GMT 19/20 Lecture 23 29/5/20

F.-F. closure/compactness theorem

Let T_n be integral currents (in $\mathbb{R}^d$, Ω, ...) s.t.

$$\mathbb{M}(T_n);\ \mathbb{M}(\partial T_n) \leq C < +\infty,$$

Then T_n converge (up to subseq.) to T integral

[proof is hard]

Boundary Rectifiability Theorem

Let T be rectifiable current with integral multiplicity (in $\mathbb{R}^d$, Ω, ...) s.t. $\mathbb{M}(\partial T) < +\infty$.

Then ∂T is rectif. + integral multiplicity ($\Longrightarrow$ T is integral).

[proof is hard]

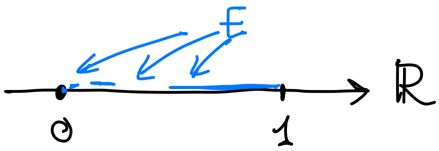
Rem

- There are T rectif. + integral multiplicity with $\mathbb{M}(\partial T) = +\infty$.

standard orient. of $\mathbb{R}$, $e = 1$

In $\mathbb{R}$, let $T = [E, e, 1]$ where

$$E = \bigcup_{n=0}^{\infty} \left[\frac{1}{2\cdot 4^n}, \frac{1}{4^n}\right]$$



(prove that $\mathbb{M}(\partial T) = +\infty$, that is,

$$\sup_{\phi \in C_c^\infty} \langle \partial T, \phi \rangle = +\infty \quad)$$

- F.-F. Th. can be restated as follows:

  Let T_n be rectifiable k-currents with integral multiplicity s.t.

  $$\mathbb{M}(T_n), \ \mathbb{M}(\partial T_n) \leqslant C < +\infty .$$

  Then, up to subseq., $T_n \longrightarrow T$ rectif. + integral multiplicity.

- F.F. Th. can be ~~improved~~ as follows (Ambrosio-Kirchheim, Jerrard):

  Let T_n be rectifiable k-currents with $|m_n| \geqslant \delta > 0$ s.t.

  multipl. of T_n

  $$\mathbb{M}(T_n), \ \mathbb{M}(\partial T_n) \leqslant C < +\infty .$$

  Then, up to subseq., $T_n \longrightarrow T$ rectif. + multiplicity m s.t. $|m| \geqslant \delta$.

Approximation of currents

(two reasons for having good approx. results!)

<u>Def.</u> A k-polyhedral current (or chain) in $\mathbb{R}^d$ is a current of the form

$$T := \sum_i [S_i, \tau_i, m_i]$$

finite sum!!

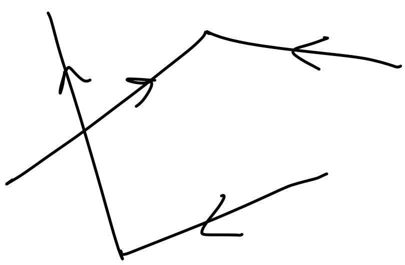
where: S_i is a k-dim. simplex (convex envelope of $k+1$ affinely indep. points in $\mathbb{R}^d$)

τ_i is a constant orientation of S_i

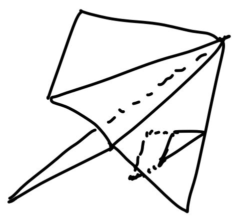
m_i is a constant multiplicity in $\mathbb{R}$ or $\mathbb{Z}$

<u>Ex.</u>

$k=1$



$k=2$



d-dim. manifold $\subset \mathbb{R}^d$

(Slightly diff. def. if $\mathbb{R}^d$ is replaced by Ω or ...)

Theorem (approx. by polyhedral chains)

(i) If T is an integral current in $\mathbb{R}^d$ (....)
then $\exists$ T_n integral polyhedral chains
s.t.

$$T_n \longrightarrow T \quad \& \quad \mathbb{M}(T_n) \longrightarrow \mathbb{M}(T)$$

$$\partial T_n \longrightarrow \partial T \quad \& \quad \mathbb{M}(\partial T_n) \longrightarrow \mathbb{M}(\partial T)$$

(ii) If T is normal the same holds
with T_n real polyhedral chains.

Rems

- For statement (ii) it is essential that $\mathbb{M}$ is defined using mass and comass norm for k-vectors and k-covectors. (and not the Euclidean norms)

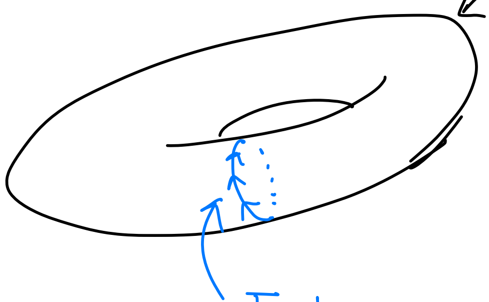
  Think about this case: $T = \tau . \mu$
  where $\mu = \mathcal{L}^d \llcorner Q$, $Q = [0,1]^d$,
  τ is a constant non-simple k-vector.
  ($2 \leq k \leq d-2$)
  To construct T_n (by hands) you will

need to write $\tau = \sum \lambda_i \tau_i$

with τ_i <u>simple</u> and $|\tau| \cong \sum \lambda_i |\tau_i|$

- You can improve the approx. result in many ways. E.g., if $\partial T = 0$ then you can require that $\partial T_n = 0$

  And also that T_n is COBORDANT to T, that is $T_n - T$ is a boundary ($T_n - T = \partial U$)



- Approximation with regular surfaces is delicate.

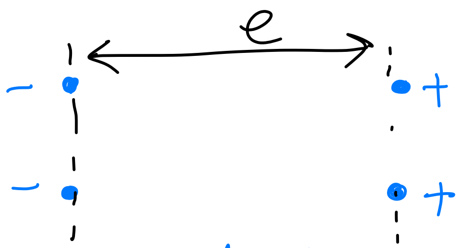
<u>Hint</u> There exist manifold M and polyh. k-chain T in M which is not

cobordant to any k-dim. surface in M.

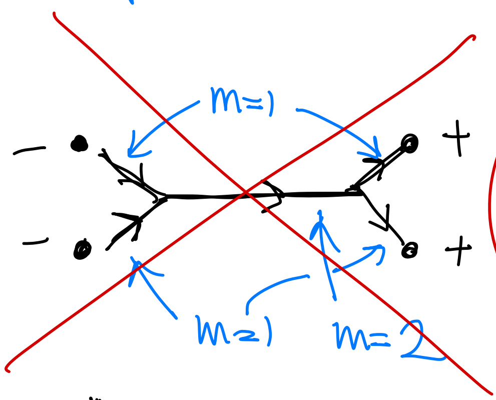
- In all this theory, mass generalizes the volume "with multiplicity" of polyhedral chains, which is not the volume.

This has some consequences

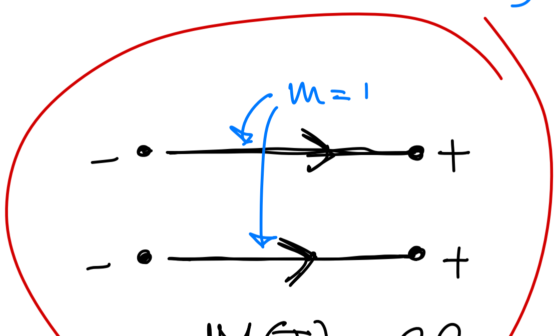
<u>Ex 1</u> $\quad \partial T_0 :=$ sum of four Diracs



Sol. of Plateau Pb. with boundary ∂T_0 is



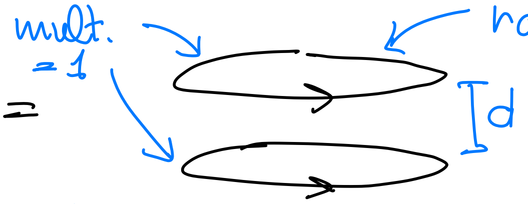
$\mathbb{M}(T) > 2e$



$\mathbb{M}(T) = 2e$

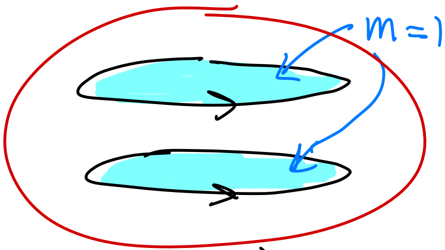
Ex 2

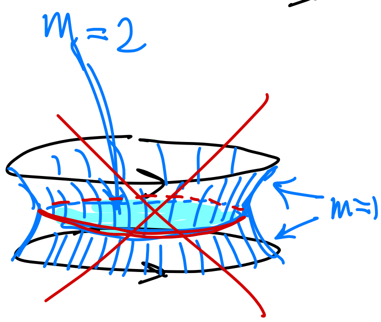
$\partial T_0 =$

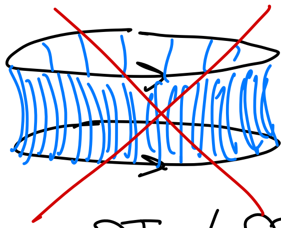


in $\mathbb{R}^3$

sol. of Pl. Pr. $\qquad d \ll r$



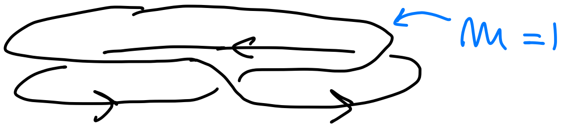


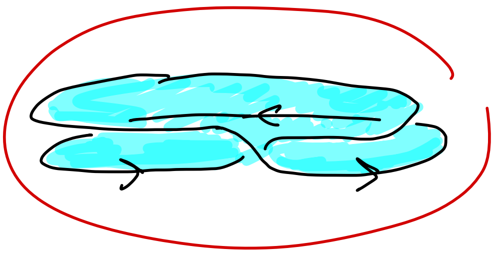


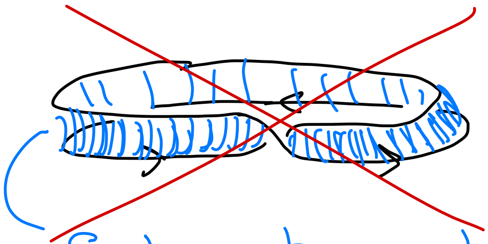
$\mathbb{M}(T) = 2\pi r^2 \qquad \mathbb{M}(T) > 2\pi r^2 \qquad \partial T \neq \partial T_0$

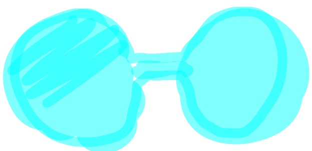
Ex 3

∂T_0









S is not orientable

— there is no current supp. by S with boundary ∂T_0

Next lectures

with some details

- Constancy lemma
- elementary operations on currents
  (product, push-forward, homothopy formula)
- flat distance (useful tool)
- Polyhedral deformation theorem
  (basic tool for approximation results)

just sketched

- Slicing, characterization of rectifiability by slicing, proof of closure th. and bdry rectif. th.

Constancy lemma and related results

Const. Lemma (basic version)

Let T be a d-dimensional current in $\mathbb{R}^d$ and $\partial T = 0$. Then $T = [\mathbb{R}^d, \underset{\substack{\uparrow \\ e_1 \wedge e_2 \wedge \dots \wedge e_d \\ \text{standard orient. of } \mathbb{R}^d}}{e}, m]$ ← constant

Proof

I associate to T a distribution Λ on $\mathbb{R}^d$:

$$\langle \Lambda, \underset{\substack{\cap \\ \mathcal{C}_c^\infty(\mathbb{R}^d)}}{\varphi} \rangle := \langle T ; \varphi . \underbrace{dx_1 \wedge \dots \wedge dx_d}_{dx} \rangle$$

Then

distributional deriv.

$$\partial T = 0 \underset{(\Longleftarrow)}{\Longrightarrow} \underset{\substack{\shortparallel \\ \left(\frac{\partial \Lambda}{\partial x_1}, \dots, \frac{\partial \Lambda}{\partial x_d}\right)}}{D\Lambda} = 0$$

Take indeed $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $i = 1, \dots, d$, and let

$$\omega := \phi \underset{\substack{\shortparallel \\ \bigwedge_{j \neq i} dx_j}}{\widehat{dx_i}} \in \mathcal{D}^{d-1}(\mathbb{R}^d)$$

Then

$$0 \overset{\partial T = 0}{=} \langle \partial T, \omega \rangle = \langle T, d\omega \rangle$$

$$d\omega = d\phi \wedge \widehat{dx_i} = \sum_j \frac{\partial \phi}{\partial x_j} . dx_j \wedge \widehat{dx_i} = \frac{\partial \phi}{\partial x_i} (-1)^{i-1} \underset{\substack{dx_1 \wedge \dots \wedge dx_d \\ \shortparallel}}{dx}$$

$$= \left\langle T, \frac{\partial \phi}{\partial x_i} (-1)^{i-1} dx \right\rangle$$

$$= \left\langle \Lambda, (-1)^{i-1} \frac{\partial \phi}{\partial x_i} \right\rangle$$

$$= \left\langle \frac{\partial \Lambda}{\partial x_i}, (-1)^{i} \phi \right\rangle$$

$$= \left\langle (-1)^{i} \frac{\partial \Lambda}{\partial x_i}, \phi \right\rangle$$

then $(-1)^i \dfrac{\partial \Lambda}{\partial x_i} = 0 \quad \forall i$

Known fact: $D\Lambda = 0 \Rightarrow \Lambda$ is (represented by) a constant m, that is

$$\langle \Lambda, \phi \rangle = \int_{\mathbb{R}^d} m \cdot \phi \, dx$$

$$\langle \Lambda, \phi \rangle = \langle T, \phi \, dx \rangle$$

$$\int_{\mathbb{R}^d} m \cdot \phi \, dx = \int_{\mathbb{R}^d} m \langle \phi \, dx ; e \rangle \, dx = \langle [\mathbb{R}^d, e, m], \phi \, dx \rangle$$

$\square$

By-product of this proof

Proposition 1

Let T be a d-current on $\mathbb{R}^d$ (or Ω, or M^d)

$\mathbb{R}^d$
$\cup$

d-dimens. manifold

with $\mathbb{M}(\partial T) < +\infty$.

Then $T = [\cancel{\mathbb{R}^d}, e, u]$ with

or Ω or M^d

$u \in BV_{loc}(\cancel{\mathbb{R}^d})$

or Ω or M^d