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Let me recall how the direct method works:
to find (prove the existence of) a minimizer of a certain functional $F$ which is naturally defined on a space $\tilde{X}$ of "regular" objects (functions, surfaces, ...) one constructs a suitable class of "generalized" objects $X$ (endowed with a suitable topology) and an extension of $F$ to $X$ such that $F$ is e.s.c. and coercive on $X$.
This implies the existence of a minimizer of $F$ on $X$, and hopefully one can show that such minimizer actually belongs to $\tilde{X}$ (regularity theory).

It is important to notice that in general the Plateau Pb. may have no solution in the class of regular surfaces (e.g., surfaces of class $C^1$).

This means that the solution actually that one finds within the class of integral currents (or any other class of "generalized" surfaces) cannot be really considered "the" solution of P.P.

However one can prove some partial regularity that show that such solutions are actually regular surfaces except for a singular set of lower dimension.

Regularity theory for minimal surfaces (actually, minimal currents) is quite difficult and by far not yet completed.
In this course I will not prove any regularity except the minimal one, namely that in the end we obtain a closed rectifiable set.

## 1.3 Examples of non regular minimal surfaces

That is, example of boundaries $\Gamma$ for which Plateau Pb. admits no solution $\Sigma$ of class $C^1$

### Example 1 $(n=4, d=2)$

In $\mathbb{R}^4 \simeq \mathbb{R}^2 \times \mathbb{R}^2$ let

$$\Gamma := (S^1 \times \{0\}) \cup (\{0\} \times S^1) \quad \longleftarrow \text{union of 2 disjoint circles}$$

$S^1 := \{x \in \mathbb{R}^2 \text{ s.t. } |x|=1\}$

and

$$\Sigma := (D^2 \times \{0\}) \cup (\{0\} \times D^2) \quad \longleftarrow \text{union of 2 discs intersecting at } (0,0)$$

$D^2 := \{x \in \mathbb{R}^2 \text{ s.t. } |x| \leq 1\}$

Note that $\Gamma$ is a smooth curve, $\Sigma^* := \Sigma \setminus \{(0,0)\}$ is a smooth surface but $\Sigma$ is NOT, and more precisely $\Sigma$ is not smooth at $(0,0)$, which is therefore a singular point.

Then $\Sigma$ is the unique solution of P.P. with boundary $\Gamma$.

This sentence is ill-formulated because I never defined the class of admissible surfaces $\Sigma$ and what it means that $\Gamma$ is the boundary of $\Sigma$ (or that $\Sigma$ spans $\Gamma$).

The precise statement is:

intended in any reasonable sense

> The infimum $m$ of area$(\Sigma)$ among all oriented surfaces $\Sigma$ of class $C^1$ with boundary $\Gamma$ is NOT attained.
>
> i.e., area$(\Sigma_n) \to m$
>
> Moreover there exists a minimizing sequence of admissible surfaces $\Sigma_n$ that converge to $\Sigma$ in Hausdorff distance, and area$(\Sigma_n) \longrightarrow m = \text{area}(\Sigma) = \text{area}(\Sigma^*)$.

**Example 2** $(n=4,\ d=2)$

In $\mathbb{R}^4 \simeq \mathbb{C}\times\mathbb{C}$ let

$$\Gamma := \{(w^3, w^2) \text{ s.t. } w \in S^1 \subset \mathbb{C}\}$$

and

$$\Sigma := \{(w^3, w^2) \text{ s.t. } w \in D^2 \subset \mathbb{C}\}$$

Note that $\Gamma$ is a smooth (connected) curve,
$\Sigma$ is homeomorphic to the disc $D^2$,
$\Sigma^* := \Sigma \setminus \{(0,0)\}$ is a smooth surface, but $\Sigma$ is NOT.
More precisely $\Sigma$ is not smooth at $(0,0)$.

Then $\Sigma$ is the unique solution of P.P. with boundary $\Gamma$.
(The precise meaning being the same as in Example 1.)

**Example 3** $(n=2m,\ d=2k$ with $1\leq k<m)$

Let
- $U$ be an open set in $\mathbb{C}^k$.
- $f: U \to \mathbb{C}^m$ be holomorphic. ($\simeq$ complex analytic)
- $D$ be a compact domain in $U$ with smooth boundary.

Then let

$$\Gamma := \{f(w) \text{ s.t. } w \in \partial D\}$$

and

$$\Sigma := \{f(w) \text{ s.t. } w \in D\}$$

$\nabla f(w) \in \mathbb{C}^{k\times m}$ is the complex gradient

Thus $\Gamma$ is a smooth $2k-1$ dimensional surface if $f$ is injective on $\partial D$ and $\boxed{\operatorname{rank}(\nabla f(w)) = k}$ $\forall\, w \in \partial D$,
and $\Sigma$ is smooth is $f$ is injective on $D$ and $\operatorname{rank}(\nabla f(w)) = k$ $\forall\, w \in D$. (But $\Sigma$ may be not smooth at $x = f(w)$ if $\operatorname{rank}(\nabla f(w)) < k$.)

Then $\Sigma$ is the unique solution of P.P. with boundary $\Gamma$.

## Example 4 ($n = 2m$ with $m \geq 4$; $d = n-1$)

In $\mathbb{R}^n \simeq \mathbb{R}^m \times \mathbb{R}^m$ let

$$\Gamma := \left\{ (x,y) \in \mathbb{R}^m \times \mathbb{R}^m \text{ s.t. } |x| = |y| = 1 \right\} = S^{m-1} \times S^{m-1}$$

and

$$\Sigma := \left\{ (x,y) \in \mathbb{R}^m \times \mathbb{R}^m \text{ s.t. } |x| = |y| \leq 1 \right\}$$

Then $\Gamma$ is a smooth surface, $\Sigma^* := \Sigma \setminus \{(0,0)\}$ is a smooth surface, but $\Sigma$ is NOT. More precisely, $\Sigma$ is singular at $(0,0)$.

Then $\Sigma$ is the unique solution of P.P. with boundary $\Gamma$.

(The precise meaning being the same as in Example 1.)

## Remarks

- Example 3 covers both Example 1 and Example 2 and is actually a particular case of a more general class of examples, that of (singular) complex surfaces with (complex) dimension $k$ in $\mathbb{C}^m$.

- The proof of the minimality of $\Sigma$ in Example 3 is an (easy) consequence of Wirtinger inequality for the Kähler form.

- In Example 4, $\Sigma = C \cap D^n$ where $D^n$ is the closed ball $\overline{B(0,1)}$ in $\mathbb{R}^n \simeq \mathbb{R}^m \times \mathbb{R}^m$ and $C$ is the so-called Simons' cone:
$$C := \{ (x,y) \in \mathbb{R}^m \times \mathbb{R}^m \text{ s.t. } |x| = |y| \}.$$

- The first proof of the minimality of $\Sigma$ in Example 4 is due to E. Bombieri, E. De Giorgi and E. Giusti (1969).
  A simple proof was given by G. De Philippis and E. Paolini (2009).

- For $2 \leq d \leq n-2$ the regularity theory for solutions of Plateau Problem in the framework of integral currents states that a solution can be represented as a closed set $\Sigma$ which is smooth (analytic) out of the boundary $\Gamma$ and a closed singular set $\mathrm{Sing}(\Sigma)$ with (Hausdorff) dimension at most $d-2$.
  Example 3 shows (for $n, d$ even) that this bound cannot be improved.

- For $d = n-1$ a much stronger result holds, since the dimension of $\mathrm{Sing}(\Sigma)$ is at most $d-7$.
  (Thus, for $d \leq 6$, $\mathrm{Sing}(\Sigma) = \emptyset$ and $\Sigma$ is analytic away from $\Gamma$.)
  Example 4 shows (for $n=8$, $d=7$) that this bound cannot be improved.

# 2 Recap of basic measure theory

We will use only few classes of measures

## Outer measures

That is, $\sigma$-subadditive measures on a given set $X$.
Outer measures are relevant (to us) only because they are easier to construct than $\sigma$-additive measures.

## Borel measures

That is, $\sigma$-additive positive measures on the Borel $\sigma$-algebra $\mathcal{B}(X)$ of a topological space $X$.

## Real- or vector-valued Borel measures

That is, $\sigma$-additive measures on $\mathcal{B}(X)$ with values in $\mathbb{R}$ or in a normed space $E$.

## Important remarks!

Except for outer measures, which are naturally defined on all subsets of a given ambient set $X$, and will have a very limited role in this course, I will only consider measure defined on the Borel $\sigma$-algebra $\mathcal{B}(X)$ of some topological space $X$.
Moreover I assume from start that:

$X$ is a locally compact, separable metric space. $\longrightarrow$ Basic example: open set in $\mathbb{R}^n$

I also assume that, unless I say otherwise:

sets, maps and functions are always Borel measurable.

Note indeed that for most applications in Analysis there is no need to consider larger classes of functions or sets.

Quite often in the next lectures the term "measure" will refer to positive finite Borel measures.

## $\boxed{2.1}$ Borel measures

### Notation and terminology

Given $\mu, \lambda$ measures on $X$, $F \subset X$, $\rho : X \to [0,+\infty]$, $f : X \to X'$ (all Borel) then:

F supports $\mu$ means $\mu(X \setminus F) = 0$;

$\mathrm{supp}(\mu)$ = support of $\mu$ = smallest closed set that supports $\mu$;

$\lambda \ll \mu$ = "$\lambda$ is absolutely continuous w.r.t. $\mu$", that is, $\lambda(E) > 0 \Rightarrow \mu(E) > 0$;

$\lambda \perp \mu$ = "$\lambda$ and $\mu$ are mutually singular", that is, $\lambda$ and $\mu$ are supported on two disjoint (Borel) sets (it does NOT mean $\mathrm{supp}(\mu) \cap \mathrm{supp}(\lambda) = \emptyset$);

$M(\mu) = \|\mu\| := \mu(X)$ = mass of $\mu$;

$\rho\mu$ = measure given by $[\rho\mu](E) := \int_E \rho \, d\mu$, note that $M(\rho\mu) = \|\rho\|_{L^1(\mu)}$;

$\mu \llcorner F$ = restriction of $\mu$ to $F$, $[\mu \llcorner F](E) := \mu(E \cap F)$.

$f_\# \mu$ = push-forward of $\mu$ according to $f$, that is, measure on $X'$ given by $[f_\# \mu](E) := \mu(f^{-1}(E))$.

$\mathcal{M}^+(X)$ = space of positive finite measures on $X$.

### Basic results

Theorem 1 If $\mu$ is locally finite then it is regular, that is, for every (Borel) set $E$ there holds

$$\mu(E) = \inf\{\mu(A) \text{ s.t. } A \text{ open}, A \supset E\}$$
$$= \sup\{\mu(K) \text{ s.t. } K \text{ compact}, K \subset E\}$$

## Theorem 2 (Lebesgue + Radon - Nikodym)

Let $\lambda, \mu$ be locally finite measures. Then

Lebesgue decomposition

(a) $\lambda = \lambda_a + \lambda_s$ with $\lambda_a \ll \mu$ & $\lambda_s \perp \mu$ and this decomposition is unique

Radon-Nikodym Th.

(b) $\lambda_a = \rho \cdot \mu$ for some $\rho \in L^1_{loc}(\mu)$, $\rho \geq 0$

If in addition $X \subset \mathbb{R}^n$ or $\mu$ is asymptotically doubling

(c) for $\mu$-a.e. $x$ there holds

Lebesgue Differentiation Th.

$$\rho(x) = \lim_{r\to 0} \frac{\lambda(\overline{B(x,r)})}{\mu(\overline{B(x,r)})} = \lim_{r\to 0} \frac{\lambda_a(\overline{B(x,r)})}{\mu(\overline{B(x,r)})}$$

Recall that $\mu$ is asymptotically doubling if

$$\limsup_{r\to 0} \frac{\mu(\overline{B(x,2r)})}{\mu(\overline{B(x,r)})} < +\infty \quad \text{for } \mu\text{-a.e. } x$$

## Remarks

The function $\rho$ in (b) is called Radon-Nikodym density of $\lambda_a$ w.r.t. $\mu$.
Statement (c) shows that $\rho$ can be recovered from the values of $\lambda$ and $\mu$ on balls.
It does not matter whether closed or open balls.

Concerning the validity of (c), note that there exist a compact metric space $X$ and finite measures $\lambda, \mu$ on $X$ such that $\lambda \neq \mu$ but $\lambda(B) = \mu(B)$ for every ball $B$. Possibly replacing $\mu$ with $\frac{1}{2}(\lambda+\mu)$ we further obtain that $\lambda \ll \mu$ and then $\lambda = \rho\mu$ by (b), and $\rho$ is NOT $\mu$-a.e. equal to 1.
Thus (c) does not hold.

**Theorem 3** (Points of $L^p$-approximate continuity)
Let $\mu$ be a locally finite measure on $X$, and assume that either $X \subset \mathbb{R}^n$ or $\mu$ is asymptotically doubling.
Let $f \in L^p_{loc}(\mu)$ with $1 \leq p < +\infty$. Then

$$\lim_{r \to 0} \fint_{\overline{B(\bar{x},r)}} |f(x) - f(\bar{x})| \, d\mu(x) = 0 \quad \text{for } \mu\text{-a.e. } \bar{x}.$$

The symbol $\fint_X$ stands for average over $X$: $\fint_X g \, d\mu := \frac{1}{\mu(X)} \int_X g \, d\mu$.

**Corollary 4**
Take $X$ and $\mu$ as in Theorem 2 and $f \in L^1_{loc}(\mu)$. Then

$$\lim_{r \to 0} \fint_{\overline{B(\bar{x},r)}} f(x) \, d\mu(x) = f(\bar{x}) \quad \text{for } \mu\text{-a.e. } \bar{x}.$$

In particular if we take $f = 1_E$ with $E$ Borel we obtain

$$\boxed{\Theta_\mu(E,\bar{x})} := \lim_{r \to 0} \frac{\mu(E \cap \overline{B(\bar{x},r)})}{\mu(\overline{B(\bar{x},r)})} = \begin{cases} 1 & \text{for } \mu\text{-a.e. } \bar{x} \in E, \\ 0 & \text{for } \mu\text{-a.e. } \bar{x} \in X \setminus E. \end{cases}$$

$\mu$-density of $E$ at $\bar{x}$

**Remarks**
Theorem 3 and Corollary 4 hold even if $f$ takes values in a finite dimensional normed space.
They hold even if takes values in a certain class of (infinite dimensional) Banach space, which includes separable Hilbert spaces (but not all Banach spaces).

Later in this course I will prove statement (c) of Theorem 1, and Theorem 2.