GMT 19/20 lecture 18 18/5/20

Review of basic multilinear algebra

Setting: V (real) vector space, V^* dual of V

Def A <u>k-linear</u>, <u>alternating form</u> (or k-covector) on V is a function $\alpha : V^k \longrightarrow \mathbb{R}$ s.t.

(i) α is linear in each variable,

(ii) α is alternating, that is:

$$\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) =$$

$$= \operatorname{sgn}(\sigma)\, \alpha(v_1, \dots, v_k)$$

$\forall\, v_1, \dots, v_k \in V$, $\forall\, \sigma \in S_k := \{$permut. of the set $\{1, \dots, k\}\}$

$$\boxed{\text{The space of all } k\text{-covector is denoted by } \Lambda^k(V)}$$

Remarks

- (ii) is equivalent to say that α change sign if I swap v_i with v_j for some $i \neq j$.

  $\alpha(v_1, \dots, v_k) = 0$ if $v_i = v_j$ for some $i \neq j$

- $\Lambda^k(V)$ is a linear space

- $\Lambda^1(V) = V^*$

- It is convenient to set $\Lambda^0(V) := \mathbb{R}$

- $\dim(\Lambda^k(V)) = 1$ if $\dim(V) = k$

  (already known as part of the charact. of determinant of $k \times k$ matr.)

- $\alpha(V_1, \dots, V_k) = 0$ if $V_1, \dots, V_k$ are linearly dependent

  (assume V_k is linear comb. of $v_1, \dots, v_{k-1}$: $v_k = \sum_{j=1}^{k-1} \alpha_j v_j$

  $$\alpha(\dots) = \alpha\left(v_1, \dots, v_{k-1}, \sum_{i=1}^{k-1} \alpha_i v_i\right) =$$
  $$= \sum_{i=1}^{k-1} \alpha(v_1, \dots, v_{k-1}, v_i) = 0.)$$

- In partic. $\Lambda^k(V) = \{0\}$ if $k > \dim V$

Def. (exterior product).

Let $\alpha \in \Lambda^h(V)$, $\beta \in \Lambda^k(V)$.

Then $\alpha \wedge \beta$ is the element of $\Lambda^{h+k}(V)$ given by

$$\alpha \wedge \beta (v_1, \dots, v_{h+k}) =$$

$$= \frac{1}{h!\,k!} \sum_{\sigma \in S_{h+k}} \operatorname{sgn}(\sigma)\, \alpha\left(v_{\sigma(1)}, \dots, v_{\sigma(h)}\right) \cdot \beta\left(v_{\sigma(h+1)}, \dots\dots v_{\sigma(h+k)}\right)$$

Remarks

- $\wedge : \Lambda^h(V) \times \Lambda^k(V) \longrightarrow \Lambda^{h+k}(V)$ is linear in each factor,
- associative $\left((\alpha_1 \wedge \alpha_2) \wedge \alpha_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3) \right)$
- $\wedge$ is NOT commutative but
$$\beta \wedge \alpha = (-1)^{hk}\, \alpha \wedge \beta$$

$\Longrightarrow$ if h is odd $\alpha \wedge \alpha = 0$

- $\lambda \in \mathbb{R} \simeq \Lambda^0(V)$ then $\lambda \wedge \alpha = \lambda \alpha$

Let now $e_1, \dots, e_n$ be a basis of V

(order is important!)

$\dim(V) = n$ is finite

Then $e_1^*, \dots, e_n^*$ is the corresponding "dual basis", i.e. basis of V^*, namely

$$e_i^*(e_j) = \delta_{ij} \qquad \forall i,j$$

$\Big(e_i^*(v) :=$ i_{th} coordinate of v w.r.t. to the basis $e_1, \dots, e_n \Big)$

$\underline{i} = (i_1, \dots, i_k)$ is a multiindex.

$$I_{n,k} := \{ \underline{i} \text{ s.t. } 1 \le i_1 < i_2 < \dots < i_k \le n \}$$

$\forall\ \underline{i} \in I_{n,k}$ we set $e_{\underline{i}}^* := \underbrace{e_{i_1}^* \wedge \dots \wedge e_{i_k}^*}_{\in\, \Lambda^k(V)}$

Theorem 1 For $k>0$, $\{e^*_{\underline{i}} : \underline{i} \in I_{n,k}\}$ is a basis of $\Lambda^k(V)$, and indeed $\forall \alpha \in \Lambda^k(V)$

$$\alpha = \sum_{\underline{i} \in I_{n,k}} \alpha_{\underline{i}} \, e^*_{\underline{i}}$$

where $\alpha_{\underline{i}} := \alpha(e_{i_1}, \dots, e_{i_k}) =: \alpha(e_{\underline{i}})$.

In particular

$$\dim(\Lambda^k(V)) = \begin{cases} 0 & \text{if } k=0 \text{ or } k>n \\ \# I_{n,k} = \binom{n}{k} & \text{if } 0<k\leq n \end{cases}$$

Lemma 2

$\forall v_1, \dots, v_k \in V$ let M be the $n \times k$ matrix whose j-th column is given by the coordinate of v_j w.r.t. to the basis.

$\forall \underline{i} \in I_{n,k}$

$$e^*_{\underline{i}}(v_1, \dots, v_k) = \det(M_{\underline{i}})$$

$k \times k$ minor of M
given by the
rows $i_1, \dots, i_k$

In particular

$$e^*_{\underline{i}}(e_{j_1},\dots,e_{j_k}) = \delta_{\underline{i}\,\underline{j}}$$

$$\parallel$$

$$e^*_{\underline{i}}(e_{\underline{j}})$$

Proof By induction on k + expansion of determinant

Lemma 3 Let $\alpha \in \Lambda^k(V)$.

If $\alpha(e_{\underline{i}}) = 0 \quad \forall \underline{i} \in I_{n,k}$

then $\alpha \equiv 0$.

Proof $\alpha(e_{\underline{i}}) = 0 \;\forall\; \underline{i} \in I_{n,k}$

$\Rightarrow \alpha(e_{i_1},\dots,e_{i_k}) = 0 \;\forall \underline{i} \in \{1,\dots,n\}^k$

Use linearity of α to write $\alpha(v_1,\dots,v_k)$ as linear combination of $\alpha(e_{\underline{i}})$ with $\underline{i} \in \{1,\dots,n\}^k$. $\square$

Proof of Th. 1

Step 1 $\{e_{\underline{i}}^* : \underline{i} \in I_{n,k}\}$ are linearly independent.

$\Uparrow$

$$e_{\underline{i}}^*(e_{\underline{i}}) = 1 \quad \text{but} \quad e_{\underline{j}}^*(e_{\underline{i}}) = 0 \quad \forall \underline{j} \neq \underline{i}$$

Step 2

$$\alpha = \sum_{\underline{i}} \underbrace{\alpha(e_{\underline{i}})}_{\alpha(e_{i_1}, \dots, e_{i_k})} e_{\underline{i}}^*$$

because

$$\beta := \alpha - \sum_{\underline{i}} \alpha(e_{\underline{i}})\, e_{\underline{i}}^*$$

satisfies $\beta(e_{\underline{j}}) = 0 \quad \forall \underline{j} \in I_{n,k}$

$$\beta(e_{\underline{j}}) = \alpha(e_{\underline{j}}) - \sum_{\underline{i}} \alpha(e_{\underline{i}}) \cancel{e_{\underline{i}}^*(e_{\underline{j}})}$$

$$e_{\underline{i}}^*(e_{\underline{j}}) = \delta_{\underline{i}\,\underline{j}}$$

$$= \alpha(e_{\underline{j}}) - \alpha(e_{\underline{j}})$$

$\square$

Special case

$V = \mathbb{R}^n$.

Then $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{R}^n$, and we write dx_i instead of e_i^* (in agreement with the differential notatation)

and dx_i instead of e_i^*.

Computations in terms of the basis

$$\alpha = dx_1 \wedge dx_2 + 2 dx_3 \wedge dx_4$$

$$\alpha \wedge \alpha = (dx_1 \wedge dx_2 + 2 dx_3 \wedge dx_4) \wedge$$
$$\wedge (dx_1 \wedge dx_2 + 2 dx_3 \wedge dx_4)$$
$$= dx_1 \wedge dx_2 \wedge dx_1 \wedge dx_2$$
$$+ 2 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$
$$+ 2 dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2$$
$$+ 4 dx_3 \wedge dx_4 \wedge dx_3 \wedge dx_4$$

$$= -\overset{0}{\cancel{dx_1 \wedge dx_1}} \wedge \overset{0}{\cancel{dx_2 \wedge dx_2}}$$

$$+\ 2\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

$$+\ 2\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

$$+\ 0$$

$$= 4\,(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$$

<u>Proposition</u> (Cauchy-Binet formula)

Given $A, B \in \mathbb{R}^{n \times k}$ with $1 \leq k \leq n$ then

$$\underbrace{\det(A^t B)}_{k \times k \text{ matrix}} = \sum_{\underline{i} \in I_{n,k}} \det(A_{\underline{i}}) \det(B_{\underline{i}})$$

$$\overset{n}{k\,\boxed{A^t}}\ \ \overset{k}{\boxed{B}}\,n$$

$k \times k$ minor of A corresp. to rows $i_1, \dots, i_k$

In particular $\det(A^t A) = \sum_{\underline{i} \in I_{n,k}} (\det(A_{\underline{i}}))^2$

(used for the Jacobian ...)

Proof Let $\alpha(v_1,\dots,v_k) := \det(A^t V)$

(V = $n\times k$ matrix with columns $v_1,\dots,v_k$)

Check that α is k-linear + alternating

then $\alpha = \sum_{\underline{i}} \alpha(e_{\underline{i}})\, e^*_{\underline{i}}$

then

$$\det(A^tV) = \alpha(v_1,\dots,v_k) = \sum_{\underline{i}} \alpha(e_{\underline{i}})\; e^*_{\underline{i}}(v_1,\dots,v_k)$$

$$\alpha(e_{\underline{i}}) = \det(A^t E_{\underline{i}}) = \det(A^t_{\underline{i}}) = \det(A_{\underline{i}}), \qquad e^*_{\underline{i}}(v_1,\dots,v_k) = \det(V_{\underline{i}})$$

Back to V linear space

Simple k-vectors in V

On V^k $(k \geq 1)$ define the equivalence relation

general k-vectors will be defined later

$$(v_1, \dots, v_k) \sim (\tilde{v}_1, \dots, \tilde{v}_k)$$

means that

$$\alpha(v_1, \dots, v_k) = \alpha(\tilde{v}_1, \dots, \tilde{v}_k) \quad \forall\, \alpha \in \Lambda^k(V)$$

Temporarily we write equivalence classes a $[v_1, \dots, v_k]$ (later $v_1 \wedge \dots \wedge v_k$)

and write $0 = [0, \dots, 0] = \{(v_1, \dots, v_k) \text{ s.t. } \alpha(v_1, \dots, v_k) = 0\ \forall \alpha\}$

Prop 5 (i) $(v_1, \dots, v_k) \sim (0, \dots, 0)$

iff. $v_1, \dots, v_k$ are linearly dependent

(ii) If $(v_1, \dots, v_k) \sim (\tilde{v}_1, \dots, \tilde{v}_k) \not\sim (0, \dots, 0)$

then $\mathrm{span}(v_1, \dots, v_k) = \mathrm{span}(\tilde{v}_1, \dots, \tilde{v}_k)$

moreover the change-of-base matrix

M (that is $\tilde{v}_i = \sum_j M_{ij} v_j \quad \forall\, i, j$)

has $\det(M) = 1$