GMT 19/20 Lecture 22 28/5/20

(Relevant) classes of currents (continuation)

1] currents with finite mass, $\mathbb{M}(T) < +\infty$

$\Rightarrow$ $T = \tau\mu$ (not geometrically relevant)

2] normal currents: $\mathbb{M}(T), \mathbb{M}(\partial T) < +\infty$

(geometrically relevant)

3] Rectifiable currents.

Let E k-dim. rectif. set in $\mathbb{R}^d$.

Let τ be an orientation of E.

i.e., $\tau : E \longrightarrow \Lambda_k(\mathbb{R}^d)$ s.t. $\tau(x)$ is an orientation of $T_x^W E$ for $\mathcal{H}^k$-a.e. x

that is, $\tau(x) = \tau_1(x) \wedge \dots \wedge \tau_k(x)$ is simple, $|\tau(x)| = 1$, $\mathrm{span}(\tau(x)) := \mathrm{span}\{\tau_1(x), \dots, \tau_k(x)\} = T_x^W E$

Let $m \in L^1(\mathcal{H}^k \llcorner E)$ be a "multiplicity".

Then let $T = [E, \tau, m]$ be the current

$$(*) \quad \langle T, \omega \rangle := \int_E \langle \omega(x), \tau(x) \rangle \cdot m(x) \, d\mathcal{H}^k(x)$$

or equiv. $T = \tau \cdot m \cdot \mathcal{H}^k \llcorner E$

If T can be written as in $(*)$ for some E, τ, m, we say that T is rectifiable.

If moreover m takes values in $\mathbb{Z}$, we say that T has integral multiplicity.

Rem.

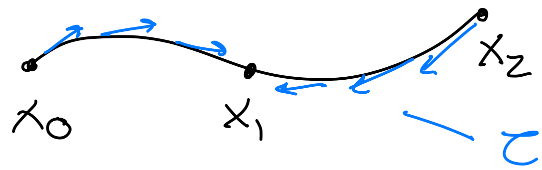
- If $T = [E, \tau, m]$ then $M(T) = \int_E |m(x)| \, d\mathcal{H}^k(x)$
$$= \|m\|_{L^1(\mathcal{H}^k \llcorner E)}$$
- Given T rectif., E, τ, m are NOT uniquely determined.

  However, if you additionally require that $m > 0$ $\mathcal{H}^k$-a.e., then E, τ, m are uniquely determined (up to $\mathcal{H}^k$-null sets) (this is an ex.)
- Note that the dimension of T is the same as the dimension of E.
- Natural example: Let S be a k-dim. oriented surface in $\mathbb{R}^d$ with $\mathcal{H}^k(S) < +\infty$.

Then $T_S := [S, \tau_S, 1]$ is rectifiable.

$$\langle T_S, \omega \rangle := \int_S \langle \omega(x), \xi(x) \rangle \, d\mathcal{H}^k(x)$$

- More interesting example: E = curve of class C^1 in $\mathbb{R}^2$



τ discontinuous orientation

Then $T := [E, \tau, 1]$ is a rectif. 1-current (with integral multipl.)

and $\quad \partial T = 2\delta_{x_1} - \delta_{x_0} - \delta_{x_2}$.

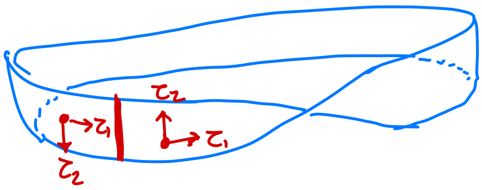
- More general: let E be ~~curve~~ are of class C^1 in $\mathbb{R}^2$ ... with continuous orientation τ.

  Let $m : E \longrightarrow \mathbb{R}$ be piecewise C^1.

  Compute the boundary of $T := [E, \tau, m]$

- If S is the Möbius strip in $\mathbb{R}^3$ you can choose a disc. orient. τ so that $T = [S, \tau, 1]$ is 1-dim. current

What is ∂T?



4 | Integral currents

We say that T is an integral k-current if both T and ∂T are rectifiable with integral multiplicity for $k \geq 1$ (for $k=0$, T is rectif. with integral multiplicity that is

$$T = \sum_i m_i \delta_{x_i}, \quad m_i \in \mathbb{Z}).$$

$\leftarrow$ finite sum

Thus $\exists$ E, τ, m ; E', τ', m' s.t.

$$T = [E, \tau, m], \quad \partial T = [E', \tau', m']$$

There should be a geometric rel. between E and E'. But it is only known for $k=d$.

Federer – Fleming Compactness Theorem

Let T_n be sequence of integral k-currents in $\mathbb{R}^d$ $(0 \leq k \leq d)$ s.t.

(*) $$\boxed{\mathbb{M}(T_n),\ \mathbb{M}(\partial T_n) \leq C < +\infty,}$$

Then, up to subseq., T_n converges to T integral k-current.

<u>Corollary</u> (Existence of solution of Plateau Problem for integral currents)

Let T_0 be an integral current in $\mathbb{R}^d$.
Then the minimum

$$\min\{\mathbb{M}(T) : T \text{ integral},\ \partial T = \partial T_0\}$$

is achieved.

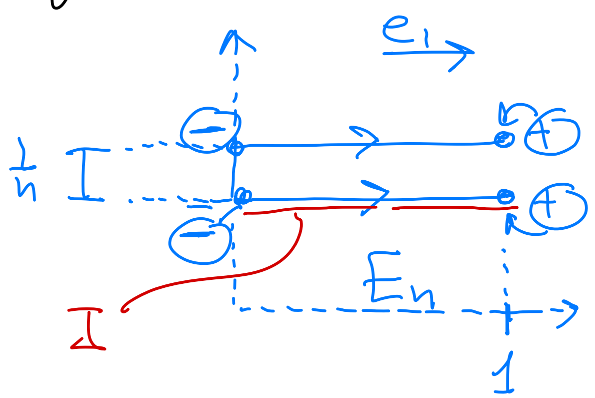
<u>Proof</u> Let T_n be a minimizing seq.
then $\mathbb{M}(T_n) \leq \mathbb{M}(T_0) < +\infty$, $\mathbb{M}(\partial T_n) = \mathbb{M}(\partial T_0) < +\infty$.
$T_n \to T$ by F.F. T is a minimizer. $\square$

Remarks

- The proof of F.F. compactness is hard.
- Under (*) we already know that (up to subseq.) T_n converges to a normal current T (soft statement). The hard part is proving that T is an integral current !!
  (Thus F-F compactness theorem is often called "F-F closure theorem".)
- There is no counterpart of F.-F. for rectifiable sets or rectifiable measures.
- The ass. (*) in F-F are the natural ones for application to Plateau Problem.

Examples (showing the assumptions in F.-F. theorem are all needed)

1) F.-F. does not hold if we replace integral currents with integral currents with multiplicity 1,



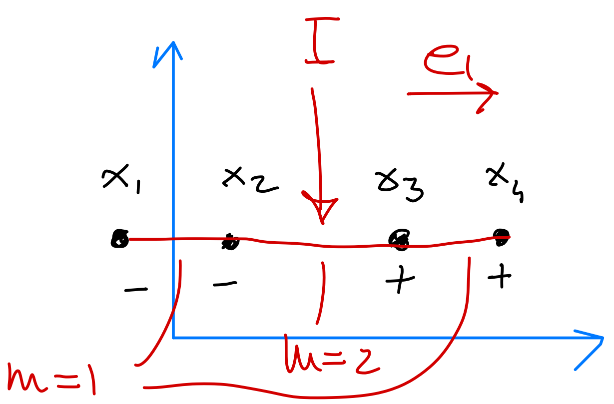
$T_n := [E_n, e_1, 1]$

$\partial T_n :=$ sum of four Dirac masses

Then $\mathbb{M}(T_n) = 2$, $\mathbb{M}(\partial T_n) = 4$

$T_n \longrightarrow T := [I, e_1, 2]$

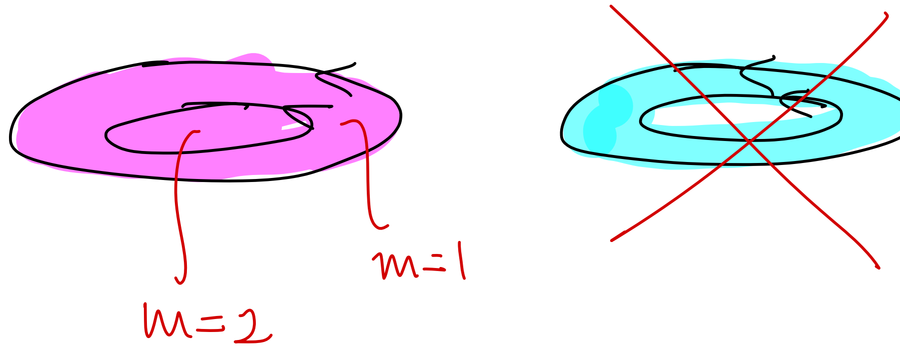
Also solutions of Pl. Pb. may have multiplicity different from 1



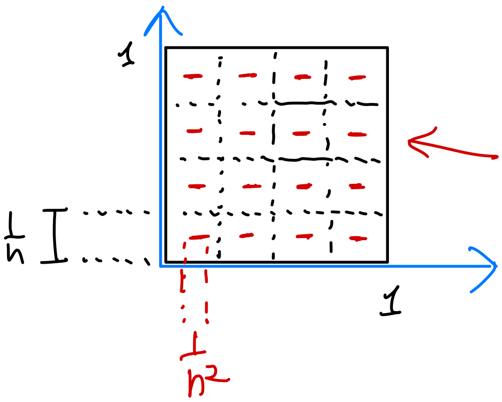
$\partial T_0 := \delta_{x_4} + \delta_{x_3} - \delta_{x_2} - \delta_{x_1}$

Then sol. of P.P. is

$T = [I, e_1, m]$



2) Let T_n be 1-current in $\mathbb{R}$ given by $T_n := [E_n, e_1, 1]$ ← integral



$\underrightarrow{e_1}$

$\mathbb{M}(T_n) = 1$

$\mathbb{M}(\partial T_n) = 2n^2$

$E_n :=$ union of n^2 horiz. segments with length $\frac{1}{n^2}$

NOT integral

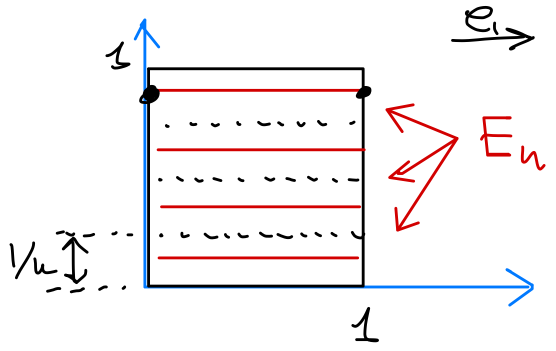
Then $T_n \longrightarrow T := e_1 \cdot \mu$

with $\mu = \mathcal{L}^2 \llcorner Q$ with $Q := [0,1]^2$.

(prove it)

This proves that the ass. $\mathbb{M}(\partial T_n) \leq C < +\infty$ in F.-F. Th. is needed!

3] $T_n := [E_n, e_1, \frac{1}{n}]$ 1-current in $\mathbb{R}^2$



$\mathbb{M}(T_n) = 1$

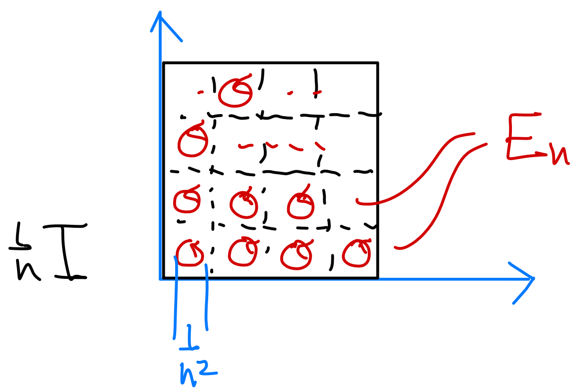
$\mathbb{M}(\partial T_n) = 2$

Then $T_n \longrightarrow T = e_1 \mu$

and $\mu := \mathcal{L}^2 \llcorner Q$. which is not even rectifiable.

This shows that in F.-F. Th the ass. of integral multiplicity is need!

4] $T_n := [E_n, \tau_n, 1]$ 1-current in $\mathbb{R}^2$



$\mathbb{M}(T_n) = \pi$

$\partial T_n = 0$

$\mathcal{H}^1 \llcorner E_n \longrightarrow \pi \cdot \mathcal{L}^2 \llcorner Q$

What is $\lim_{n \to \infty} T_n = ?$