GMT 19/20

Lecture 3, 19/3/20

Review of basic measure theory (continued)

<u>Theorem</u> (Hahn–Lebesgue–Radon–Nikodym)

Given μ, λ positive finite Borel measures on X .....

Then

- $\lambda = \lambda_a + \lambda_s$ with $\lambda_a \ll \mu$, $\lambda_s \perp \mu$
- $\lambda_a = f\mu$ with $f \in L^1(\mu)$

Moreover, if $X \subset \mathbb{R}^n$ <u>or</u> μ has the A.D.P. then

$$f(x) := \lim_{r \to 0} \frac{\lambda(\overline{B(x,r)})}{\mu(\overline{B(x,r)})} \quad \text{for } \mu\text{-a.e. } x$$

Theorem (points of L^p-approx. cont.)

Assume that $X \subset \mathbb{R}^n$ or

μ has the A.D.P. $\rightarrow$ $\boxed{\limsup_{r\to 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < +\infty \text{ for } \mu\text{-a.e. } x}$

Take $f \in L^p(X)$

with $1 \leq p < +\infty$.

Then for μ-a.e. $\bar{x} \in X$

$$\lim_{r\to 0} \fint_{B(\bar{x},r)} |f(x) - f(\bar{x})|^p \, d\mu(x) = 0$$

Corollary 1) $\displaystyle \lim_{r\to 0} \fint_{B(\bar{x},r)} f(x)\, d\mu(x) = f(\bar{x})$ (for μ-a.e. $\bar{x}$)

2) If $f = \mathbb{1}_E$ then

$$\Theta_\mu(E,\bar{x}) = \lim_{r\to 0} \frac{\mu(E\cap B(\bar{x},r))}{\mu(B(\bar{x},r))} = \begin{cases} 1 & \text{for } \mu\text{-a.e. } \bar{x} \in E \\ 0 & \text{for } \mu\text{-a.e. } \bar{x} \notin E \end{cases}$$

μ-density of E at $\bar{x}$

Vector-valued measures

Let μ be a Borel positive meas. on X

Let F be a finite dimensional normed space ($F = \mathbb{R}$, $F = \mathbb{R}^n$ ....)

Let $f \in L^1(X, F)$.

Then $\lambda = f\mu$ is the F-valued Borel measure on X defined by

$$\lambda(E) = [f\mu](E) = \int_E f \, d\mu$$

Theorem Every F-valued measure λ on X can be represented as above for suitable μ and f

Notation Let $\lambda = f\mu$ be an F-valued measure on X.

$$|\lambda| := |f| \cdot \mu$$

$$\|\lambda\| = M(\lambda) := |\lambda|(X) = \int_X |f| \, d\mu = \|f\|_1$$

The case where F is a Banach space is relevant but more delicate!

Preparation for Riesz theorem.

Let X be compact (for the time being)

Let $\lambda = f\mu$ be as before

$\forall g \in \mathcal{C}(X, F^*)$ let

dual of F

$$T_\lambda(g) := \cancel{\int_X g\, d\lambda} := \int_X \underbrace{\langle g(x), f(x)\rangle}_{\substack{\text{duality pairing of } F^* \text{ and } F \\ \text{action of } g(x)\in F^* \text{ on } f(x)\in F}} d\mu(x)$$

$F = \mathbb{R}^n$ with eucl. norm then $F^* \simeq \mathbb{R}^n$ identif. by scalar product and $\langle g(x), f(x)\rangle$ is the scalar product of $\mathbb{R}^n$

Proposition T_λ is a bounded linear functional on $\mathcal{C}(X, F^*)$ and

$$\|T_\lambda\| = M(\lambda)$$

$$\left(\sup_{|g|\le 1} \int \langle g, f\rangle \, d\mu = \|f\|_1 \right)$$

Denote by $\mathcal{M}(X,F)$ the space of all F-valued measures on X endowed with the norm $\|M(\lambda)=\|\lambda\|$

This is a Banach space.

$$\boxed{\begin{array}{l}\underline{\text{Riesz Theorem}}\\ \lambda \longmapsto T_\lambda \text{ is a } \underline{\text{surjective}} \\ \text{isometry of } \mathcal{M}(X,F) \text{ into } (\mathcal{C}(X,F^*))^*\end{array}}$$

The non-trivial part is the surjectivity.

Consequences of Riesz theorem

1| The identif. of $\mathcal{M}(X,F)$ and $(\mathcal{C}(X,F^*))^*$ induces a weak* topology on $\mathcal{M}(X,F)$.

If $\lambda_n \xrightarrow{*} \lambda$ (that is, if $T_{\lambda_n} \xrightarrow{*} T_\lambda$, that is $\int_X g\, d\lambda_n \to \int_X g\, d\lambda$ $\forall g \in \mathcal{C}(X,F^*)$) we say that λ_n converge to λ in the sense of measures, and we simply write
$$\lambda_n \longrightarrow \lambda$$

2| this topology is metrizable on bounded subsets of $\mathcal{M}(X,F)$

This is the only relevant topology on measures!!

3] We use this topology also dealing with ~~real~~ positive valued meas.

4] Compactness

If (λ_n) is a bounded seq. of measures on $\mathcal{M}(X, F)$, then, up to subseq., $\lambda_n \longrightarrow \lambda$.

Proof immediate Corollary of Banach-Alaoglu theorem

5] If X is locally compact (and not compact) everything works provided you replace $\mathcal{C}(X, F^*)$ with $\mathcal{C}_0(X, F^*) = \{ g : X \to F^* \text{ cont. s.t. } g(x) \xrightarrow[x \to \infty]{} 0 \}$

6| (μ_n) positive finite measures on X s.t. $\mu_n \to \mu$ and X is compact

(that is, $\int_X g\, d\mu_n \xrightarrow[n\to+\infty]{} \int_X g\, d\mu \quad \forall g \in C(X)$)

Then (i) $\forall A$ open in X

$$\liminf_{n\to\infty} \mu_n(A) \geq \mu(A)$$

$\forall g : X \to [0,+\infty]$ l.s.c. then

$$\lim_{n\to\infty} \int_X g\, d\mu_n \geq \int_X g\, d\mu$$

(ii) $\forall C$ closed in X

$$\limsup_{n\to+\infty} \mu_n(C) \leq \mu(C)$$

(iii) $\forall E$ Borel s.t. $\mu(\partial E) = 0$

$$\lim_{n\to\infty} \mu_n(E) = \mu(E)$$

Remark Let $X = B^d \subset \mathbb{R}^d$, $\mu_n := \delta_{x_n}$ with $x_n \in \mathbb{R}^d$, $x_n \to x$.
Then $\mu_n = \delta_{x_n} \to \delta_x$.

Let $X = \mathbb{R}^d$, $\mu_n = \delta_{x_n}$, $|x_n| \to +\infty$

then $\mu_n \to 0$

Outer measures

Let μ be an outer measure on the set X

Definition a set $E \subset X$ is μ-measur. (in the sense of Caratheodory) if

$$\mu(F) = \mu(F \cap E) + \mu(F \setminus E) \quad \forall F \subset X$$

$\leq$ holds for free by subadditivity

or

$$\mu = \mu \llcorner E + \mu \llcorner (X \setminus E)$$

Proposition The class $\mathcal{M}_\mu$ of all μ measurable sets is a σ-algebra and μ is σ-additive on $\mathcal{M}_\mu$

The proof is an exercise.

Theorem (Caratheodory)

If X is a metric space and μ is an outer measure on X which is additive on distant sets that is

$$\mu(E \cup E') = \mu(E) + \mu(E')$$

if $\operatorname{dist}(E, E') := \inf_{\substack{x \in E \\ x' \in E'}} d(x, x') > 0$

Then $\mathcal{M}_\mu$ contains Borel sets.

Proof is non-trivial.
But it is easy if X = Cantor set (or more generally, a totally disconn. metric space)

Examples

1) X arbitrary, $\mu(A) := \# A$

<u>then</u> $\mathcal{M}_\mu = 2^X$

2) X arbitrary, $\mu(A) := \begin{cases} 1 & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}$

(μ is an outer measure!)

$\mathcal{M}_\mu = ?$

3) $X = \mathbb{R}^d$, $\mu :=$ Lebesgue outer meas.

then $\mathcal{M}_\mu = \{$Lebesgue meas. sets$\}$.

exercise?

$(\mathcal{L}^d)^*(E) := \inf \left\{ \sum \mathrm{vol}(R_i) \;\middle|\; \underbrace{\{R_i\}}_{\text{rectangles}} \text{ covering of } E \right\}$

E is Leb. meas. if $\forall \varepsilon > 0 \; \exists$ A open, C closed s.t. $C \subset E \subset A$, $\mu(A \setminus C) \leq \varepsilon$.