GMT 19/20

Lecture 4 20/3/20

<u>Recall</u> μ outer measure on X

$\mathcal{M}_\mu$ class of μ-meas. sets

$\to$ $\mathcal{M}_\mu$ is a σ-algebra and μ is σ-add on $\mathcal{M}_\mu$

$\to$ Caratheodory Theorem

If X is metric and μ is additive on <u>distant sets</u> then $\mathcal{M}_\mu$ contains Borel sets

Next step: construct MEANINGFUL outer measures that are additive on distant sets!

Carathéodory Construction

(simplified version that fits our need!)

X metric space

$\mathcal{F}$ family of subsets of X, $\emptyset \in \mathcal{F}$

$\rho : \mathcal{F} \mapsto [0,\infty]$ "gauge function"

$(\rho(\emptyset) = 0)$

For every $\delta \in (0,+\infty]$ and every $E \subset X$ define

- $$\psi_\delta(E) := \inf \left\{ \sum_i \rho(E_i) \;\middle|\; \begin{array}{c} \{E_i\} \subset \mathcal{F} \text{ count. cover of } E \\ \& \\ \operatorname{diam}(E_i) \le \delta \end{array} \right\}$$

$\boxed{\text{Rem } \inf \emptyset = +\infty}$

- $$\psi(E) := \sup_{\delta > 0} \psi_\delta(E) = \lim_{\delta \to 0} \psi_\delta(E)$$

Rem If δ decreases, $\psi_\delta(E)$ increases ...

Proposition

- $\forall \delta > 0$, ψ_δ is an outer measure

  and $\psi_\delta(E \cup E') \overset{\le \ (\text{for free})}{=} \psi_\delta(E) + \psi_\delta(E')$

  if $\operatorname{dist}(E, E') > \delta$

- ψ is an outer measure additive on distant sets

(ψ is σ-add. on Borel sets!!)

Proof is an exercise.

Example 1 Lebesgue measure

$X = \mathbb{R}^d$, $\mathcal{F} = \{ R = I_1 \times \cdots \times I_d \}$

with $I_1, \dots, I_n$ intervals in $\mathbb{R}$

$$\rho(R) = \mathrm{vol}_d(R) = \prod_{i=1}^{d} |I_i|$$

length of I_i

In this case ψ_δ does not depend on δ

$(\mathcal{L}^d := \psi = \psi_\delta = \psi_\infty)$

$\mathcal{L}^d := \psi_\delta$

proof based on the following remark:

$\forall \delta > 0$, a rect. R can be written as $R = \bigcup R_j$ so that

$\mathrm{diam}(R_j) \leq \delta$ & $\mathrm{vol}_d(R) = \sum_j \mathrm{vol}_d(R_j)$

Ex $\mathcal{L}^n(E) = \inf \{ \lambda(A) \mid A \text{ open}, A \supset E \}$

& $\lambda(A) = \sup \{ \sum_i \mathrm{vol}_d(R_i) \mid \dots \}$

Example 2 | Hausdorff Measure

X metric space

$d \in [0, +\infty)$

$\mathcal{F} := \mathcal{P}(X) = 2^X$

$\rho(E) := (\operatorname{diam} E)^d$

If $d = 0$

$\rho(E) := \begin{cases} 0 & \text{if } E = \emptyset \\ 1 & \text{otherwise} \end{cases}$

$\mathcal{H}^d_\delta(E) := c_d \cdot \psi_\delta(E)$, $\mathcal{H}^d(E) = c_d \cdot \psi(E)$

Hausdorff pre-meas.

renormalization constant

d-dimens. Hausd. meas

$c_d := \dfrac{\alpha_d}{2^d}$ for $d \in \mathbb{N}$

where $\alpha_d :=$ volume of unit ball in $\mathbb{R}^d$

List of Remarks and Facts

1) $\cdot \ \mathcal{H}^0(E) = \# E$

$\cdot \ \mathcal{H}^0_\infty(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ 1 & \text{if } E \neq \emptyset \end{cases} \left(= \rho(E) \right)$

2) Why is $\mathcal{H}^d$ called "d-dimensional"?

If $X = \mathbb{R}^m$ then $\mathcal{H}^d(\lambda E) = \lambda^d \mathcal{H}^d(E)$

(basic scaling prop. of $\mathcal{H}^d$)

3) If $f: X \to Y$ is Lipschitz

then

$$(*) \quad \mathcal{H}^d(f(E)) \leq (\mathrm{Lip}(f))^d \cdot \mathcal{H}^d(E)$$

Follows from $\mathrm{diam}(f(E)) \leq \mathrm{Lip}(f) \cdot \mathrm{diam}(E)$

If $f: X \to Y$ is an isometry then

$$\mathcal{H}^d(f(E)) = \mathcal{H}^d(E)$$

apply $*$ to f and f^{-1} and use that
$\mathrm{Lip}(f) = \mathrm{Lip}(f^{-1}) = 1$

4] If $X = \mathbb{R}^d$ then $\mathcal{H}^d = \mathcal{H}^d_\delta = \mathcal{L}^d$ $(\forall \delta \in (0,+\infty])$

Here the choice of renorm. constant is e s s e n t i a l !!!

The proof is delicate! (uses covering theorems and isodiametric inequality)

$\longrightarrow$ LATER

<u>Rem</u> Both $\mathcal{H}^d$ and $\mathcal{L}^d$ are translation invariant measures on $\mathbb{R}^d$ (+ locally finite) then general statement on Haar measures implies that $\mathcal{H}^d = c\,\mathcal{L}^d$ for some constant c.

The delicate part is to find c!

5| If $E \subset S$ = d-dim. surface of class C^1 in $\mathbb{R}^m$
(S = d-dim. Riemann. Manifold, C^1)
then $\mathcal{H}^d(E)$ = d-dimensional volume of E
(e.g., the one computed using the area formula)

(Unfort. we do not have area formula yet)

6| There are several notions of d-dimensional measure of a set E in $\mathbb{R}^m$ (in part. for d integer)

But they all agree on surfaces of dim. d (and class C^1, and even Lipschitz)

Theorem Fix d integer

let S d-dim. surface of class C^1 in $\mathbb{R}^m$ (or d-dim. Riem. manifold of class C^1)

let λ be a measure on S s.t.

Rem $\mathcal{H}^d$ has this property

(*) $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$\forall f : U$ open in $S \longrightarrow \mathbb{R}^d$ with isometry defect $\leq \delta$

$$\left(\frac{1}{1+\delta} |x-x'| \leq |f(x) - f(x')| \leq (1+\delta)|x-x'| \quad \forall x, x' \in U \right)$$

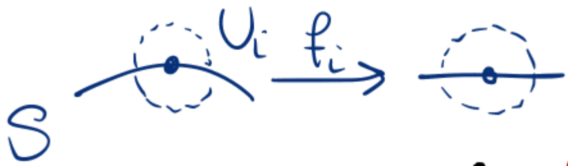
there holds

$$\frac{1}{1+\varepsilon} \lambda(E) \leq \mathcal{L}^d(f(E)) \leq (1+\varepsilon)\lambda(E) \quad \forall E \subset U$$

Then λ is unique.

<u>Proof</u> Take λ, λ' as above, $E \subset S$ (E Borel), we claim $\lambda(E) = \lambda'(E)$.

- Fix ε and take δ as in $(*)$
- Cover S with open sets U_i s.t. $\exists\, f_i : U_i \longrightarrow \mathbb{R}^d$ with isom. def. $\leq \delta$



disjoint union

- Write $E = \dot\bigcup E_i$ with $E_i \subset U_i$

$$\frac{1}{1+\varepsilon} |f_i(E_i)| \leq \lambda(E_i) \; ; \; \lambda'(E_i) \leq (1+\varepsilon) |f_i(E_i)|$$

$$\Downarrow$$

$$\frac{1}{(1+\varepsilon)^2} \lambda(E_i) \leq \lambda'(E_i) \leq (1+\varepsilon)^2 \lambda(E_i)$$

$$\Downarrow$$

$$\frac{1}{(1+\varepsilon)^2} \lambda(E) \leq \lambda'(E) \leq (1+\varepsilon)^2 \lambda(E)$$

$$\Downarrow \quad \text{send } \varepsilon \text{ to } 0$$

$$\lambda'(E) = \lambda(E) \,. \qquad \square$$

7 Refinement of $\mathcal{F}$ in the construction of $\mathcal{H}^d$.

$\mathcal{H}^d$ is NOT changed if we replace $\mathcal{F} = \mathcal{P}(X)$ in the def. of $\mathcal{H}^d$ by

- $\mathcal{F} = \{\text{closed sets}\}$

  because $\operatorname{diam}(\overline{E}) = \operatorname{diam}(E)$

- $\mathcal{F} = \{\text{open sets}\}$

  because $\forall E \subset X \ \forall \varepsilon > 0 \ \exists$ A open, $A \supset E$, $\operatorname{diam}(A) \leq \operatorname{diam}(E) + \varepsilon$

If $X = \mathbb{R}^m$ or a normed space

- $\mathcal{F} = \{\text{convex sets}\}$

  convex hull of E

  because $\operatorname{diam}(\operatorname{Conv}(E)) = \operatorname{diam}(E)$

same for "convex and open" or "convex and open"

However the value of $\mathcal{H}_d$ changes if

- $\mathcal{F} = \{ \text{balls} \}$

! In this case you get what is called "spherical Hausd. meas." $\mathcal{H}^d_S$

In general a set E is NOT contained in a ball with same diameter

However $\mathcal{H}^d_S$ agrees with $\mathcal{H}^d$ on d-dim. surface of class C^1

for the proof use that $\mathcal{H}^d_S = \mathcal{L}^d$ on $\mathbb{R}^d$