GMT 19/20, lecture 13, 30/4/20

Lipschitz functions (and maps) cont.'d.

<u>Setting</u> : $f : X \to Y$ $\qquad \mathrm{Lip}(f) := \ldots\ldots$

metric space

<u>Properties of Lip. maps</u>

- Compactness (Arzelà-Ascoli Th.)
- Extension properties
  - <u>McShane lemma</u>
  - (Kirszbraun Theorem)
- Differentiability
  - <u>Rademacher Theorem</u>

    Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz.
    Then f is differentiable at a.e. $x \in \mathbb{R}^n$.

Huge difference with cont./Hölder functions

Radem. Th is a corollary of:

Th. 1 If $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is Lipschitz (open)
then $f \in W^{1,\infty}_{loc}(\Omega, \mathbb{R}^m)$.

Th. 2 If $f \in \mathcal{E} \cap W^{1,p}_{loc}(\Omega)$ (open in $\mathbb{R}^n$) with $p > n$.
Then f is diff. a.e. in Ω
(and the grad. of f agrees a.e. with the distrib. gradient — then we use ∇f to denote both).

Sketch of Proof of Th. 1

For $n = m = 1$, and $\Omega = \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz

Then

$$Df = \lim_{h \to 0} \frac{f - \tau_h f}{h}$$

distrib. derivative of f

in the sense of distrib.

If f is Lip. then RHS are unif. bounded
$\Longrightarrow$ RHS converge weakly* in L^∞ $\square$

Sketch of Proof of Th. 2 $(m=1)$

Let ∇f be the distrib. (weak) gradient of f.

Fix $B = B(\bar{x}, r) \subset\subset \Omega$.

1) Poincaré ineq.: $\displaystyle\sup_{x\in B} |f(x) - \underbrace{m}_{\text{av. of } f \text{ on } B}| \leq \underbrace{C}_{=C(B)} \left(\int_B |\nabla f|^p\right)^{\frac{1}{p}}$

2) $\displaystyle \underbrace{\mathrm{osc}(f, \bar{B})}_{=B(\bar{x},r)} \leq \cancel{C(B)} = \cancel{C(r)} = C\cdot r \left(\fint_B |\nabla f|^p dx\right)^{\frac{1}{p}}$

guess by dim. analysis, proof by change of variables

3) $\displaystyle |f(\bar{x}+h) - f(\bar{x})| \leq C|h| \left(\fint_B |\nabla f|^p\right)^{\frac{1}{p}}$

4) $\displaystyle |f(\bar{x}+h) - f(\bar{x}) - v\cdot h| \leq C|h| \left(\fint_B |\nabla f - v|^p\right)^{\frac{1}{p}}$

v: Vector in $\mathbb{R}^n$

Assume that $\boxed{\nabla f \text{ is } L^p\text{-approx. cont. at } \bar{x}}$ (true for a.e. $\bar{x}$!!) then

$$\frac{|f(\bar{x}+h) - f(\bar{x}) - \nabla f(\bar{x})\cdot h|}{|h|} \leq C \cancel{|h|} \left(\fint_B |\nabla f(x) - \nabla f(\bar{x})|^p dx\right)^{\frac{1}{p}} \xrightarrow{|h|\to 0} 0$$

that is, f is diff. at $\bar{x}$ $\square$

∘ Lusin property of Lipschitz functions

<u>Th</u> Let Ω bounded open in $\mathbb{R}^n$, $f:\Omega \to \mathbb{R}$ is continuous and a.e. diff. (with gradient ∇f)

Then $\forall \varepsilon > 0$ $\exists$ K compact $\subset \Omega$ and a C^1-function $g: \cancel{\Omega}\ \mathbb{R}^n \to \mathbb{R}$ s.t.

- $|\Omega \setminus K| \leq \varepsilon$
- $f = g$ on K ($\Rightarrow \nabla f = \nabla g$ a.e. on K)
- $\mathrm{Lip}(g) \leq \mathrm{Lip}(f)$

<u>Partial proof</u>

Let $D := \{x \in \Omega \mid f \text{ is diff. at } x \text{ with gradient } \nabla f(x)\}$. That is

$$(*) \qquad \frac{|f(x+h) - f(x) - \nabla f(x)\cdot h|}{|h|} \xrightarrow[|h| \to 0]{} 0$$

By Severini–Egorov theorem (!) + <u>Lusin</u> we find K (compact) s.t. the conv. in $(*)$ is uniform & ∇f is cont. on K

that is

$$f(x+h) = f(x) + h\,\nabla f(x) + R_x(h)$$

with $\quad |R_x(h)| \leq |h| \cdot \omega(|h|)$

modulus of cont. which DOES NOT depend on x

By Whitney extension theorem, the restriction of f to K can be extended to a C^1 function $g : \mathbb{R}^n \to \mathbb{R}$

$\square$

Lemma (Exercise)

If f, g are a.e. diff. on K (compact) and $f = g$ a.e. on K, then $\nabla f = \nabla g$ a.e. on K

Area formula

I will state many versions: the first is the simplest to prove, the last is the most general.

Th 1 (A.F. v.1)

Σ d-dim. surface of class C^1 in $\mathbb{R}^n$ parametrized by $\Phi : \Omega \longrightarrow \Sigma \subset \mathbb{R}^n$ param. of class C^1

open in $\mathbb{R}^d$

(thus Φ is C^1, bijective, proper)

Then $\forall F$ Borel set $\subset \Sigma$

$$\boxed{(*) \qquad \mathcal{H}^d(F) = \int_{\phi^{-1}(F)} J\phi(x)\, \underbrace{dx}_{\text{Lebesgue meas.}}}$$

where

$$J\phi(x) := \left|\det(d_x\phi)\right|$$

$$= \sqrt{\det\left(\nabla^t\phi(x) \cdot \nabla\phi(x)\right)}$$

$d \times m$, $n \times d$

$$= \sqrt{\sum_{M \ d\times d \text{ minor of } \nabla\phi(x)} (\det M)^2}$$

$d_x\phi$ linear map $\subset \mathbb{R}^n$

$d_x\phi : \mathbb{R}^d \to \mathrm{Tan}(\Sigma, \phi(x))$

$|\det(d_x\phi)| = |\det M|$

with M matrix ass. to $d_x\phi$ wrt orthonormal bases of $\mathbb{R}^d$, ...

Equivalent formulation of $(*)$:

$$(**)\quad \mathcal{H}^d \llcorner \Sigma = \underbrace{\phi_\#\left(J\phi \cdot \mathcal{L}^d \llcorner \Omega\right)}_{\overset{!!}{\lambda}}$$

$\phi : X \to Y$ Borel, μ measure on X, then the PUSH-FORWARD of μ acc. to ϕ is the meas. on Y defined by

$$\phi_\# \mu(E) := \mu(\phi^{-1}(E)) \qquad E \subset Y$$

Proof

$(**)$ follows from a theorem stated in lecture 4, once we show that

$\oplus$ $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall f : U$ open in $\Sigma \to \mathbb{R}^d$ of class C^1 with $\boxed{\text{isometry defect} \leq \delta}$ then

$$\frac{1}{1+\varepsilon}\lambda(E) \leq \mathcal{L}^d(f(E)) \leq (1+\varepsilon)\lambda(E) \quad \forall E \subset U$$

$$\frac{1}{1+\delta}|y-y'| \leq |f(y)-f(y')| \leq (1+\delta)|y-y'| \quad \forall y, y' \in U$$

Take $f : U$ open in $\Sigma \longrightarrow \mathbb{R}^d$ δ-isometry.

let $\tilde{U} := \phi^{-1}(U)$,

$$g := f \circ \phi : \underset{\subset \mathbb{R}^d}{\tilde{U}} \longrightarrow \mathbb{R}^d$$

then g is injective, C^1.

Moreover $d_y f$ is a δ-isometry $\forall y \in U$ that is

$$\frac{1}{1+\delta} |h| \leq |d_y f(h)| \leq (1+\delta) |h| \quad \forall h \in \mathrm{Tan}(\Sigma, y)$$

then

$$(1+\delta)^{-d} \leq |\det d_y f| \leq (1+\delta)^d$$

then

$$(1+\delta)^{-1} J\phi(x) \leq \underset{|\det \nabla g(x)|}{\overset{\shortparallel}{|\det d_x g|}} \leq (1+\delta)^d J\phi(x)$$

Take $E \subset U$, let $\tilde{E} := \phi^{-1}(E)$ then

$$|f(E)| = |g(\tilde{E})| = \int_{\tilde{E}} \det |\nabla g(x)| dx$$

and

$$(1+\delta)^{-d} \lambda(E) \leq |f(E)| \leq (1+\delta)^d \int_{\tilde{E}} J\phi = (1+\delta)^d \lambda(E)$$

Given ε, choose δ s.t. $(1+\delta)^d \le 1+\varepsilon$

then $\oplus$ holds! $\square$