to regular functions, i.e., that for every function u there exist regular functions u_h such that $u_h \to u$ in the $W^{1,p}$ norm and $F(u_h) \to F(u)$.

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1. Introduction

When dealing with minimization problems related to variational integrals, the following problem arises: let T be a topological space of weakly differentiable functions, X a sequentially dense subset of regular functions, and $F : T \to [0,\infty]$ a sequentially lower semicontinuous (integral) functional; then the infima of F on the sets T and X may not agree, i.e.,

$$\inf_{u\in T} F(u) < \inf_{u\in X} F(u) \ . \tag{1.1}$$

When this happen, we say that a *Lavrentiev phenomenon* occurs, after the name of M. Lavrentiev, who gave in 1926 the first example of such behaviour in [L], where T is the space of all absolutely continuous functions u on the interval $[a,b]$ such that $u(a) = \alpha$, $u(b) = \beta$, X is the subset of all functions of class C^1, F is an integral functional of the form $F(u) = \int_a^b f(t,u,u')\,dt$, and (1.1) holds for suitable f, α, β.

On the other hand, the Lavrentiev phenomenon is forestalled if we replace F with the relaxed functional $\overline{F} : T \to [0,\infty]$ given by

$$\overline{F}(u) = \inf\big\{ \liminf_{h\to\infty} F(u_h) : \ (u_h) \subset X, \ u_h \to u \big\} \ . \tag{1.2}$$

It is easy to verify that $\overline{F}$ agrees with F on X and

$$\inf_{u\in T} \overline{F}(u) = \inf_{u\in X} F(u) \ . \tag{1.3}$$

Then it is natural to study when $\overline{F}$ and F agree on the whole T: in general we have that $\overline{F} \geq F$ on T and it may happen that equality does not hold. In this case we say that F has a *gap from X to T*. We remark that when there is no gap, by (1.3) there is no Lavrentiev phenomenon. Moreover, since F is lower semicontinuous, equality $\overline{F}(u) = F(u)$ holds if and only if there exists a sequence $(u_h) \subset X$ such that $F(u_h) \to F(u)$ (and we say that (u_h) *approximates u in energy*) and $u_h \to u$ in T.

Typically, these problems have been studied when T is a subset of a Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$ endowed with the weak topology or the L^p strong topology, X is a subset of regular functions, and F is a variational integral of the form $F(u) = \int_\Omega f(x, u, Du)\, dx$, where Ω is an open subset of $\mathbb{R}^h$, $p \geq 1$, h and n are positive integers, and $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{h\times n} \to [0, \infty]$.

Many results are available in this case, giving examples of gap and Lavrentiev phenomenon, or showing that under suitable assumptions on the integrand f, they never occur (many examples and references may be found in [D], section 3.4.3, [DA] and [BuM], see [BaM] and [BuM] for the case $h = 1$, $n \geq 1$, [S], [Bu], [DA], [CEDA] and [CPSG] for the case $h \geq 1$, $n = 1$, [AM], [BCL] and [GMS] for the case $h, n \geq 1$).

In this paper we consider the case of one-dimensional autonomous functionals, i.e., $h = 1$, $n \geq 1$, $T = W^{1,p}(I, \mathbb{R}^n)$ with $I =]0, 1[$ and

$$F(u) = \int_I f(u, u')\, dt \ .$$

In particular we show that under very mild assumptions on f there is no gap from the set X of all Lipschitz functions on I, to $W^{1,p}$ (endowed with any topology weaker than the norm topology). By a previous remark, this is essentially an approximation result, and indeed we prove that every function $u \in W^{1,p}$ can be approximated in energy by a sequence of Lipschitz functions which converges to u in the $W^{1,p}$ norm (Theorems 2.2 and 2.4, see also Corollaries 2.3, 2.5, and Remarks 2.8, 2.9).

We remark that Theorem 2.2 is weaker than Theorem 2.4, but its proof is very simple and relies essentially on a regularity result for minimizers of one-dimensional integral functionals established by Clarke and Vinter [CV]; on the other hand the proof of Theorem 2.4 is more complicated but gives an explicit construction of the approximating sequence and allows a more general result.

Finally we recall that in the case of autonomous integral functionals of the form

$$F(u) = \int_\Omega f(u, Du)\, dx$$

with $\Omega \subset \mathbb{R}^h$, $h > 1$, and $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, $n > 1$, the gap may occur even for very regular integrands (see [AM]). It remains still open the case $h > 1$, $n = 1$, in the sense that it is not clear whether gap or Lavrentiev phenomenon may occur in this case.

2. Statement and Proof of the Results

In the following, I is the open interval $]0,1[$, and $\overline{I}$ the closed interval $[0,1]$. For every Lebesgue measurable set $B \subset \mathbb{R}$, $|B|$ denotes the Lebesgue measure of B.

Let p be a real number in $[1,\infty[$ and let n be a positive integer; as usual $L^p(I,\mathbb{R}^n)$ is the (Banach) space of all p-summable functions from I into $\mathbb{R}^n$, and $W^{1,p}(I,\mathbb{R}^n)$ is the Sobolev space of all functions from I into $\mathbb{R}^n$ with p-summable distributional derivative, endowed with the norm $\|u\|_{W^{1,p}} = \|u\|_p + \|u'\|_p$ (we write $L^p(I)$ and $W^{1,p}(I)$ when $n = 1$).

For every integer k in $[0,\infty]$, $C^k(\overline{I},\mathbb{R}^n)$ is the space of all functions $u : \overline{I} \to \mathbb{R}^n$ of class C^k on I, such that $D^h u$ admits a continuous extension to $\overline{I}$ for every h with $h \le k$ (as usual, we write $C(\overline{I},\mathbb{R}^n)$ when $k = 0$).

We recall that every Sobolev function $u : I \to \mathbb{R}^n$ agrees a.e. on I with a continuous function on $\overline{I}$: we shall always identify u and this continuous representative.

Definition 2.1. Let $f : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$ be a Borel function, and for every $u \in W^{1,1}(I,\mathbb{R}^n)$, let $F(u)$ be the integral functional

$$F(u) := \int_I f(u,u')\,dt \ . \tag{2.1}$$

We say that a sequence (u_h) approximates u *in energy* when $F(u_h)$ converge to $F(u)$.

In Theorems 2.2 and 2.4 we show that under suitable hypotheses on f, each u may be approximated in energy by a sequence of more regular functions (u_h) which converges to u with respect to some prescribed topology.

Let's start from the first approximation result.

Theorem 2.2. *Assume that f is finite valued, continuous and convex in the second variable (i.e., $f(u,\cdot)$ is a finite convex function on $\mathbb{R}^n$ for every $u \in \mathbb{R}^n$). Then, for every $p \in]1,\infty[$ and every $u \in W^{1,p}(I,\mathbb{R}^n)$, there exists a sequence of functions in $C^1(\overline{I},\mathbb{R}^n)$ which converge to u in the $W^{1,p}$ norm and approximate u in energy.*

Once proved the C^1 approximation in the previous theorem, we can easily obtain a slight improvement in the regularity of approximating functions:

Corollary 2.3. *In Theorem 2.2, the approximating sequence may be taken in $C^\infty(\overline{I},\mathbb{R}^n)$, and not only in C^1.*

Proof of Theorem 2.2.

Let $u_0 \in W^{1,p}$ be fixed. For every ε with $1 \ge \varepsilon > 0$, take $v_\varepsilon \in C^1(\overline{I},\mathbb{R}^n)$ such that

$$\|u_0 - v_\varepsilon\|_{W^{1,p}} \le \varepsilon \ . \tag{2.2}$$

We note that the function $(u,s) \mapsto f(u,s) + |s|^p$ satisfies the hypothesis of Theorem 3.11, and then (cf. Definition 3.6) for every $\varepsilon > 0$ there exists a function $f_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty[$ of class C^∞ and convex in the second variable such that

$$f(v,s) + |s|^p - \varepsilon \le f_\varepsilon(v,s) \le f(v,s) + |s|^p \quad \forall (v,s) \in \mathbb{R}^n \times \mathbb{R}^n . \tag{2.3}$$

Finally, for every $u \in W^{1,p}(I,\mathbb{R}^n)$ we set

$$G_\varepsilon(u) := \int_I \Big[f_\varepsilon(u,u') + \frac{1}{\varepsilon}|u - v_\varepsilon|^p \Big]\, dt\ . \tag{2.4}$$

Each G_ε is a weakly lower semicontinuous and coercive functional on $W^{1,p}(I,\mathbb{R}^n)$ because f is continuous, convex in the second variable and has growth p (cf. (2.3)), and then well-known theorems ensure that there exists a minimum point u_ε. We claim that the functions u_ε belong to C^1, $u_\varepsilon \to u$ in $W^{1,p}$ and $F(u_\varepsilon) \to F(u_0)$ when $\varepsilon \to 0$.

The integrand of G_ε is $g_\varepsilon(t,v,s) = f_\varepsilon(v,s) + \varepsilon^{-1}|v - v_\varepsilon(t)|^p$. Then it is a function of class C^1 on $I \times \mathbb{R}^n \times \mathbb{R}^n$ and

$$\frac{\partial g_\varepsilon}{\partial t}(t,v,s) = \frac{p}{\varepsilon}|v - v_\varepsilon|^{p-1}\big[\mathrm{sgn}(v_\varepsilon(t) - v)\big] \cdot \big[v_\varepsilon'(t)\big]$$

(where $\mathrm{sgn}(x) = x/|x|$ for every $x \ne 0$), and then

$$\int_I \Big| \frac{\partial g_\varepsilon}{\partial t}(t,u_\varepsilon,u_\varepsilon') \Big|\, dt \le \frac{p}{\varepsilon}\|u_\varepsilon - v_\varepsilon\|_\infty^{p-1}\|v_\varepsilon'\|_1 < \infty\ .$$

Hence, by Proposition 3.1 of [CV], u_ε belongs to $C^1(\overline{I},\mathbb{R}^n)$.

Now we want to show that $u_\varepsilon \to u_0$ in $W^{1,p}$ and $F(u_\varepsilon) \to F(u_0)$ as $\varepsilon \to 0$. For every ε, (2.2), (2.3) and (2.4) yield

$$\begin{aligned} G_\varepsilon(u_\varepsilon) \le G_\varepsilon(u_0) &\le \int_I \Big[f(u_0,u_0') + |u_0'|^p + \frac{1}{\varepsilon}|u_0 - v_\varepsilon|^p \Big]\, dt \\ &\le F(u_0) + \|u_0'\|_p^p + \varepsilon^{p-1} \end{aligned} \tag{2.5}$$

and then the values $G_\varepsilon(u_\varepsilon)$ are bounded by the constant $C = F(u_0) + \|u_0'\|_p^p + 1$ for every $\varepsilon \in]0,1]$. Hence, recalling that $|s|^p \le f_\varepsilon(u,s) + \varepsilon$ (cf. (2.3)) and the definition of G_ε,

$$\|u_\varepsilon'\|_p^p + \frac{1}{\varepsilon}\|u_\varepsilon - v_\varepsilon\|_p^p \le G_\varepsilon(u_\varepsilon) + \varepsilon \le C + 1\, .$$

Then $\|u_\varepsilon'\|_p$ is bounded and $\|u_\varepsilon - v_\varepsilon\|_p \to 0$ as $\varepsilon \to 0$, and since $\|v_\varepsilon - u_0\|_p \to 0$, we obtain that u_ε weakly converge to u_0 in $W^{1,p}(I,\mathbb{R}^n)$.

Finally, from (2.3) and (2.5) we obtain

$$F(u_\varepsilon) + \|u'_\varepsilon\|_p^p - \varepsilon \le \int_I f_\varepsilon(u_\varepsilon, u'_\varepsilon)\, dt$$
$$\le G_\varepsilon(u_\varepsilon) \le F(u_0) + \|u'_0\|_p^p + \varepsilon^{p-1}$$

and then

$$\limsup_{\varepsilon\to 0} \left[F(u_\varepsilon) + \|u'_\varepsilon\|_p^p\right] \le F(u_0) + \|u'_0\|_p^p \ .$$

Taking into account that F is weakly lower semicontinuous, this means

$$\lim_{\varepsilon\to 0} F(u_\varepsilon) = F(u_0) \quad \text{and} \quad \lim_{\varepsilon\to 0} \|u'_\varepsilon\|_p = \|u'_0\|_p \ ,$$

and since L^p is an uniformly strictly convex space for $p > 1$, the second equality yields that u_ε converge to u_0 strongly in $W^{1,p}(I, \mathbb{R}^n)$. □

Proof of Corollary 2.3.

Since the integrand f is continuous, F is continuous with respect to the C^1 norm and then every function in $C^1(I, \mathbb{R}^n)$ can be approximated in energy by any sequence of C^∞ functions which converge to it in the C^1 norm. Thereafter Theorem 2.2 and a diagonal argument show that every function u in $W^{1,p}(I, \mathbb{R}^n)$ (with $p > 1$) can be approximated in energy by a sequence of functions in $C^\infty(\overline{I}, \mathbb{R}^n)$ which converge to u in the $W^{1,p}$ norm. □

In Theorem 2.4, we improve the basic approximation result given in Theorem 2.2. In particular we weaken the hypotheses on f, by assuming that it only satisfies condition (B) below. Notice that in this case f is allowed to assume the value $+\infty$ in large zones (see also Remark 2.9) and F may be not lower semicontinuous.

Theorem 2.4. *Assume that the following condition holds:*

(B) *for every $r > 0$ there exists $c > 0$ such that f is bounded on $B_r \times B_c$.*

(where B_ρ denotes the closed ball of $\mathbb{R}^n$ with center 0 and radius ρ). Then, for every $p \in [1, \infty[$ and every $u \in W^{1,p}(I, \mathbb{R}^n)$, there exists a sequence of Lipschitz functions which converge to u in the $W^{1,p}$ norm and approximate u in energy.

As for Theorem 2.2, also in Theorem 2.4 the regularity of the approximating functions can be improved under very mild assumptions.

Corollary 2.5. *In Theorem 2.4, when the integrand f is bounded on bounded sets, the approximating sequence can be taken in $C^1(\overline{I}, \mathbb{R}^n)$ (and not only Lipschitz); and moreover, when f is continuous, it can be taken in $C^\infty(\overline{I}, \mathbb{R}^n)$.*

We recall some elementary facts we shall need in the proof of Theorem 2.4 and Corollary 2.5.

Lemma 2.6. *Let $g : I \to [0, \infty]$ be a (Lebesgue) measurable function and let B_h be a sequence of measurable subsets of I such that $|I \setminus B_h| \to 0$. Then*

$$\int_{B_h} g\,dt \to \int_I g\,dt\ .$$

Proof. Possibly passing to subsequences, we may assume that the characteristic functions of B_h converge to 1 a.e. in I and then Fatou's lemma yields $\liminf \int_{B_h} g\,dt \geq \int_I g\,dt$. Moreover, since g is positive and $B_h \subset I$ for every n, we have that $\int_{B_h} g\,dt \leq \int_I g\,dt$, and this concludes the proof. □

Lemma 2.7. *Let $\phi_h : I \to \mathbb{R}$ be a sequence of Lipschitz functions such that $\phi'_h \geq 1$ a.e. for every h and $\phi_h(t) \to t$ as $h \to \infty$ for every $t \in I$. Then $f(\phi_h) \to f$ in $L^p(I)$ for every $f \in L^p(\mathbb{R})$.*

Proof. Let T_h be the linear operator which maps each function $f : \mathbb{R} \to \mathbb{R}$ into the function $f(\phi_h) : I \to \mathbb{R}$. Then a simple computation shows that T_h maps $L^p(\mathbb{R})$ into $L^p(I)$ and $\|T_h\| \leq 1$. Moreover $T_h f \to f$ in $L^p(I)$ whenever f is a continuous function with compact support (i.e. $f \in C_c(\mathbb{R})$) and since $C_c(\mathbb{R})$ is a dense subspace of $L^p(\mathbb{R})$, the same holds for every $f \in L^p(\mathbb{R})$. □

Proof of Theorem 2.4.

Take $r > \|u\|_\infty$: by hypothesis (B) we may find $c > 0$ and $M < \infty$ such that

$$f(v, s) \leq M \quad \text{whenever } |v| \leq r \text{ and } |s| \leq c. \tag{2.6}$$

Notice that since u belongs to $W^{1,1}$, well-known theorems (see for instance section 3.10 in [Z]) shows that for every positive integer n there exist a Lipschitz function $v_h : \overline{I} \to \mathbb{R}^n$ and an open set A_h such that

$$v_h = u \text{ and } v'_h = u' \text{ in } I \setminus A_h, \tag{2.7a}$$

$$|A_h| \leq 1/h\ . \tag{2.7b}$$

Moreover, it is not difficult to see that the functions v_h can be taken such that

$$v_h \text{ is affine on each connected components of } A_h, \tag{2.7c}$$

$$v_h(0) = u(0). \tag{2.7d}$$

What happens if we use the functions v_h to approximate u? One can easily prove that $v_h \to u$ in $W^{1,p}$, and by (2.7a) it holds

$$F(v_h) = \int_{I \setminus A_h} f(u, u')\,dt + \int_{A_h} f(v_h, v'_h)\,dt\ .$$

The first integral in the right term of this equality converges to $F(u)$ because $|A_h| \to 0$ (cf. Lemma 2.6), but we cannot prove that the second integral converges to 0, because we have no good estimates of it (unless we assume that f has p-growth).

So we have to consider a different approximating sequence which we shall obtain from v_h by modifying the "bad" set A_h. More precisely, our idea is to "fatten" each connected component of A_h so that v_h become an affine function with growth less than c in this component, and then we use inequality (2.6) to obtain suitable estimates.

We write each A_h as the (countable) disjoint union of all connected components: $A_h = \cup_k I_{h,k}$ where $I_{h,k} =]a_{h,k}, a'_{h,k}[$ for every k. Thus we set

$$\begin{aligned} \alpha_{h,k} &:= |I_{h,k}| = a'_{h,k} - a_{h,k} \ , \\ \beta_{h,k} &:= v_h(a'_{h,k}) - v_h(a_{h,k}) \ . \end{aligned} \tag{2.8}$$

By (2.7c) we obtain that for every h, k,

$$v'_h = \frac{\beta_{h,k}}{\alpha_{h,k}} \quad \text{in } I_{h,k} \ , \tag{2.9}$$

and since $a_{h,k}, a'_{h,k}$ belongs to $I \setminus A_h$, (2.7a) yields

$$\beta_{h,k} = u(a'_{h,k}) - u(a_{h,k})$$

and then

$$\sum_k |\beta_{h,k}| \leq \sum_k \int_{I_{h,k}} |u'| \, dt = \int_{A_h} |u'| \, dt < \infty \ . \tag{2.10}$$

Now, taking into account that the measure of A_h converges to 0 as $h \to \infty$ and u' is a summable function, the last term in (2.10) converges to 0 and then

$$\Big[\sum_k |\beta_{h,k}| \Big] \to 0 \quad \text{as } h \to \infty. \tag{2.11}$$

For every h, let $\phi_h \in W^{1,1}(I)$ be such that:

$$\phi_h(0) = 0 \tag{2.12a}$$

$$\phi'_h = \begin{cases} 1 & \text{in } I \setminus A_h, \\ \dfrac{|\beta_{h,k}|}{c\, \alpha_{h,k}} \vee 1 & \text{in } I_{h,k} \text{ for every } k \end{cases} \tag{2.12b}$$

(where c is the constant taken in the beginning of this proof, cf. (2.6)). Taking into account (2.10),

$$\int_I |\phi'_h| \, dt \leq 1 + \frac{1}{c} \sum_k |\beta_{h,k}| < \infty \ .$$

Statement 1: the sequence ϕ_h converges to the identity function $\mathrm{Id}(t) = t$ in $W^{1,1}(I)$.

Since $\phi_h(0) = 0$ for all n, it is enough to prove that ϕ'_h converge to the constant function 1 in the L^1 norm. In fact we have (cf. (2.11))

$$\int_I |\phi'_h - 1|\, dt = \sum_k \Big[\frac{|\beta_{h,k}|}{c\,\alpha_{h,k}} \vee 1 - 1\Big]\, \alpha_{h,k} \le \frac{1}{c} \sum_k |\beta_{h,k}| \to 0$$

Statement 2: the measure of $\phi_h(A_h)$ converges to 0 as $h \to \infty$. We have indeed

$$\begin{aligned} |\phi_h(A_h)| &= \sum_k |\phi_h(I_{h,k})| = \sum_k \Big(\frac{|\beta_{h,k}|}{c} \vee \alpha_{h,k}\Big) \\ &\le \sum_k \alpha_{h,k} + \frac{1}{c} \sum_k |\beta_{h,k}| = |A_h| + \frac{1}{c}\sum_k |\beta_{h,k}| \to 0 \end{aligned}$$

(we used (2.7*b*) and (2.11)).

Formula (2.12*b*) shows that $\phi'_h \ge 1$ a.e., and then each ϕ_h is a strictly increasing function on I and $\phi_h(I) \supset I$. Hence there exists a left inverse $\phi_h^{-1} : I \to I$ which is a 1-Lipschitz function and by (2.12) satisfies

$$\phi_h^{-1}(0) = 0 \tag{2.13a}$$

$$(\phi_h^{-1})' = \begin{cases} 1 & \text{in } I \setminus \phi_h(A_h), \\ \dfrac{c\,\alpha_{h,k}}{|\beta_{h,k}|} \wedge 1 & \text{in } \phi_h(I_{h,k}) \text{ for every } k. \end{cases} \tag{2.13b}$$

Finally we set for every h

$$u_h = v_h(\phi_h^{-1}) \;, \tag{2.14}$$

and we claim that this sequence of functions satisfies our requirements. First of all we note that each u_h is a Lipschitz function, being a superposition of Lipschitz functions, and from (2.7*a*), (2.9), (2.13) and the usual chain-rule we deduce

$$u_h(0) = u(0) \tag{2.15a}$$

$$u'_h = v'_h(\phi_h^{-1})\,(\phi_h^{-1})' = \begin{cases} u'(\phi_h^{-1}) & \text{in } I \setminus \phi_h(A_h), \\ c\dfrac{\beta_{h,k}}{|\beta_{h,k}|} \bigvee \dfrac{\beta_{h,k}}{\alpha_{h,k}} & \text{in } \phi_h(I_{h,k}) \text{ for every } k. \end{cases} \tag{2.15b}$$

Statement 3: the sequence u_h converge to u in $W^{1,p}(I, \mathbb{R}^n)$.

Since $u_h(0) = u(0)$ for all h, it is enough to prove that the convergence of derivatives in the L^p norm. Thus we have

$$\|u'_h - u'\|_p \le$$

$$\le \underbrace{\Big(\int_{I \setminus \phi_h(A_h)} |u'_h - u'|^p dt\Big)^{1/p}}_{P_h^1} + \underbrace{\Big(\int_{I \cap \phi_h(A_h)} |u'_h|^p dt\Big)^{1/p}}_{P_h^2} + \underbrace{\Big(\int_{I \cap \phi_h(A_h)} |u'|^p dt\Big)^{1/p}}_{P_h^3} .$$

We claim that both P_h^1, P_h^2 and P_h^3 converge to 0.

About P_h^1, from (2.15b) we get

$$P_h^1 = \int_{I\setminus\phi_h(A_h)} |u'(\phi_h^{-1}) - u'|^p dt \ ,$$

and if we apply the change of variable $s = \phi_h^{-1}(t)$, taking into account that $(\phi_h^{-1})' = 1$ a.e. in $I \setminus \phi_h(A_h)$ (see (2.13b)), we get

$$P_h^1 = \int_{\phi_h^{-1}(A_h)\setminus A_h} |u' - u'(\phi_h)|^p ds \leq \int_I |u' - u'(\phi_h)|^p ds$$

(we have set $u' = 0$ in $\mathbb{R} \setminus I$) and then $P_h^1 \to 0$ by Lemma 2.7 and Statement 1.

About P_h^2, from (2.15b) we get $|u_h'| \leq c$ a.e. in $I \cap \phi_h(A_h)$ and Statement 2 yields

$$P_h^2 = \int_{I\cap\phi_h(A_h)} |u_h'|^p dt \leq c^p |\phi_h(A_h)| \to 0 \ .$$

Finally, Statement 2 and the fact that $|u'|^p$ is summable yield $P_h^3 \to 0$.

Statement 4: $F(u_h) \to F(u)$. We write

$$F(u) = \underbrace{\int_{I\setminus\phi_h(A_h)} f(u_h, u_h')\, dt}_{P_h^1} + \underbrace{\int_{I\cap\phi_h(A_h)} f(u_h, u_h')\, dt}_{P_h^2} \ .$$

From (2.15b) we get

$$P_h^1 = \int_{I\setminus\phi_h(A_h)} f\big(u(\phi_h^{-1}), u'(\phi_h^{-1})\,(\phi_h^{-1})'\big)\, dt$$

and if we apply the change of variable $s = \phi_h^{-1}(t)$, recalling that $(\phi_h^{-1})' = 1$ a.e. in $I \setminus \phi_h(A_h)$, we get

$$P_h^1 = \int_{\phi_h^{-1}(I)\setminus A_h} f(u, u')\, ds \ . \tag{2.16}$$

Now we recall that $|A_h| \to 0$ and, taking into account that $(\phi_h^{-1})' = 1$ in $I \setminus \phi_h(A_h)$ and Statement 2, we get

$$|\phi_h^{-1}(I)| = \int_I (\phi_h^{-1})' dt \geq |I \setminus \phi_h(A_h)| \to 1 \ .$$

So we have shown that the sets $\phi_h^{-1}(I) \setminus A_h$ converge to I in the sense of Lebesgue measure, and then we may apply Lemma 2.6 to equality (2.16) and we get

$$P_h^1 \to \int_I f(u, u')\, ds = F(u) \ .$$

Now it remains to prove that P_h^2 converges to 0. Since $u_h \to u$ in $W^{1,p}(I,\mathbb{R}^n)$ and we have taken r so that $\|u\|_\infty < r$, $\|u_h\| < r$ for n large enough. Moreover $|u'_h| \le c$ a.e. in $\phi_h(A_h)$ by (2.15b), and taking into account (2.6) and Statement 2, we get

$$P_h^2 \le M\,|\phi_h(A_h)| \to 0$$

and the proof is completed. □

Proof of Corollary 2.5.

We say that a sequence of Lipschitz functions u_h on I τ_∞-converge to u when the functions u_h converge to u uniformly, and the derivatives u'_h are uniformly bounded in L^∞ and converge to u' almost everywhere in I (and then $u_h \to u$ in $W^{1,p}$ for every $p < \infty$).

We claim that when f is bounded on bounded sets, every Lipschitz function $u : I \to \mathbb{R}^n$ may be approximated in energy by a sequence of functions $C^1(\overline{I},\mathbb{R}^n)$ which τ_∞ converge to u. Then Theorem 2.4 and a diagonal argument show that every $u \in W^{1,p}(I,\mathbb{R}^n)$ may be approximated in energy by a sequence of C^1 functions which converge to u in norm.

Notice that since u is Lipschitz, for every positive integer h there exist $u_h \in C^1(\overline{I},\mathbb{R}^n)$ and an open set A_h such that $|A_h| \le 1/h$, $u_h = u$ and $u'_h = u'$ in $I \setminus A_h$, $\|u_h\|_{W^{1,\infty}} \le \|u\|_{W^{1,\infty}}$.

A simple computation shows that the functions u_h τ_∞ converge to u and approximate u in energy.

Furthermore, when f is continuous, the same argument used in the proof of Corollary 2.3 shows that every $u \in W^{1,p}(I,\mathbb{R}^n)$ may be approximated in energy by a sequence in $C^\infty(\overline{I},\mathbb{R}^n)$ which converge to u in norm. □

Finally, we add some remarks on the statements and the proofs of previous theorems.

Remark 2.8. Notice that in Theorems 2.2 and 2.4 (and Corollaries 2.3 and 2.5), we can replace the space $W^{1,p}(I,\mathbb{R}^n)$ with the subspace T of all $u \in W^{1,p}$ such that $u(0) = \alpha$ and $u(1) = \beta$ with $\alpha, \beta \in \mathbb{R}^n$ fixed, and every function $u \in T$ can still be approximated in energy by a sequence of Lipschitz functions in T which converge to u in the $W^{1,p}$ norm.

Remark 2.9. A careful examination of the proof shows that the result of Theorem 2.4 can be slightly improved.

Let f be the integrand of the functional F. Let D be a subset of $\mathbb{R}^n$, $\mathscr{F}$ a family of functions from I to $\mathbb{R}^n$, and k, m positive real numbers: we say that $\mathscr{F}$ (k,m)*-connects* D when all the functions $v \in \mathscr{F}$ are k-Lipschitz, all the functions $t \mapsto f(v(t), v'(t))$ with $v \in \mathscr{F}$ are uniformly integrable on I, and for every $y_1, y_2 \in D$ there exist $v \in \mathscr{F}$ and $x_1, x_2 \in I$ such that $y_1 = v(x_1)$, $y_2 = v(x_2)$, and $|x_1 - x_2| \le m|y_1 - y_2|$. Then we have the following generalization of Theorem 2.4.

Let u be a function which belongs to $W^{1,p}(I,\mathbb{R}^n)$ for some $p \in [1,\infty[$, and assume that there exists a family $\mathscr{F}$ which (k,m)-connects the image of u for some k,m; then u can be approximated in energy by a sequence of Lipschitz functions which converge to u in the $W^{1,p}$ norm.

(We remark that condition (B) in Theorem 2.4 yields that for every $r > 0$, the set B_r of all $y \in \mathbb{R}^n$ such that $|y| \le r$ is $(c, 1/c)$ connected by the family of all affine functions $v : I \to \mathbb{R}^n$ with $|v'| \le c$.)

We may apply this statement when we study the functional $F(u)$ with the constraint $u \in T$, where T is the space of all $u \in W^{1,p}$ such that $u(t) \in M$ for every $t \in [0,1]$, and M is a closed Lipschitz submanifold of $\mathbb{R}^n$: if we assume that f is finite, then every $u \in T$ can be approximated in energy by Lipschitz functions in T which converge to u in the $W^{1,p}$ norm.

The proof can be achieved by applying the previous statement to the auxiliary functional

$$G(u) = F(u) + \int_I g(u)\, dt$$

where g satisfies $g(y) = 0$ when $y \in M$, $g(y) = +\infty$ otherwise.

3. Appendix: Approximation of Convex Functions

Let f be a convex function on some convex open set; thus the following problem may arise: can f be uniformly approximated by smooth convex functions? In this section we prove that this is true (Theorem 3.5), and then we examine what happens when we try to approximate a continuous function of two (vector) variables which is convex in the second variable only (Theorem 3.11).

In the following, n, k are positive integers, Ω is a convex open subset of $\mathbb{R}^n$, and A an open subset of $\mathbb{R}^k$. By *convex function* we mean a convex function which takes values in $]-\infty,\infty]$, and by *finite convex function* a convex function which takes real values only. By *smooth function* we mean a function of class C^∞.

When f_1, f_2 are real valued functions, $f_1 \vee f_2$ is the supremum function, i.e., $[f_1 \vee f_2](x) := \sup\{f_1(x), f_2(x)\}$ for all x. In general, when F is a family of real functions, we define $[\vee F](x) := \sup\{f(x) : f \in F\}$ for all x.

Definition 3.1. We denote by $\mathscr{C}(\Omega)$ the class of all finite convex functions on Ω, by $\mathscr{C}_s(\Omega)$ the subclass of all smooth functions, and by $\mathscr{F}(\Omega)$ the class of all $f : \Omega \to \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $g \in \mathscr{C}_s(\Omega)$ which satisfies

$$f - \varepsilon \le g \le f\ . \tag{3.1}$$

We claim that $\mathscr{F}(\Omega) = \mathscr{C}(\Omega)$ (Theorem 3.5). The proof is divided in several steps.

Proposition 3.2. $\mathscr{C}_s(\Omega) \subset \mathscr{F}(\Omega) \subset \mathscr{C}(\Omega)$.

Proof. The inclusion $\mathscr{C}_s(\Omega) \subset \mathscr{F}(\Omega)$ is trivial, and by Definition 3.1, every function f in $\mathscr{F}(\Omega)$ agrees with the supremum of all $g \in \mathscr{C}_s(\Omega)$ such that $g \le f$. Hence f is convex. □

Lemma 3.3. *Let $g_1, g_2 \in \mathscr{C}_s(\Omega)$ be given. Then for every $\varepsilon > 0$ there exists $g \in \mathscr{C}_s(\Omega)$ such that*

$$g_1 \vee g_2 - \varepsilon \le g \le g_1 \vee g_2 \ . \tag{3.2}$$

Moreover g may be taken so that

$$g(x) = g_1(x) \vee g_2(x) - \varepsilon \quad \text{when} \quad |g_1(x) - g_2(x)| \ge \varepsilon \ . \tag{3.3}$$

Proof. Take a smooth convex function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(t) = |t| - \varepsilon \quad \text{when} \quad |t| \ge \varepsilon \ . \tag{3.4}$$

The convexity assumption yields $|t| - \varepsilon \le \phi(t) \le 0$ for every $t \in [-\varepsilon, \varepsilon]$ and then, for all $t \in \mathbb{R}$,

$$|t| - \varepsilon \le \phi(t) \le |t| \ . \tag{3.5}$$

Moreover

$$-1 \le \phi'(t) \le 1 \tag{3.6}$$

because ϕ' is nondecreasing. Now we set

$$g := \frac{g_1 + g_2 + \phi(g_1 - g_2)}{2} \ .$$

Since g_1, g_2, ϕ are smooth, g is smooth too, and

$$\begin{aligned} 2\, D^2 g = [1 + \phi'(g_1 - g_2)]\, D^2 g_1 + [1 - \phi'(g_1 - g_2)]\, D^2 g_2 + \\ + [\phi''(g_1 - g_2)]\, D(g_1 - g_2) \otimes D(g_1 - g_2) \ . \end{aligned} \tag{3.7}$$

Now, for every $x \in \Omega$, $D^2 g_1(x)$ and $D^2 g_2(x)$ are positively semidefinite matrices because g_1 and g_2 are convex functions, and $D(g_1 - g_2) \otimes D(g_1 - g_2)$ is positively semidefinite because this is true for all matrices of the form $v \otimes v$ with $v \in \mathbb{R}^n$. Moreover $[1 + \phi']$ and $[1 - \phi']$ are non-negative by (3.6) and $[\phi'']$ is non-negative because ϕ is convex. Hence (3.7) shows that $D^2 g(x)$ is always a positively semidefinite matrix , and g is convex.

Finally, taking into account that $g_1 \vee g_2 = \frac{1}{2}\big(g_1 + g_2 + |g_1 - g_2|\big)$, (3.5) and (3.4) yield (3.2) and (3.3) respectively. □

Proposition 3.4. *Let F be a subset of $\mathscr{F}(\Omega)$ such that $\vee F < \infty$ everywhere. Then $\vee F$ belongs to $\mathscr{F}(\Omega)$.*

Proof. The proof is divided in three steps.

Step 1.

When F consist of two elements f_1, f_2 only, $f_1 \vee f_2$ is a finite convex function and this statement is an immediate consequence of Lemma 3.3. The same conclusion holds in general when F is finite.

Step 2.

Assume that F is countable, i.e., $F = \{f_h\}_{h\geq 0}$, and set $f := \vee_h f_h$; then f is a finite convex function by hypothesis. Let $\varepsilon > 0$ be fixed: we claim that there exists $g \in \mathscr{C}_s(\Omega)$ such that (3.1) holds.

Let $\Omega_0 \subset \Omega_1 \subset \Omega_2 \dots$ be a sequence of open sets relatively compact in Ω which cover Ω.

By Step 1 we may assume that $f_0 \leq f_1 \leq f_2 \leq \dots$ and hence, since each Ω_k is relatively compact in Ω, for every integer $k \geq 0$ there exists h_k such that

$$f - \frac{\varepsilon}{2^{k+1}} \leq f_{h_k} \quad \text{in } \Omega_k.$$

Since f_{h_k} belongs to $\mathscr{F}(\Omega)$, by Definition 3.1 there exists $g_k \in \mathscr{C}_s(\Omega)$ such that

$$f_{h_k} - \frac{\varepsilon}{2^{k+1}} \leq g_k \leq f_{h_k} \quad \text{in } \Omega$$

and then, taking into account that $f_{h_k} \leq f$,

$$f - \frac{\varepsilon}{2^k} \leq g_k \text{ in } \Omega_k \text{ and } g_k \leq f \text{ in } \Omega. \tag{3.8}$$

Now we define by induction on k a sequence $\{g'_k\}_{k\geq 0} \subset \mathscr{C}_s$ such that

(i) $f - \Big(3 - \dfrac{1}{2^{k-1}}\Big)\varepsilon \leq g'_k$ in Ω_k and $g'_k \leq f$ in Ω,

(ii) $g'_k = g'_{k-1} - \frac{\varepsilon}{2^{k+1}}$ in Ω_{k-1} (for $k > 1$).

We set $g'_1 = g_1$. If g'_k satisfying (i) and (ii) is given, then we apply Lemma 3.3, with $\varepsilon/2^{k+2}$ instead of ε, to the functions

$$h_1 := g'_k \quad \text{and} \quad h_2 := g_{k+1} - \Big(3 - \frac{1}{2^{k-1}} + \frac{1}{2^{k+2}}\Big)\varepsilon\ ,$$

and we find $g'_{k+1} \in \mathscr{C}_s$ satisfying (3.2) and (3.3). We claim that g'_{k+1} satisfies (i) and (ii).

We have indeed $g'_{k+1} \leq h_1 \vee h_2$ (cf. (3.2)) and $h_1, h_2 \leq f$ (cf. (i) and (3.8)), and then $g'_{k+1} \leq f$ in Ω. Moreover $g'_{k+1} \geq h_2 - \varepsilon/2^{k+2}$ in Ω (cf. (3.2)) and $h_2 = g_{k+1} - \big(3 - \frac{1}{2^{k-1}} + \frac{1}{2^{k+2}}\big)\varepsilon$. Hence, taking into account (3.8) we obtain $g'_{k+1} \geq f - \big(3 - \frac{1}{2^k}\big)\varepsilon$ in Ω_{k+1} and then (i) holds for g'_{k+1}.

Taking into account (ii) and (3.8), for all $x \in \Omega_k$

$$
\begin{aligned}
h_1 = g'_k &\geq f - \Big(3 - \frac{1}{2^{k-1}}\Big)\varepsilon \\
&\geq g_{k+1} - \Big(3 - \frac{1}{2^{k-1}} + \frac{1}{2^{k+1}}\Big)\varepsilon + \frac{\varepsilon}{2^{k+1}} = h_2 + \frac{\varepsilon}{2^{k+1}}
\end{aligned}
$$

and since $g'_{k+1}(x) = h_1(x) \vee h_2(x) - \varepsilon/2^{k+2}$ whenever $|h_1(x) - h_2(x)| \geq \varepsilon/2^{k+2}$ (cf. (3.3)), we have that

$$g'_{k+1} = h_1 - \frac{\varepsilon}{2^{k+2}} = g'_k - \frac{\varepsilon}{2^{k+2}} \quad \text{in } \Omega_k$$

and (ii) holds for g'_{k+1} too.

Finally we note that $\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \ldots$ and (ii) yield that when h goes to ∞, the sequence g'_h converge in each Ω_k to $g'_k + \varepsilon/2^{k+1}$. Hence, taking into account that the sets Ω_k cover Ω, the sequence g'_h converge in every point to a convex function g which agrees with $g'_k + \varepsilon/2^{k+1}$ in each Ω_k, and then it is smooth, and (i) yields

$$f - 3\varepsilon \leq g \leq f \quad \text{in } \Omega.$$

Step 3.

Assume that F is arbitrarily taken, and let f be the supremum of all functions in F. By hypothesis f is a finite convex function, and a simple topological argument shows that f is the supremum of some countable subfamily of F. Hence f belongs to $\mathscr{F}(\Omega)$ by Step 2. □

Theorem 3.5. $\mathscr{F}(\Omega) = \mathscr{C}(\Omega)$.

Proof. By Proposition 3.2, it is enough to prove that every finite convex function f belongs to $\mathscr{F}(\Omega)$. Thus we note that f is the supremum of all functions $g \in \mathscr{C}_s(\Omega)$ which satisfy $g \leq f$ because $\mathscr{C}_s(\Omega)$ contains all affine functions, and then it is enough to apply Proposition 3.4. □

Now, let A be an open subset of $\mathbb{R}^k$ and let $f : A \times \Omega \to \mathbb{R}$ be a continuous function which is convex in the second variable (i.e., $f(t, \cdot)$ is a convex function on Ω for every $t \in A$). Then f can be uniformly approximated by smooth functions which are convex in the second variable if some growth hypotheses are fulfilled (see Proposition 3.7 and Theorem 3.11). As for the previous case, we begin with some definitions.

Definition 3.6. We denote by $\mathscr{C}(A, \Omega)$ the class of all continuous functions $f : A \times \Omega \to \mathbb{R}$ which are convex in the second variable, by $\mathscr{C}_s(A, \Omega)$ the subclass of all smooth functions, and by $\mathscr{F}(A,\Omega)$ the class of all continuous functions $f : A\times\Omega \to \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $g \in \mathscr{C}_s(A,\Omega)$ which satisfies

$$f - \varepsilon \le g \le f\ . \tag{3.9}$$

Now, Propositions 3.2 and 3.4 may be generalized as follows.

Proposition 3.7. *The following statements hold:*

(i) $\mathscr{C}_s(A,\Omega) \subset \mathscr{F}(A,\Omega) \subset \mathscr{C}(A,\Omega)$,

(ii) for every $F \subset \mathscr{F}(A,\Omega)$ *such that* $\vee F$ *is a finite continuous function on* $A\times\Omega$, $\vee F$ *belongs to* $\mathscr{F}(A,\Omega)$,

(iii) a function f *in* $\mathscr{C}(A,\Omega)$ *belongs to* $\mathscr{F}(A,\Omega)$ *if and only if*

$$f = \bigvee \left\{ g \in \mathscr{C}_s(A,\Omega) :\ g \le f \right\}\ . \tag{3.10}$$

Proof. The proofs of statements (i) and (ii) are the same of Propositions 3.2 and 3.4 respectively. Statement (iii) follows immediately from (ii). □

Remark 3.8. What is not trivial in this case is to prove equality (3.10). And indeed it does not hold for all functions in $\mathscr{C}(A,\Omega)$. In particular, when $\Omega = \mathbb{R}$ and $f(t,x) = v(t)\,x$ with $v : A \to \mathbb{R}$ a non-smooth continuous function, then f belongs to $\mathscr{C}(A,\mathbb{R})$, and for every function $g \in \mathscr{C}(A,\mathbb{R})$ such that $|f-g|$ is bounded, we have $g(t,x) = g(t,0)+v(t)\,x$, and then g is not smooth. Hence f cannot be uniformly approximated by smooth functions in $\mathscr{C}(A,\mathbb{R})$, and therefore (3.10) cannot hold.

However we have the following result.

Lemma 3.9. *Let* $f : A\times\Omega \to [0,\infty]$ *be a lower semicontinuous function which is convex in the second variable and satisfies*

$$f(t,x) \ge |x|^p \quad \text{for all } (t,x) \in A\times\Omega \tag{3.11}$$

for some $p > 1$*. Then* (3.10) *holds.*

Proof. Let $f' : A\times\mathbb{R}^n \to]-\infty,\infty]$ be the maximal lower semicontinuous function which is convex in the second variable and satisfies $f' \le f$ in $A\times\Omega$. Then f' satisfies (3.11) in $A\times\mathbb{R}^n$ and $f' = f$ in $A\times\Omega$. Thereafter we may assume $\Omega = \mathbb{R}^n$.

Of course it is enough to show that for any $(t_0,x_0) \in A\times\mathbb{R}^n$ and any $a < f(t_0,x_0)$ there exists $g \in \mathscr{C}_s(A,\mathbb{R}^n)$ such that $a \le g(t_0,x_0)$ and $g \le f$ everywhere. With no loss in generality we may assume that $x_0 = 0$. Then we set $b := f(t_0,x_0)$.

Since f is convex in the second variable, there exists $v \in \mathbb{R}^n$ such that

$$f(t_0,x) \ge b + v\cdot x \qquad \text{for all } x \in \mathbb{R}^n. \tag{3.12}$$

Since $p > 1$, there exists a compact set $K \subset \mathbb{R}^n$ such that

$$|x|^p \ge b + v\cdot x \quad \text{for all } x \in \mathbb{R}^n \setminus K. \tag{3.13}$$

Taking into account (3.12) and the fact that K is compact and $b > a$, we may find an open neighborhood I of t_0 such that

$$f(t,x) \geq a + v \cdot x \quad \text{for all } t \in I,\ x \in K. \tag{3.14}$$

Now, let $\phi : A \to [0,1]$ be a smooth function such that $\phi = 0$ out of I and $\phi(t_0) = 1$. Thus we set

$$g(t,x) := \phi(t)(a + v \cdot x) \quad \text{for all } t, x.$$

The function g is smooth, convex in the second variable and $g(t_0, 0) = a < f(t_0, 0)$, and moreover $0 \vee (a + v \cdot x) \geq g(t,x)$ everywhere. Then $f \geq g$ in $I \times K$ by (3.14), $f \geq g$ in $A \times (\mathbb{R}^n \setminus K)$ by (3.11) and (3.13), and $f \geq g$ in $(A \setminus I) \times \mathbb{R}^n$ because f is positive and $g = 0$ in $(A \setminus I) \times \mathbb{R}^n$. Thus we have proved that $f \geq g$ everywhere. □

Remark 3.10. Lemma 3.9 may be easily generalized to all lower semicontinuous functions $f : A \times \Omega \to]-\infty, \infty]$ which are convex in the second variable and satisfy

$$f(t,x) \geq -a(t) + b(t)\,\phi(x) \quad \text{for all } (t,x) \in A \times \Omega \tag{3.15}$$

for some superlinear convex function ϕ and continuous positive functions a, b on A.

We remark that when Ω is bounded, and $f : A \times \Omega \to]-\infty, \infty]$ is lower semicontinuous and convex in the second variable, $f(t, \cdot)$ is bounded from below for any $t \in A$. Then it may be proved that (3.15) holds with $\phi(x) = |x|^2$, $b = 1$ and a suitably chosen a, and hence also (3.10) holds.

Theorem 3.11. *Let f be a function in $\mathscr{C}(A,\Omega)$ such that* (3.11) *holds for some $p > 1$ (more in general, such that* (3.15) *holds for some superlinear convex function ϕ and some continuous positive functions a, b on A). Then f belongs to $\mathscr{F}(A,\Omega)$ and in particular this holds for every $f \in \mathscr{C}(A,\Omega)$ when Ω is bounded.*

Proof. Apply Proposition 3.7 (iii) and Lemma 3.9 (and Remark 3.10). □

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