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Classical Solutions of Nonautonomous Riccati Equations Arising in Parabolic Boundary Control Problems

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Abstract. An abstract linear-quadratic regulator problem over finite time horizon is considered; it covers a large class of linear nonautonomous parabolic systems in bounded domains, with boundary control of Dirichlet or Neumann type. The associated differential Riccati equation is studied from the point of view of semigroup theory; it is shown to have a classical, explicitly represented solution for very general final data; weighted Hölder regularity results for the optimal pair are deduced.

Key Words. Optimal control, Riccati equation, Boundary control, Parabolic systems.

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0. Introduction

Let H, U be complex Hilbert spaces; for fixed $T > 0$ we consider the following linear-quadratic regulator problem:

$$\begin{cases} \text{minimize} \\ \quad J(u) := \displaystyle\int_0^T \{(M(t)y(t) \mid y(t))_H + (N(t)u(t) \mid u(t))_U\}\, dt \\ \qquad\qquad + (P_T y(T) \mid y(T))_H & (0.1) \\ \text{over all controls } u \in L^2(0,T;U) \text{ subject to the state equation} \\ \quad y(t) = U(t,0)x - \displaystyle\int_0^t U(t,r)A(r)G(r)u(r)\, dr, \qquad t \in [0,T]; & (0.2) \end{cases}$$

here $\{M(t)\}$ and P_T are positive, bounded, self-adjoint operators in H, $\{N(t)\}$ are positive, bounded, self-adjoint operators in U, x is an element of H, each $A(t)$ generates an analytic semigroup $\{e^{\tau A(t)}\}$ in H, $\{U(t,s)\}$ is the evolution operator associated to $\{A(t)\}$, and $G(t)$ is the "Green map" relative to $A(t)$. More precise assumptions on $\{A(t)\}$, $\{G(t)\}$, $\{M(t)\}$, $\{N(t)\}$, and P_T are listed in Section 1.

The state equation (0.2) represents a large class of linear parabolic nonautonomous initial-boundary value problems, with boundary controls of Dirichlet or Neumann type: see Section 9 below for some typical examples. Looking for a pointwise feedback optimal control for problem (0.1)–(0.2), the main step is the study of the associated Riccati equation, whose integral version is

$$P(t) = U(T,t)^* P_T U(T,t) + \int_t^T U(r,t)^* \times [M(r) - P(r)A(r)G(r)N(r)^{-1}G(r)^* A(r)^* P(r)]U(r,t)\,dr, \tag{0.3}$$

and whose differential version is

$$\begin{cases} P'(t) + A(t)^* P(t) + P(t)A(t) \\ \quad = -M(t) + P(t)A(t)G(t)N(t)^{-1}G(t)^* A(t)^* P(t), \\ P(T) = P_T. \end{cases} \tag{0.4}$$

The Ricatti equation and its corresponding control problem in the autonomous case have been widely studied by several people and the whole theory is, more or less, complete: we quote, among others, [B], [LT1], [F1], [DI1], [F2], [F4], [LT3], and [LT4]. Two different approaches are available: (i) the variational method, which starts from the Euler equation for the cost functional and yields explicit formulas which express in terms of the data both the optimal pair and the Riccati operator, and (ii) the dynamic programming method, which solves directly the Riccati equation and obtains, through the Riccati operator, a feedback formula for the optimal control in terms of the optimal state. Both methods are carefully described in the survey papers [LT2] and [BDDM].

Only a few papers deal with the nonautonomous control problem (0.1)–(0.2); [Li] and [DS] are based on variational techniques, whereas in [DI2] and [AFT] the dynamic programming approach is used.

In [AFT] it was shown that under certain abstract assumptions, which are naturally fulfilled in the concrete parabolic problems of Section 9, (0.3) has a unique global solution $P(\cdot)$, where $P(t)$ is a positive, bounded, self-adjoint operator for each $t \in [0,T[$, provided the final datum P_T is suitably regular; consequently one is able to find an optimal pair $(\hat{u},\hat{y})$ for problem (0.1)–(0.2) in the space $L^2(0,T;U) \times L^2(0,T;H)$. On the other hand, in the autonomous case the minimal assumption on P_T is more general and in addition the optimal pair turns out to enjoy some regularity properties, as shown in [LT1], [LT3], and [LT4].

Thus our main goal here is to extend as far as possible the results of [LT1], [LT3], and [LT4] to the nonautonomous situation. To this purpose we were not able to repeat, for a general choice of P_T, the direct proof of existence and uniqueness of mild solutions of (0.3), given in [AFT] by means of the dynamic programming technique; here we follow instead the variational approach of [LT1] and [LT3], adapting and refining it according to the nonautonomous situation, through the extensive use of the nonautonomous theory of abstract parabolic equations developed in [AT1], [AT2], [A1], [AT3], [A2], and [AFT]. In fact not only do we generalize to this situation almost all statements of [LT1], [LT3], and [LT4], but we even improve some of them.

Here is a list of our main results:

(i) We show that the solution of the Riccati equation (0.3) is in fact classical, i.e., $P(\cdot)$ is continuously differentiable as an $\mathcal{L}(H)$-valued function and satisfies (0.4) in the sense of $\mathcal{L}(H)$, provided the operator $A(t)^*P(t) + P(t)A(t)$ is replaced by its bounded extension $\Lambda(t)P(t)$ (see Section 7 for details). When $A(t) \equiv A$ this result was known in the case of distributed control, see [D], and in the case of boundary control, under additional regularity assumptions on P_T (see [LT4]). We also derive some weighted Hölder regularity results for the optimal pair.

(ii) We take the final datum P_T essentially in the largest possible class, as the counterexample in [F3] and the remarks in Section 7 of [LT3] show; in addition we prove that the Riccati operator $P(t)$ is always strongly continuous at $t = T$, and give necessary and sufficient conditions in order that $P(t) \to P_T$ in $\mathcal{L}(H)$ as $t \to T^-$.

We believe that the results of this paper can be generalized to cover nonautonomous control problems over infinite time horizon, thus improving those of [AT4] and [A3].

We now describe the contents of the following sections. Section 1 contains the list of our assumptions; in Section 2 the control problem is properly posed and some basic operators are introduced. Section 3 is devoted to the preliminary study of the optimal pair; in Section 4 some pointwise estimates for the optimal pair are proved. In Section 5 we introduce the state operator $\varphi(t, s)$ and the Riccati operator $P(t)$, recalling their elementary properties; Section 6 concerns the differentiability of $\varphi(t, s)$. In Section 7 we define the unbounded operator $\Lambda(t)$, acting in the space of bounded self-adjoint operators in H, and describe its properties, whereas in Section 8 we show that $\Lambda(t)P(t)$ is well defined as a bounded operator in H, and that $P(t)$ solves the differential Riccati equation in the sense of $\mathcal{L}(H)$. Finally in Section 9 we describe some concrete examples and show that our abstract assumptions are fulfilled there, thus giving a motivation for them. There are also two appendices: in Appendix A some useful function spaces, often involved in this paper, are described, and Appendix B contains a short survey on strict and classical solutions of abstract nonautonomous parabolic equations.

We are forced to omit the proofs of the statements of Section 6, because of their length and technical complexity, which would have enlarged the size of this paper too much. A detailed proof of such statements, as well as further remarks and related results, can be found in [AT5].

We conclude this section by listing some notations. If X is a Banach space and $I \subseteq \mathbb{R}$ is an interval, we use the usual Lebesgue spaces $L^p(I, X)$, $1 \le p \le \infty$, and the usual Hölder spaces $C^\gamma(I, X)$, $C^{k+\gamma}(I, X)$ ($\gamma \in]0, 1[$, $k \in \mathbb{N}$); when $\gamma = 0$ we write $C(I, X)$ instead of $C^0(I, X)$.

If X, Y are Banach spaces, $\mathcal{L}(X, Y)$ is the space of bounded linear operators $T\colon X \to Y$ (and we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$). If H is a Hilbert space, $\Sigma(H)$ is the space of self-adjoint operators $T \in \mathcal{L}(H)$ and $\Sigma^+(H)$ is the space of self-adjoint operators $T \in \mathcal{L}(H)$ which are positive, i.e., $(Tx \mid x)_H \ge 0$ for each $x \in H$. If H is a Hilbert space and T is a linear operator in H, we denote by D_T, $\sigma(T)$, and $\rho(T)$ the domain of T, the spectrum of T, and the resolvent set of T; we denote by T^* the adjoint operator of T (whenever it exists).

Finally if $m \in \mathbb{N}^+$ and Ω is a bounded open set of $\mathbb{R}^n$, we use the Lebesgue and Hölder spaces of $\mathbb{C}^m$-valued functions $[L^p(\Omega)]^m$ and $[C^{k+\gamma}(\bar{\Omega})]^m$ ($k \in \mathbb{N}, \gamma \in]0,1[, p \in [1,\infty]$), and the usual Sobolev spaces $[W^{\gamma,p}(\Omega)]^m$, $[W^{\gamma,p}(\partial\Omega)]^m$ $p \in [1,\infty[, \gamma \in \mathbb{R}$), and $[W_0^{\gamma,p}(\Omega)]^m$ ($p \in [1,\infty[, \gamma \in]1/p,\infty[$).

1. Assumptions

We list here our abstract assumptions.

Hypothesis 1.1. *For each $t \in [0,T]$, $A(t) : D_{A(t)} \subseteq H \to H$ is a closed linear operator generating an analytic semigroup $\{e^{\tau A(t)}, \tau \geq 0\}$; in particular there exist $M > 0$ and $\theta \in]\pi/2, \pi[$ such that*

$$\|[\lambda - A(t)]^{-1}\|_{\mathcal{L}(H)} \leq M(1+|\lambda|)^{-1}, \qquad \forall \lambda \in \overline{S(\theta)}, \quad \forall t \in [0,T], \tag{1.1}$$

where $S(\theta) = \{z \in \mathbb{C} : |\arg z| < \theta\}$.

Hypothesis 1.2. *There exist $N > 0$ and $\rho, \mu \in]0,1]$ with $\delta := \rho + \mu - 1 \in]0,1[$, such that*

$$\begin{aligned} &\|A(t)[\lambda - A(t)]^{-1}[A(t)^{-1} - A(s)^{-1}]\|_{\mathcal{L}(H)} \\ &\qquad + \|A(t)^*[\lambda - A(t)^*]^{-1}\left[[A(t)^*]^{-1} - [A(s)^*]^{-1}\right]\|_{\mathcal{L}(H)} \\ &\quad \leq N|t-s|^{\mu}(1+|\lambda|)^{-\rho}, \qquad \forall \lambda \in \overline{S(\theta)}, \qquad \forall t,s \in [0,T]. \end{aligned} \tag{1.2}$$

Hypothesis 1.3. *$\{U(t,s), 0 \leq s \leq t \leq T\}$ is the evolution operator relative to $\{A(t), t \in [0,T]\}$; in particular,*

$$\begin{aligned} &\|[-A(t)]^{\eta}U(t,s)[-A(s)]^{-\gamma}\|_{\mathcal{L}(H)} + \|[-A(s)^*]^{\eta}U(t,s)^*[-A(t)^*]^{-\gamma}\|_{\mathcal{L}(H)} \\ &\quad \leq M_{\eta\gamma}[1 + (t-s)^{\gamma-\eta}] \qquad \text{for} \quad 0 \leq s < t \leq T, \quad \eta, \gamma \in [0,1]. \end{aligned} \tag{1.3}$$

Hypothesis 1.4. *The number $\delta = \rho + \mu - 1$ is such that*

$$\begin{aligned} &\|[-A(t)]^{\eta}U(t,s)[-A(s)]^{-\gamma} - [-A(\tau)]^{\eta}U(\tau,s)[-A(s)]^{-\gamma}\|_{\mathcal{L}(H)} \\ &\qquad \leq N_{\gamma\eta}(t-\tau)^{\delta}[1 + (\tau-s)^{\gamma-\eta-\delta}] \\ &\qquad\quad \text{for} \quad 0 \leq s < \tau \leq t \leq T, \quad \eta, \gamma \in [0,1], \end{aligned} \tag{1.4}$$

$$\begin{aligned} &\|[-A(\sigma)^*]^{\eta}U(t,\sigma)^*[-A(t)^*]^{-\gamma} - [-A(s)^*]^{\eta}U(t,s)^*[-A(t)^*]^{-\gamma}\|_{\mathcal{L}(H)} \\ &\qquad \leq N_{\gamma\eta}(\sigma-s)^{\delta}[1 + (t-\sigma)^{\gamma-\eta-\delta}] \\ &\qquad\quad \text{for} \quad 0 \leq s \leq \sigma < t \leq T, \quad \eta, \gamma \in [0,1], \end{aligned} \tag{1.5}$$

all operators being strongly continuous with respect to t, τ, σ, and s.

Hypothesis 1.5. *For each $t \in [0,T]$, $G(t) \in \mathcal{L}(U,H)$ and there exists $\alpha \in]\delta, \frac{1}{2}[$ such that*

$$[-A(\cdot)]^{\alpha}G(\cdot) \in C^{\delta}([0,T], \mathcal{L}(U,H)). \tag{1.6}$$

Hypothesis 1.6. *We have* $M(\cdot) \in C^\delta([0,T],\Sigma^+(H))$, $N(\cdot) \in C^\delta([0,T],\Sigma^+(U))$, *and there exists* $\nu > 0$ *such that* $N(t) \geq \nu$, *i.e.*, $(N(t)u|u)_U \geq \nu\|u\|_U^2$ *for each* $u \in U$ *and* $t \in [0,T]$.

Hypothesis 1.7. $P_T \in \Sigma^+(H)$ *and in addition the linear operator* $P_T^{1/2}L_{0T}\colon D(L_{0T}) \subseteq L^2(0,T;U) \to H$ *is closed* (*the operators* L_{sT} *are defined in* (2.7) *below*).

Remark 1.8. (i) Hypotheses 1.1 and 1.2 arise naturally in the study of the Cauchy problem for abstract linear nonautonomous parabolic equations in Hilbert spaces; they are fulfilled in several concrete nonautonomous parabolic problems with homogeneous data at the boundary (see Section 2 of [AFT]). They allow us to construct the evolution operator relative to the family $\{A(t)\}$, and to prove its properties: in fact, Hypotheses 1.3 and 1.4 are consequences of the previous ones. This was shown in Proposition 2.8(iv) and Corollary 2.10 of [AFT].

(ii) Hypothesis 1.5 concerns the smoothness of the abstract "Green map" $G(t)$, whose realization in concrete problems yields the lifting of the nonzero datum at the boundary, i.e., transforms the nonhomogeneous initial-boundary value problem into a homogeneous one by a modification of the right member of the equation. These assumptions hold true in the examples of [AFT] and [A3] (compare with Theorem 9.3 below), possibly with some $\alpha \geq \frac{1}{2}$. However, we note that the smaller α is, the harder is the situation: in particular, when $\alpha < \frac{1}{2}$ (the "nonsmoothing case" of [LT3]) the optimal pair will have a singularity at $t = T$. Hence we assume this to be the case; of course when $\alpha \geq \frac{1}{2}$ better results could be proved. We also remark that the restriction $\delta < \alpha$ can always be fulfilled just by choosing a smaller δ, which is possible in view of Proposition A.4(iii) in Appendix A below.

(iii) Hypotheses 1.6 and 1.7 are regularity conditions on the data of the control problem: the former is a standard one and might be weakened by allowing a moderate degree of unboundedness of $M(\cdot)$ (compare with Section 2.3 of [F4]); the latter is sufficient, as in [LT3], to prove existence of the optimal control, and to define the Riccati operator $P(t)$, and in addition it allows us to prove that $P(t)$ solves the differential Riccati equation (0.4). We note that, due to the closedness of L_{0T} (see Section 2), Hypothesis 1.7 is automatically satisfied if P_T has a bounded inverse; we also remark that it might be weakened by assuming that the operator $P_T^{1/2}L_{0T}$ is just closable (see Remark 3.2 below).

2. The State Equation

We follow closely [LT1] and [LT3], adapting their method to the nonautonomous case. We consider the control problem (0.1)–(0.2) with initial point $s \in [0,T[$:

$$\begin{cases} \text{minimize} \\ \quad J_s(u) := \displaystyle\int_s^T \{(M(t)y(t) \mid y(t))_H + (N(t)u(t) \mid u(t))_U\}\,dt \\ \qquad\qquad + (P_T y(T) \mid y(T))_H & (2.1) \\ \text{over all controls } u \in L^2(s,T;U) \text{ subject to the state equation} \\ \quad y(t) = U(t,s)x - \displaystyle\int_s^t U(t,r)A(r)G(r)u(r)\,dr, \qquad t \in [s,T]. & (2.2) \end{cases}$$

We introduce the linear operator L_s which represents the integral term in the state equation (2.2):

$$(L_s u)(t) := -\int_s^t U(t,r)A(r)G(r)u(r)\,dr, \qquad t \in [s,T[, \tag{2.3}$$

or, more exactly,

$$(L_s u)(t) = \int_s^t \big[[-A(r)^*]^{1-\alpha}U(t,r)^*\big]^* \big[[-A(r)]^\alpha G(r)\big]u(r)\,dr, \qquad t \in [s,T[.$$

By Hypotheses 1.3 and 1.5 it is easy to see that

$$\|L_s\|_{\mathcal{L}(L^2(s,T,U),L^2(s,T,H))} \le c(T-s)^\alpha; \tag{2.4}$$

in addition the adjoint operator L_s^* is defined for each $v \in L^2(s,T,H)$ by

$$(L_s^* v)(t) = -G(t)^*A(t)^* \int_t^T U(r,t)^* v(r)\,dr, \qquad t \in [s,T[, \tag{2.5}$$

or, more exactly,

$$(L_s^* v)(t) = \big[[-A(t)]^\alpha G(t)\big]^* \int_t^T [-A(t)^*]^{1-\alpha}U(r,t)^* v(r)\,dr, \qquad t \in [s,T[,$$

and, of course,

$$\|L_s^*\|_{\mathcal{L}(L^2(s,T,H),L^2(s,T,U))} \le c(T-s)^\alpha. \tag{2.6}$$

As L_s acts on L^2 functions, when $\alpha \le \frac{1}{2}$ it is not true in general that $(L_s u)(T)$ is meaningful as an element of H. Thus we set

$$\begin{cases} D(L_{sT}) := \{u \in L^2(s,T,U) : T \text{ is a Lebesgue point for } L_s u\}, \\ L_{sT}(u) := \lim\limits_{r\to 0^+} \dfrac{1}{r}\displaystyle\int_{T-r}^T (L_s u)(t)\,dt. \end{cases}$$

We remark that $D(L_{sT})$ is dense in $L^2(s,T,U)$, since in particular when $u \in C([s,T],U)$ we have $L_s u \in C([s,T],H)$ and

$$L_{sT}(u) = -\int_s^T U(T,r)A(r)G(r)u(r)\,dr.$$

Moreover, using Hypotheses 1.3 and 1.4 it is not difficult to see that L_{sT} is a closable operator, and its closure is the operator, still denoted by L_{sT}, defined by

$$\begin{cases} D(L_{sT}) := \Big\{u \in L^2(s,T,U) : \\ \qquad\qquad \displaystyle\int_s^T [-A(T)]^{-\eta}U(T,r)A(r)G(r)u(r)\,dr \in D([-A(T)]^\eta)\Big\}, \\ L_{sT}(u) := -[-A(T)]^\eta \displaystyle\int_s^T [-A(T)]^{-\eta}U(T,r)A(r)G(r)u(r)\,dr, \end{cases} \tag{2.7}$$

where $\eta > \frac{1}{2} - \alpha$ is a fixed number.

The adjoint of L_{sT} is the operator L^*_{sT} given by

$$\begin{cases} D(L^*_{sT}) = \{y \in H : G(\cdot)^* A(\cdot)^* U(t,\cdot)^* y \in L^2(s,T;U)\}, \\ L^*_{sT} y = -G(\cdot)^* A(\cdot)^* U(T,\cdot)^* y; \end{cases} \tag{2.8}$$

clearly it holds that

$$D(L^*_{sT}) \begin{cases} = H & \text{if} \quad \alpha > \frac{1}{2}, \\ \supseteq D\left([-A(T)^*]^\eta\right), \quad \forall \eta > \frac{1}{2} - \alpha, & \text{if} \quad \alpha \leq \frac{1}{2}. \end{cases}$$

Proposition 2.1. *Under Hypotheses* 1.1–1.5 *and* 1.7, *let the operator* L_{sT} *be defined by* (2.7). *Then*:

(i) $D(L^*_{sT}) = D(L^*_{0T})$ *and* $L^*_{sT} y = L^*_{0T} y_{|]s,T[}$, $\forall y \in D(L^*_{0T})$, $\forall s \in [0,T[$;
(ii) $\lim_{s \to T^-} \|L^*_{sT} y\|_{L^2(s,T,U)} = 0$, $\forall y \in D(L^*_{0T})$.

Proof. It is a standard consequence of (2.7) and (2.8). □

Next, due to the presence of the possibly undefined vector $y(T)$ in the cost functional, we rewrite J_s in the following way:

$$J_s(u) := \begin{cases} \displaystyle\int_s^T \{(M(t)y(t) \mid y(t))_H + (N(t)u(t) \mid u(t))_U\}\, dt \\ \qquad +(P_T y(T) \mid y(T))_H \qquad \text{if} \quad u \in D(L_{sT}), \\ +\infty \qquad \text{if} \quad u \in L^2(s,T,U) - D(L_{sT}). \end{cases} \tag{2.9}$$

By (2.3) the state equation (2.2) can be rewritten as

$$y(t) = U(t,s)x + (L_s u)(t), \tag{2.10}$$

and using Hypothesis 1.7 it is an easy task to verify that the functional J_s is strictly convex and continuous in $L^2(s,T,U)$. Thus for the control problem (2.9)–(2.10) a unique optimal pair $(\hat{y},\hat{u}) \in L^2(s,T,H) \times D(L_{sT})$ exists for each fixed $s \in [0,T[$ and $x \in H$; we denote it by $(\hat{y}(\cdot,s;x), \hat{u}(\cdot,s;x))$. By (2.10) and (2.7) we have

$$\hat{y}(t,s;x) = \begin{cases} U(t,s)x + L_s[\hat{u}(\cdot,s;x)](t) & \text{if} \quad 0 \leq s \leq t < T, \\ U(T,s)x + L_{sT}[\hat{u}(\cdot,s;x)] & \text{if} \quad 0 \leq s < t = T. \end{cases} \tag{2.11}$$

In addition, uniqueness implies that

$$\hat{y}(t,s;x) = \hat{y}(t,r;\hat{y}(r,s;x)) \qquad \text{for} \quad 0 \leq s \leq r \leq t \leq T, \tag{2.12}$$

$$\hat{u}(t,s;x) = \hat{u}(t,r;\hat{y}(r,s;x)) \qquad \text{for} \quad 0 \leq s \leq r \leq t < T. \tag{2.13}$$

3. The Optimal Pair

Following again [LT1] and [LT3] we want to get some representation formulas for the optimal control $\hat{u}(\cdot,s;x)$. By Hypothesis 1.7, $P_T^{1/2} L_{0T}$ is a closed operator with domain $D(L_{0T})$; hence it is clear that for each $s \in [0,T[$ the operator $P_T^{1/2} L_{sT}$ is closed too.

Then setting, for each $s \in [0, T[$,

$$\begin{cases} X_s := D(L_{sT}), \\ (u \mid v)_{X_s} := (N(\cdot)u \mid v)_{L^2(s,T,U)} + (P_T L_{sT} u \mid L_{sT} v)_H, \end{cases}$$

X_s is a Hilbert space; by Hypothesis 1.6 we have the continuous inclusions $X_s \subseteq L^2(s, T, U) \subseteq X_s^*$ and, for each $s \in [0, T[$,

$$\begin{aligned} \|u\|_{L^2(s,T,U)} &\le \nu^{-1/2}\|u\|_{X_s}, \qquad \forall u \in X_s, \\ \|u\|_{X_s^*} &\le \nu^{-1/2}\|u\|_{L^2(s,T,U)}, \qquad \forall u \in L^2(s, T, U). \end{aligned} \tag{3.1}$$

By definition of X_s we also have

$$\|P_T^{1/2} L_{sT}\|_{\mathcal{L}(X_s,H)} \le 1, \qquad \|L_{sT}^* P_T^{1/2}\|_{\mathcal{L}(H,X_s^*)} \le 1, \qquad \forall s \in [0, T[. \tag{3.2}$$

Plugging the state equation (2.10) into the cost functional (2.9) we obtain the following expression for J_s:

$$\begin{aligned} J_s(u) = \int_s^T &\{(M(t)[U(t,s)x + (L_s u)(t)] \mid U(t,s)x + (L_s u)(t))_H \\ &+ (N(t)u(t) \mid u(t))_U\}\, dt \\ &+ (P_T[U(T,s)x + L_{sT}u] \mid U(T,s)x + L_{sT}u)_H, \qquad \forall u \in X_s. \end{aligned} \tag{3.3}$$

The optimal control $\hat{u}(\cdot, s; x)$ solves the Euler equation

$$\begin{cases} \hat{u}(\cdot, s; x) \in X_s, \\ \left[\dfrac{d}{dh} J_s(\hat{u}(\cdot, s; x) + hv)\right]_{h=0} = 0, \qquad \forall v \in X_s, \end{cases}$$

i.e.,

$$\begin{cases} \hat{u}(\cdot, s; x) \in X_s, \\ \displaystyle\int_s^T \{(M(t)[U(t,s)x + (L_s \hat{u}(\cdot, s; x))(t)] \mid (L_s v)(t))_H \\ \qquad + (N(t)\hat{u}(t, s; x) \mid v(t))_U\}\, dt \\ \quad + (P_T[U(T,s)x + (L_{sT}\hat{u}(\cdot, s; x))] \mid L_{sT} v)_H = 0, \qquad \forall v \in X_s. \end{cases} \tag{3.4}$$

Hence we get, using (2.5), (2.11),

$$\begin{aligned} &(P_T \hat{y}(T, s; x) \mid L_{sT} v)_H \\ &\quad = -\int_s^T \{L_s^*[M(\cdot)\hat{y}(\cdot, s; x)](t) + N(t)\hat{u}(t, s; x) \mid v(t))_U\, dt, \qquad \forall v \in X_s; \end{aligned}$$

this implies $P_T \hat{y}(T, s; x) \in D(L_{sT}^*)$ and

$$\begin{aligned} &(L_{sT}^* P_T \hat{y}(T, s; x) + L_s^*[M(\cdot)\hat{y}(\cdot, s; x)] + N(\cdot)\hat{u}(\cdot, s; x) \mid v)_{L^2(s,T,U)} = 0, \\ &\quad \forall v \in X_s. \end{aligned} \tag{3.5}$$

As X_s is dense in $L^2(s, T, U)$, we conclude that

$$\hat{u}(\cdot, s; x) = -N(\cdot)^{-1}\left[L_{sT}^* P_T \hat{y}(T, s; x) + L_s^*[M(\cdot)\hat{y}(\cdot, s; x)]\right] \tag{3.6}$$

as an element of $L^2(s,T,U)$; in addition, by (3.5) and (2.11) we get that

$$\begin{aligned}(N(\cdot) + L_{sT}^* P_T L_{sT} + L_s^* M(\cdot) L_s)\hat{u}(\cdot, s; x) \\ = -L_{sT}^* P_T U(T,s)x - L_s^* M(\cdot) U(\cdot, s)x\end{aligned} \tag{3.7}$$

as an element of X_s^*.

We again imitate [LT1] and [LT3] by introducing the operator

$$\Lambda_{sT} w := N(\cdot)w(\cdot) + L_{sT}^* P_T L_{sT} w + L_s^* M(\cdot) L_s w, \qquad w \in X_s; \tag{3.8}$$

clearly, $\Lambda_{sT} \in \mathcal{L}(X_s, X_s^*)$. Moreover, Λ_{sT} is an unbounded operator in $L^2(s,T,U)$ with domain

$$D(\Lambda_{sT}) = \{w \in X_s : P_T L_{sT} w \in D(L_{sT}^*)\}. \tag{3.9}$$

Proposition 3.1. *Under Hypotheses* 1.1–1.7 *the operator* Λ_{sT} *is one-to-one from* $D(\Lambda_{sT})$ *onto* $L^2(s,T,U)$ *and from* X_s *onto* X_s^*, *with bounded inverse in both cases; in particular,* $\Lambda_{sT}\colon D(\Lambda_{sT}) \subseteq L^2(s,T;U) \to L^2(s,T;U)$ *is closed.*

Proof. Define, for $w, x \in X_s$,

$$\begin{aligned}a(w,z) &:= (w \mid z)_{X_s} + (M(\cdot)L_s w \mid L_s z)_{L^2(s,T,H)} \\ &= (N(\cdot)w \mid z)_{L^2(s,T,U)} + (P_T L_{sT} w \mid L_{sT} z)_H \\ &\quad + (M(\cdot)L_s w \mid L_s z)_{L^2(s,T,H)}.\end{aligned} \tag{3.10}$$

Obviously,

$$a(w,z) = [\Lambda_{sT} w](z), \qquad \forall w, z \in X_s, \tag{3.11}$$

$$a(w,z) = (\Lambda_{sT} w \mid z)_{L^2(s,T,U)}, \qquad \forall w \in D(\Lambda_{sT}), \quad \forall z \in X_s. \tag{3.12}$$

The bilinear form $a(\cdot,\cdot)$ is clearly continuous and coercive on X_s: hence by the Lax–Milgram theorem for each $f \in X_s^*$ there exists a unique $\eta \in X_s$ such that $a(\eta, z) = f(z)$ for each $z \in X_s$; by (3.11) this means $\Lambda_{sT}\eta = f$ as an element of X_s^*, i.e., Λ_{sT} is one-to-one from X_s onto X_s^*. Moreover we get

$$\|\Lambda_{sT}^{-1}\|_{\mathcal{L}(X_s^*, X_s)} \le 1. \tag{3.13}$$

In particular, for each $y \in L^2(s,T,U) \subseteq X_s^*$ there exists a unique $\eta \in X_s$ such that, by (3.12),

$$(\Lambda_{sT}\eta \mid z)_{L^2(s,T,U)} = (y \mid z)_{L^2(s,T,U)}, \qquad \forall z \in X_s;$$

consequently it is easy to see that

$$|(P_T L_{sT}\eta \mid L_{sT} z)_H| \le c(y,\eta)\|z\|_{L^2(s,T,U)},$$

so that $P_T L_{sT}\eta \in D(L_{sT}^*)$, i.e., $\eta \in D(\Lambda_{sT})$ and

$$(\Lambda_{sT}\eta \mid z)_{L^2(s,T,U)} = a(\eta, z) = (y \mid z)_{L^2(s,T,U)}, \qquad \forall z \in X_s. \tag{3.14}$$

By density, we conclude that $\Lambda_{sT}\eta = y$ as an element of $L^2(s, T, U)$, i.e., Λ_{sT} is one-to-one from $D(\Lambda_{sT})$ onto $L^2(s, T, U)$; moreover, choosing $z = \eta$ in (3.14) we readily get

$$\|\Lambda_{sT}^{-1}\|_{\mathcal{L}(L^2(s,T,U))} \leq 1. \tag{3.15}$$

The proof is complete. □

Using the operator Λ_{sT} we can rewrite (3.7) as

$$\hat{u}(\cdot, s; x) = -\Lambda_{sT}^{-1}[L_{sT}^* P_T U(T, s)x + L_s^* M(\cdot)U(\cdot, s)x], \tag{3.16}$$

which expresses the optimal control in terms of the initial state x.

Remark 3.2. Assuming in Hypothesis 1.7 that $P_T^{1/2} L_{0T}$ is only closable, the above arguments apply just by replacing the operators $P_T^{1/2} L_{sT}$ by their closed extensions $\overline{P_T^{1/2} L_{sT}}$. The functional J_s can be extended finitely to any $u \in D(\overline{P_T^{1/2} L_{sT}})$, since (3.3) can be rewritten as

$$\begin{aligned} J_s(u) = \int_s^T &\Big\{\big(M(t)[U(t, s)x + (L_s u)(t)] \mid U(t, s)x + (L_s u)(t)\big)_H \\ &\quad + (N(t)u(t) \mid u(t))_U\Big\}\, dt \\ &+ \|P_T^{1/2}[U(T, s)x + (\overline{P_T^{1/2} L_{sT}})u]\|_H^2, \qquad \forall u \in D(\overline{P_T^{1/2} L_{sT}}). \end{aligned}$$

Then we have to take $X_s = D(\overline{P_T^{1/2} L_{sT}})$; the operator Λ_{sT} becomes

$$\Lambda_{sT} w := N(\cdot)w(\cdot) + (\overline{P_T^{1/2} L_{sT}})^*(\overline{P_T^{1/2} L_{sT}})w + L_s^* M(\cdot)L_s w, \qquad w \in X_s,$$

and the formula for the optimal state is

$$\hat{u}(\cdot, s; x) = -\Lambda_{sT}^{-1}\left[(\overline{P_T^{1/2} L_{sT}})^* P_T^{1/2} U(T, s)x + L_s^* M(\cdot)U(\cdot, s)x\right].$$

Similar changes are needed in the next sections: we omit the details.

4. Pointwise Estimates for the Optimal Pair

We collect here some pointwise estimates concerning the optimal pair.

Proposition 4.1. *Under Hypotheses* 1.1–1.7, *let* $(\hat{y}(\cdot, s; x), \hat{u}(\cdot, s; x))$ *be the optimal pair relative to the control problem* (2.9)–(2.10). *Then we have*:

(i) $\|\hat{u}(\cdot, s; x)\|_{L^2(s,T,U)} \leq c_1\|\hat{u}(\cdot, s; x)\|_{X_s} \leq c_2\|x\|_H$, $\forall s \in [0, T[$;
(ii) $\|\hat{y}(\cdot, s; x)\|_{L^2(s,T,H)} \leq c\|x\|_H$, $\forall s \in [0, T[$;
(iii) $\|P_T^{1/2}\hat{y}(T, s; x)\|_H \leq c\|x\|_H$, $\forall s \in [0, T[$.

Proof (compare with Proposition 2.1 of [LT3]). (i) Easy consequence of (3.1) and (3.16), using (3.13), (3.2), Hypotheses 1.3 and 1.7, and (2.6).

(ii) It follows by (2.11) using Hypothesis 1.3 and (2.4).

(iii) It follows by (2.11), using Hypotheses 1.3 and 1.7, (3.2), and (i). □

Our next result is a refinement of the preceding estimates; it seems to be new even in the autonomous case.

Proposition 4.2. *Under Hypotheses* 1.1–1.7, *let* $(\hat{y}(\cdot, s; x), \hat{u}(\cdot, s; x))$ *be the optimal pair relative to the control problem* (2.9)–(2.10). *Then we have*:

(i) $\lim_{s\to T^-} \|\hat{u}(\cdot, s; x)\|_{X_s} = 0, \forall x \in H$;
(ii) $\lim_{s\to T^-} \|P_T^{1/2}\hat{y}(T, s; x) - P_T^{1/2}x\|_H = 0, \forall x \in H$.

Proof. (i) We write

$$\|\hat{u}(\cdot, s; x)\|_{X_s} \le \|\hat{u}(\cdot, s; x) + \Lambda_{sT}^{-1}L_{sT}^*P_Tx\|_{X_s} + \|\Lambda_{sT}^{-1}L_{sT}^*P_Tx\|_{X_s}.$$

The first term on the right-hand side tends to 0 as $s \to T^-$ since by (3.16), (3.13), (3.2), (2.6), and Hypotheses 1.3, 1.6, and 1.7 we have

$$\begin{aligned}
&\|\hat{u}(\cdot, s; x) + \Lambda_{sT}^{-1}L_{sT}^*P_Tx\|_{X_s}\\
&\quad= \| - \Lambda_{sT}^{-1}[L_{sT}^*P_T[U(T, s) - 1_H]x + L_s^*M(\cdot)U(\cdot, s)x]\|_{X_s}\\
&\quad\le \|P_T^{1/2}\|_{\mathcal{L}(H)}\|[U(T, s) - 1_H]x\|_H + c(T - s)^\alpha\|x\|_H.
\end{aligned}$$

Concerning the second term, following the proof of (5.8) of [LT3] we remark that the vector space

$$D := D(L_{0T}^*P_T^{1/2}) = \{z \in H\colon\ P_T^{1/2}z \in D(L_{0T}^*)\}$$

is dense in H, since, by Hypothesis 1.7, $L_{0T}^*P_T^{1/2}$ is the adjoint of the closed and densely defined operator $P_T^{1/2}L_{0T}$. For a fixed $\varepsilon > 0$, select $z \in D$ such that $\|P_T^{1/2}x - z\|_H < \varepsilon$; then, by (3.13), (3.2), Proposition 2.1(i), and (3.1),

$$\begin{aligned}
&\|\Lambda_{sT}^{-1}L_{sT}^*P_Tx\|_{X_s}\\
&\quad\le \|\Lambda_{sT}^{-1}L_{sT}^*P_T^{1/2}[P_T^{1/2}x - z]\|_{X_s} + \|\Lambda_{sT}^{-1}L_{sT}^*P_T^{1/2}z\|_{X_s}\\
&\quad\le c\varepsilon + \|L_{sT}^*P_T^{1/2}z\|_{X_s^*} \le c\varepsilon + \|L_{sT}^*P_T^{1/2}z\|_{L^2(s,T;U)}, \qquad \forall s \in [0, T[;
\end{aligned}$$

thus by Proposition 2.1(ii) we get that

$$\lim_{s\to T^-} \|\Lambda_{sT}^{-1}L_{sT}^*P_Tx\|_{X_s} = 0,$$

which proves (i).

(ii) We have, by (3.2),

$$\|P_T^{1/2}\hat{y}(T,s;x) - P_T^{1/2}x\|_H \le \|P_T^{1/2}[U(T,s)-1_H]x\|_H + \|P_T^{1/2}L_{sT}\hat{u}(\cdot,s;x)\|_H$$
$$\le c\|U(T,s)x - x\|_H + c\|\hat{u}(\cdot,s;x)\|_{X_s},$$

and the result follows by (i). □

The main statement of this section is the following theorem, which is a maximal regularity result for the optimal pair; the use of the spaces $Z_{\gamma,\eta}$ (see Definition A.1 of Appendix A) allows us to refine the corresponding results of Theorem 3.6(i) of [LT3].

Theorem 4.3. *Under Hypotheses* 1.1–1.7, *let* $(\hat{y}(\cdot,s;x),\hat{u}(\cdot,s;x))$ *be the optimal pair relative to the control problem* (2.9)–(2.10). *Then* $\hat{u}(\cdot,s;x)$ *and* $\hat{y}(\cdot,s;x)$ *are continuous in* $[s,T[$; *more precisely we have*:

(i) $\hat{u}(\cdot,s;x) \in Z_{1-\alpha,\delta}([s,T[,U)$ *and*

$$\|\hat{u}(\cdot,s;x)\|_{Z_{1-\alpha,\delta}([s,T[,U)} \le c\|x\|_H, \qquad \forall s \in [0,T[;$$

(ii) $\hat{y}(\cdot,s;x) \in C([s,(s+T)/2];H) \cap Z_{0,\alpha}(]s,(s+T)/2],H) \cap Z_{1-2\alpha,\alpha}([(s+T)/2,T[,H)$ *and*

$$\|\hat{y}(\cdot,s;x)\|_{L^\infty(s,(s+T)/2;H)} + \|\hat{y}(\cdot,s;x)\|_{Z_{0,\alpha}(]s,(s+T)/2],H)}$$
$$+\|\hat{y}(\cdot,s;x)\|_{Z_{1-2\alpha,\alpha}([(s+T)/2,T[,H)} \le c\|x\|_H, \qquad \forall s \in [0,T[.$$

Proof. We need some lemmas.

Lemma 4.4. *Under Hypotheses* 1.1–1.5 *and* 1.7 *let the operator* L_s *be defined by* (2.3), *and for* $p \in [1,\infty[$ *set*

$$r := \begin{cases} p/(1-\alpha p) & \text{if} \quad p < 1/\alpha, \\ \text{arbitrary} < \infty & \text{if} \quad p = 1/\alpha, \\ +\infty & \text{if} \quad p > 1/\alpha. \end{cases}$$

Then for each $s \in [0,T[$ *we have*:

(i) $\|L_s u\|_{L^r(s,T,H)} \le c\|u\|_{L^p(s,T,U)}$;
(ii) $\|L_s u\|_{C^\alpha([s,T],H)} \le c\|u\|_{L^\infty(s,T,U)}$;
(iii) $\|L_s u\|_{Z_{\gamma-\alpha,\alpha}([s,T[,H)} \le c\|u\|_{C_\gamma([s,T[,U)}$, $\forall \gamma \in [0,1]-\{\alpha\}$;
(iv) $\|L_s^* v\|_{L^r(s,T,U)} \le c\|v\|_{L^p(s,T,H)}$;
(v) $\|L_s^* v\|_{C^\delta([s,T],U)} \le c\|v\|_{L^\infty(s,T,H)}$;
(vi) $\|L_s^* v\|_{Z_{\gamma-\alpha,\delta}([s,T[,U)} \le c\|v\|_{C_\gamma([s,T[,H)}$, $\forall \gamma \in [0,1]-\{\alpha\}$.

Proof. (i) It follows by (2.3) and Hypotheses 1.3 and 1.5, by using Theorem 383 of [HLP].

(ii) Writing, for $s \le \tau < t \le T$,

$$L_s u(t) - L_s u(\tau)$$

$$= -\int_\tau^t U(t,r)A(r)G(r)u(r)\,dr$$
$$-\int_s^\tau \int_\tau^t A(q)U(q,r)A(r)G(r)u(r)\,dq\,dr$$
$$= \int_\tau^t \left[[-A(r)^*]^{1-\alpha} U(t,r)^*\right]^* [-A(r)]^\alpha G(r)u(r)\,dr$$
$$+\int_s^\tau \int_\tau^t A(q)U\left(q, \frac{q+r}{2}\right)\left[[-A(r)^*]^{1-\alpha} U\left(\frac{q+r}{2}, r\right)^*\right]^*$$
$$\times\ [-A(r)]^\alpha G(r)u(r)\,dq\,dr, \tag{4.1}$$

by Hypotheses 1.3–1.5 we easily obtain

$$\|L_s u(t) - L_s u(\tau)\|_H$$
$$\le c\left\{\int_\tau^t (t-r)^{\alpha-1}\,dr + \int_s^\tau \int_\tau^t (q-r)^{\alpha-2}\,dq\,dr\right\}\|u\|_{L^\infty(s,T,U)}$$
$$\le c(t-\tau)^\alpha \|u\|_{L^\infty(s,T,U)}.$$

(iii) For $s \le \tau < t \le T$ and $\gamma \in [0,1]$ we get, by (4.1) in the same way as before,

$$\|L_s u(t) - L_s u(\tau)\|_H \le c\left\{\int_\tau^t (t-r)^{\alpha-1}(T-r)^{-\gamma}\,dr\right.$$
$$\left.+\int_s^\tau \int_\tau^t (q-r)^{\alpha-2}\,dq(T-r)^{-\gamma}\,dr\right\}\|u\|_{C_\gamma([s,T[,U)}$$
$$\le c(t-\tau)^\alpha (T-t)^{-\gamma}\|u\|_{C_\gamma([s,T[,U)};$$

for $\gamma \ne \alpha$ the result then follows by Proposition A.2 in Appendix A below.

(iv) It follows by (2.5) and Hypotheses 1.3 and 1.5, by using Theorem 383 of [HLP].

(v) Writing, for $s \le \tau < t \le T$,

$$L_s^* v(t) - L_s^* v(\tau)$$
$$= \left[[-A(t)]^\alpha G(t) - [-A(\tau)]^\alpha G(\tau)\right]^* \int_t^T [-A(t)^*]^{1-\alpha} U(r,t)^* v(r)\,dr$$
$$+\left[[-A(\tau)]^\alpha G(\tau)\right]^* \int_t^T \left[[-A(t)^*]^{1-\alpha} U(r,t)^*\right.$$
$$\left.-[-A(\tau)^*]^{1-\alpha} U(r,\tau)^*\right] v(r)\,dr$$
$$-\left[[-A(\tau)]^\alpha G(\tau)\right]^* \int_\tau^t [-A(\tau)^*]^{1-\alpha} U(r,\tau)^* v(r)\,dr, \tag{4.2}$$

by Hypotheses 1.3–1.5 we easily obtain

$$\|L_s^* v(t) - L_s^* v(\tau)\|_U$$
$$\le c\left\{(t-\tau)^\delta \int_t^T (r-t)^{\alpha-1}\,dr + \int_t^T (t-\tau)^\delta (r-t)^{\alpha-\delta-1}\,dr\right.$$
$$\left.+\int_\tau^t (r-\tau)^{\alpha-1}\,dr\right\}\|v\|_{L^\infty(s,T,H)}$$
$$\le c(t-\tau)^\delta \|v\|_{L^\infty(s,T,H)}.$$

(vi) For $s \le \tau < t \le T$ and $\gamma \in [0,1]$ we get, by (4.2) in the same way as before,

$$\begin{aligned}
&\|L_s^* v(t) - L_s^* v(\tau)\|_U \\
&\quad \le c \left\{ (t-\tau)^\delta \int_t^T (r-t)^{\alpha-1}(T-r)^{-\gamma}\, dr \right. \\
&\qquad + \int_t^T (t-\tau)^\delta (r-t)^{\alpha-\delta-1}(T-r)^{-\gamma}\, dr \\
&\qquad \left. + \int_\tau^t (r-\tau)^{\alpha-1}(T-r)^{-\gamma}\, dr \right\} \|v\|_{C_\gamma([s,T[,H)} \\
&\quad \le c(t-\tau)^\delta (T-t)^{\alpha-\delta-\gamma} \|v\|_{C_\gamma([s,T[,H)};
\end{aligned}$$

for $\gamma \ne \alpha$ the result then follows by Proposition A.2 in Appendix A below. □

Remark 4.5. For further use we notice that if $u \in C_\gamma([s,T[,U)$, $\gamma \in [0,1]$, we have

$$\|L_s u(t)\|_H \le c(t-s)^\alpha (T-t)^{-\gamma} \|u\|_{C_\gamma([s,t[,U)} \qquad \text{for} \quad s \le t < T; \tag{4.3}$$

this follows by Lemma 4.4(iii) since $L_s u(s) = 0$.

Lemma 4.6. *Under Hypotheses* 1.1–1.7, *let* $\hat{y}(\cdot, s; x)$ *be the optimal state relative to the control problem* (2.9)–(2.10), *and let the operators* L_s *and* L_{sT} *be defined by* (2.3) *and* (2.7). *Then we have*:

(i) $L_s^* M(\cdot) U(\cdot, s)x \in C^\delta([s,T],U)$ *and*

$$\|L_s^* M(\cdot) U(\cdot, s)x)\|_{C^\delta([s,T],U)} \le c\|x\|_H;$$

(ii) $L_{sT}^* P_T \hat{y}(T,s;x) \in Z_{1-\alpha,\delta}([s,T[,U)$ *and*

$$\|L_{sT}^* P_T \hat{y}(T,s;x)\|_{Z_{1-\alpha,\delta}([s,T[,U)} \le c\|x\|_H.$$

Proof. (i) It is a consequence of Hypotheses 1.3, 1.6, and Lemma 4.4.

(ii) By Hypothesis 1.7, $P_T \hat{y}(T,s;x)$ is a well-defined element of H. Next, by (2.8), Hypotheses 1.3–1.5, and Proposition 4.1(iii) we have, for $s \le \tau \le t < T$,

$$\begin{aligned}
&\|[L_{sT}^* P_T \hat{y}(T,s;x)](t) - [L_{sT}^* P_T \hat{y}(T,s;x)](\tau)\|_U \\
&\quad \le \| \left[[-A(t)]^\alpha G(t) - [-A(\tau)]^\alpha G(\tau) \right]^* \left[[-A(t)^*]^{1-\alpha} U(T,t)^* \right] P_T \hat{y}(T,s;x)\|_U \\
&\qquad + \| \left[[-A(\tau)]^\alpha G(\tau) \right]^* \left[[-A(t)^*]^{1-\alpha} U(T,t)^* - [-A(\tau)^*]^{1-\alpha} U(T,\tau)^* \right] \\
&\qquad \times P_T \hat{y}(T,s;x)\|_U \\
&\quad \le c(t-\tau)^\delta (T-t)^{\alpha-1}\|x\|_H + c(t-\tau)^\delta (T-t)^{\alpha-\delta-1}\|x\|_H,
\end{aligned}$$

and the result follows by Proposition A.2 in Appendix A below. □

Lemma 4.7. *Under Hypotheses* 1.1–1.7, *let* $\hat{u}(\cdot, s; x)$ *be the optimal control relative to the control problem* (2.9)–(2.10), *and let the operator* L_s *be defined by* (2.3). *Then we have* $L_s^* M(\cdot) L_s \hat{u}(\cdot, s; x) \in Z_{1-\alpha,\delta}([s,T[,U)$ *and*

$$\|L_s^* M(\cdot) L_s \hat{u}(\cdot, s; x)\|_{Z_{1-\alpha,\delta}([s,T[,U)} \le c\|x\|_H.$$

Proof. We rewrite (3.7) as

$$\begin{aligned} N(\cdot)\hat{u}(\cdot, s; x) &= -L_{sT}^* P_T \hat{y}(\cdot, s; x) - L_s^* M(\cdot)U(\cdot, s)x - L_s^* M(\cdot) L_s \hat{u}(\cdot, s; x) \\ &=: F - L_s^* M(\cdot) L_s \hat{u}(\cdot, s; x). \end{aligned} \tag{4.4}$$

By Lemma 4.6 we have

$$\|F\|_{Z_{1-\alpha,\delta}([s,T[,U)} \le c\|x\|_H; \tag{4.5}$$

on the other hand, since $\hat{u}(\cdot, s; x) \in L^2(s, T; U)$ we have, by Lemma 4.4(i)–(iv) and Proposition 4.1(i),

$$\|L_s^* M(\cdot) L_s \hat{u}(\cdot, s; x)\|_{L^q(s,T,U)} \le c\|x\|_H,$$

where

$$q := \begin{cases} 2/(1-4\alpha) & \text{if} \quad \alpha < \frac{1}{4}, \\ \text{arbitrary } < \infty & \text{if} \quad \alpha = \frac{1}{4}, \\ +\infty & \text{if} \quad \alpha > \frac{1}{4}. \end{cases}$$

By (4.4) we get $\hat{u} - N(\cdot)^{-1}F \in L^q(s, T; U)$, and since $N(\cdot)^{-1} \in C^\delta([0, T], \mathcal{L}(U))$ by Hypothesis 1.6, (4.5) implies $N(\cdot)^{-1}F \in Z_{1-\alpha,\delta}([s, T[, U)$; thus, by Lemma 4.4,

$$\begin{aligned} &L_s^* M(\cdot) L_s (N(\cdot)^{-1}F) \in Z_{1-\alpha,\delta}([s, T[, U), \\ &L_s^* M(\cdot) L_s (\hat{u} - N(\cdot)^{-1}F) \in L^r(s, T; U), \end{aligned}$$

where

$$r := \begin{cases} 2/(1-8\alpha) & \text{if} \quad \alpha < \frac{1}{8}, \\ \text{arbitrary } < \infty & \text{if} \quad \alpha = \frac{1}{8}, \\ +\infty & \text{if} \quad \alpha > \frac{1}{8}. \end{cases}$$

Hence (4.4) now yields $\hat{u} - N(\cdot)^{-1}F \in L^r(s, T; U)$. After a finite number of steps, we find

$$\begin{aligned} &L_s^* M(\cdot) L_s (N(\cdot)^{-1}F) \in Z_{1-\alpha,\delta}([s, T[, U), \\ &L_s^* M(\cdot) L_s (\hat{u} - N(\cdot)^{-1}F) \in L^\infty(s, T; U), \end{aligned}$$

which finally gives $\hat{u} - N(\cdot)^{-1}F \in L^\infty(s, T; U)$. Applying once more Lemma 4.4 we obtain

$$\begin{aligned} &L_s^* M(\cdot) L_s (N(\cdot)^{-1}F) \in Z_{1-\alpha,\delta}([s, T[, U), \\ &L_s^* M(\cdot) L_s (\hat{u} - N(\cdot)^{-1}F) \in C^\delta([s, T], U), \end{aligned}$$

and the result follows. □

Let us prove Theorem 4.3. The proof of part (i) is easy: indeed, by (4.4),

$$\begin{aligned} \hat{u}(\cdot, s; x) = &-N(\cdot)^{-1} L_{sT}^* P_T \hat{y}(\cdot, s; x) - N(\cdot)^{-1} L_s^* M(\cdot) U(\cdot, s)x \\ &- N(\cdot)^{-1} L_s^* M(\cdot) L_s \hat{u}(\cdot, s; x); \end{aligned}$$

the right member of this identity belongs to $Z_{1-\alpha,\delta}([s,T[,U)$ in view of Hypothesis 1.6 and Lemmas 4.6 and 4.7, and the estimate also follows.

Let us prove part (ii): as $\|U(t,s)\|_{\mathcal{L}(H)} \le c$ for $0 \le s \le t \le T$ and

$$\begin{aligned}\|U(t,s) - U(\tau,s)\|_{\mathcal{L}(H)} &= \left\| \int_\tau^t A(q)U(q,s)\,dq \right\|_{\mathcal{L}(H)} \\ &\le c\frac{t-\tau}{\tau-s} \qquad \text{for} \quad s < \tau \le t \le T, \end{aligned} \tag{4.6}$$

Propositions A.2 and A.4 in Appendix A yield

$$\|U(\cdot,s)x\|_{Z_{0,\alpha}(]s,T],H)} \le c\|U(\cdot,s)x\|_{Z_{0,1}(]s,T],H)} \le c\|x\|_H; \tag{4.7}$$

on the other hand, by (i) and Lemma 4.4,

$$\|L_s\hat{u}(\cdot,s;x)\|_{Z_{1-2\alpha,\alpha}([s,T[,H)} \le c\|x\|_H. \tag{4.8}$$

Hence by (2.11), (4.7), and (4.8) we get, for $s < \tau \le t < T$,

$$\begin{aligned}\|\hat{y}(t,s;x) - \hat{y}(\tau,s;x)\|_H &\le c[(t-\tau)^\alpha(\tau-s)^{-\alpha} + (t-\tau)^\alpha(T-t)^{\alpha-1}]\|x\|_H \\ &\le c(t-\tau)^\alpha(\tau-s)^{-\alpha}(T-t)^{\alpha-1}\|x\|_H,\end{aligned}$$

and (ii) is proved by using again Proposition A.2 (continuity and the L^∞ estimate in $[s,(T+s)/2]$ are obvious). The proof of Theorem 4.3 is complete. □

Remark 4.8. The results of Theorem 4.3 improve the corresponding ones, relative to the autonomous case, see Theorem 3.6 of [LT3]; however, we cannot have here the analyticity of the optimal state, which is a special feature of the autonomous situation.

5. The Operators $\varphi(t,s)$, $P(t)$

We define the state operator $\varphi(t,s)$ relative to the control problem (2.9)–(2.10): we set, for $0 \le s \le t \le T$,

$$\varphi(t,s)x := \hat{y}(t,s;x), \qquad \forall x \in H. \tag{5.1}$$

Let us collect the main properties of this operator. By Theorem 4.3(ii), we have $\varphi(t,s) \in \mathcal{L}(H)$ for $0 \le s \le t < T$ and, by (2.13),

$$\varphi(t,t) = 1_H, \qquad \varphi(t,s) = \varphi(t,r)\varphi(r,s) \qquad \text{for} \quad 0 \le s \le r \le t \le T. \tag{5.2}$$

Next, by Proposition 4.1(ii)

$$\|\varphi(\cdot,s)\|_{\mathcal{L}(H,L^2(s,T,H))} \le c, \qquad \forall s \in [0,T[. \tag{5.3}$$

Proposition 5.1. *Under Hypotheses* 1.1–1.7, *let* $\varphi(t,s)$ *be defined by* (5.1). *Then*

$$\begin{aligned}&t \to \varphi(t,s)x \in C_{1-2\alpha}([s,T[,H), \qquad \forall x \in H, \quad \forall s \in [0,T[;\\ &s \to \varphi(t,s)x \in C([0,t],H), \qquad \forall x \in H, \quad \forall t \in]0,T[.\end{aligned}$$

Proof. The first assertion follows by Theorem 4.3(i). Concerning the second one, fix $t \in]0, T[$ and let $s \in [0, t]$: then we have, as $\tau \to s^+$,

$$\|\varphi(t,\tau)x - \varphi(t,s)x\|_H \le \|\varphi(t,\tau)\|_{\mathcal{L}(H)}\|x - \varphi(\tau,s)x\|_H \to 0,$$

whereas as $\tau \to s^-$, by the strong continuity of $s \to U(t,s)$ in $[0, T]$, by (4.3) and by Theorem 4.3(i),

$$\begin{aligned}
&\|\varphi(t,\tau)x - \varphi(t,s)x\|_H \\
&\quad \le \|\varphi(t,s)\|_{\mathcal{L}(H)}\|\varphi(s,\tau)x - x\|_H \\
&\quad \le \|\varphi(t,s)\|_{\mathcal{L}(H)}\left[\|U(s,\tau)x - x\|_H + \|[L_\tau \hat{u}(\cdot,\tau;x)](s)\|_H\right] \\
&\quad \le \|\varphi(t,s)\|_{\mathcal{L}(H)}\left[\|U(s,\tau)x - x\|_H + c(s-\tau)^\alpha (T-s)^{\alpha-1}\|x\|_H\right] \to 0. \qquad \square
\end{aligned}$$

In particular, by Proposition 5.1 and the Uniform Boundedness Principle we obtain

$$\begin{aligned}
&\|\varphi(t,s)\|_{\mathcal{L}(H)} \le c(s,T,\varepsilon), \\
&\qquad \forall s \in [0,T[, \quad \forall \varepsilon \in]0, T-s[, \quad \forall t \in [s, T-\varepsilon[;
\end{aligned} \tag{5.4}$$

by Theorem 4.3(ii) we also get

$$\|\varphi(t,s)\|_{\mathcal{L}(H)} \le c(T-t)^{2\alpha-1}, \qquad \forall s \in [0,T[, \quad \forall t \in [s,T[. \tag{5.5}$$

In addition, by Proposition 4.1(iii),

$$\|P_T^{1/2}\varphi(T,s)\|_{\mathcal{L}(H)} \le c, \qquad \forall s \in [0,T[. \tag{5.6}$$

We now define the Riccati operator $P(t)$: for $t \in [0, T[$ we set

$$P(t) := \int_t^T U(r,t)^* M(r)\varphi(r,t)\, dr + U(T,t)^* P_T \varphi(T,t). \tag{5.7}$$

By (5.5) and (5.6) we get $P(t) \in \mathcal{L}(H)$ for each $t \in [0, T[$ and $P \in L^\infty(0, T, \mathcal{L}(H))$; moreover, it is clear that $[-A(t)^*]^\eta P(t) \in \mathcal{L}(H)$ for each $t \in [0, T[$ and $\eta \in [0, 1[$, and by (5.5) and Hypotheses 1.3 and 1.5 the following estimates hold:

$$\|[-A(t)^*]^\eta P(t)\|_{\mathcal{L}(H)} \le c(\eta)(T-t)^{-\eta}, \qquad \forall t \in [0,T[, \quad \forall \eta \in [0,1[, \tag{5.8}$$

$$\|G(t)^* A(t)^* P(t)\|_{\mathcal{L}(H)} \le c(T-t)^{\alpha-1}, \qquad \forall t \in [0,T[. \tag{5.9}$$

The operator $P(t)$ allows us to express the optimal cost $\hat{u}(t,s;x)$ as a function of the optimal state $\hat{y}(t,s;x)$.

Proposition 5.2. *Under Hypotheses* 1.1–1.7, *let* $P(t)$ *be defined by* (5.7). *Then for* $s \in [0, T[$ *and* $t \in [s, T[$ *we have*

$$\hat{u}(t,s;x) = N(t)^{-1} G(t)^* A(t)^* P(t)\hat{y}(t,s;x), \qquad \forall x \in H. \tag{5.10}$$

We remark that, as usual, the above formula in fact means

$$\hat{u}(t,s;x) = -N(t)^{-1}\left[[-A(t)]^\alpha G(t)\right]^* [-A(t)^*]^{1-\alpha} P(t)\hat{y}(t,s;x), \qquad \forall x \in H,$$

which is meaningful by Hypothesis 1.5 *and* (5.8).

Proof. Let $0 \le s < t < T$. By (3.6), (2.8), (5.1), and (2.5) we have, for $s \le \tau \le t$ and $y \in H$,

$$\hat{u}(t,\tau;y) = N(t)^{-1}G(t)^*A(t)^* \times \left[U(T,t)^*P_T\varphi(T,\tau)y + \int_t^T U(r,t)^*M(r)\varphi(r,\tau)y\,dr\right];$$

hence choosing $\tau = t$ and $y = \varphi(t,s)x$ we get, by (5.2), (5.7), and (5.1),

$$\begin{aligned}\hat{u}(t,t;\varphi(t,s)x) &= N(t)^{-1}G(t)^*A(t)^*\left[U(T,t)^*P_T\varphi(T,s)x + \int_t^T U(r,t)^*M(r)\varphi(r,s)x\,dr\right]\\ &= N(t)^{-1}G(t)^*A(t)^*P(t)\varphi(t,s)x\\ &= N(t)^{-1}G(t)^*A(t)^*P(t)\hat{y}(t,s;x),\end{aligned}$$

and the result follows by (2.13). □

Remark 5.3. (i) As, by Theorem 4.3, $\hat{u}(\cdot,s;x) \in L^2(s,T,U) \cap Z_{1-\alpha,\delta}([s,T[,U)$, we immediately deduce that $N^{-1}G^*A^*P\varphi(\cdot,s)x$ belongs to the same space and

$$\|N^{-1}G^*A^*P\varphi(\cdot,s)x\|_{L^2(s,T,U)} + \|N^{-1}G^*A^*P\varphi(\cdot,s)x\|_{Z_{1-\alpha,\delta}([s,T[,U)} \le c\|x\|_H, \quad \forall s \in [0,T[.$$

(ii) By (5.1), (2.11), and (5.10) we may rewrite the optimal dynamics as follows:

$$\varphi(\tau,t) = U(\tau,t) - \int_t^\tau U(\tau,r)A(r)G(r)N(r)^{-1}G(r)^*A(r)^*P(r)\varphi(r,t)\,dr, \quad 0 \le t \le \tau < T. \tag{5.11}$$

Proposition 5.4. *Under Hypotheses* 1.1–1.7, *let* $P(t)$ *be defined by* (5.7). *Then for each* $x, y \in H$ *and* $t \in [0,T[$ *we have*

$$\begin{aligned}(P(t)x \mid y)_H &= \int_t^T (M(t)\varphi(\tau,t)x \mid \varphi(\tau,t)y)_H\,d\tau + (P_T\varphi(T,t)x \mid \varphi(T,t)x)_H\\ &\quad + \int_t^T (G(\tau)^*A(\tau)^*P(\tau)\varphi(\tau,t)x \mid N(\tau)^{-1}G(\tau)^*A(\tau)^*P(\tau)\varphi(\tau,t)y)_U\,d\tau,\end{aligned}$$

and, in particular, $P(t)$ *is self-adjoint and positive. Moreover,*

$$J_t(\hat{u}(\cdot,t,x)) = (P(t)x \mid x)_H, \qquad \forall t \in [0,T[, \quad \forall x \in H. \tag{5.12}$$

Proof. We have, by (5.7),

$$(P(t)x \mid y)_H = \int_t^T (M(\tau)\varphi(\tau,t)x \mid U(\tau,t)y)_H\,d\tau + (P_T\varphi(T,t)x \mid U(T,t)y)_H.$$

Now we replace $U(\tau, t)$ by its expression deduced from (5.11); the resulting double integral is convergent since the integrand is square integrable over $\Delta := \{(\tau, r) : t < r < \tau < T\}$: this follows easily by (5.3), Hypothesis 1.3, and Remark 5.3(i). Hence by the Fubini–Tonelli theorem we get

$$
\begin{aligned}
&(P(t)x \mid y)_H \\
&\quad = \int_t^T (M(\tau)\varphi(\tau, t)x \mid \varphi(\tau, t)y)_H \, d\tau \\
&\qquad + \int_t^T \int_r^T (M(\tau)\varphi(\tau, t)x \mid U(\tau, r)A(r)G(r)N(r)^{-1} \\
&\qquad\qquad \times G(r)^* A(r)^* P(r)\varphi(r, t)y)_H \, d\tau \, dr \\
&\qquad + (P_T\varphi(T, t)x \mid \varphi(T, t)y)_H + (P_T\varphi(T, t)x \mid L_{tT}\hat{u}(\cdot, t; y))_H \\
&\qquad \text{(recalling that } P_T\varphi(T, t)x \in D(L_{tT}^*)) \\
&\quad = \int_t^T (M(\tau)\varphi(\tau, t)x \mid \varphi(\tau, t)y)_H \, d\tau + (P_T\varphi(T, t)x \mid \varphi(T, t)y)_H \\
&\qquad + \int_t^T \int_r^T (G(r)^* A(r)^* U(\tau, r)^* M(\tau)\varphi(\tau, t)x \mid N(r)^{-1}G(r)^* A(r)^* \\
&\qquad\qquad \times P(r)\varphi(r, t)y)_U \, d\tau \, dr \\
&\qquad + \int_t^T (G(r)^* A(r)^* U(T, r)^* P_T\varphi(T, t)x \mid N(r)^{-1}G(r)^* A(r)^* \\
&\qquad\qquad \times P(r)\varphi(r, t)y)_U \, dr.
\end{aligned}
$$

Finally, by (5.7) and (5.2) we conclude that

$$
\begin{aligned}
&(P(t)x \mid y)_H \\
&\quad = \int_t^T (M(\tau)\varphi(\tau, t)x \mid \varphi(\tau, t)y)_H \, d\tau + (P_T\varphi(T, t)x \mid \varphi(T, t)y)_H \\
&\qquad + \int_t^T (G(r)^* A(r)^* P(r)\varphi(r, t)x \mid N(r)^{-1}G(r)^* A(r)^* P(r)\varphi(r, t)y)_U \, dr;
\end{aligned}
$$

thus $P(t) \in \Sigma(H)$. Choosing in particular $y = x$, we obtain $P(t) \geq 0$ and

$$
\begin{aligned}
&(P(t)x \mid x)_H \\
&\quad = \int_t^T (M(\tau)\hat{y}(\tau, t; x) \mid \hat{y}(\tau, t; x))_H \, d\tau + (P_T\hat{y}(T, t; x) \mid \hat{y}(T, t; x))_H \\
&\qquad + \int_t^T (N(r)\hat{u}(r, t; x) \mid \hat{u}(r, t; x))_U \, dr \\
&\quad = J_t(\hat{u}(\cdot, t : x)).
\end{aligned}
$$

□

The next result seems to be new even in the autonomous case (compare with Section 5 of [LT3].

Theorem 5.5. *Under Hypotheses* 1.1–1.7, *let* $P(t)$ *be defined by* (5.7). *Then we have*

$$
\lim_{t \to T^-} \|P(t)x - P_T x\|_H = 0, \qquad \forall x \in H.
$$

Proof. We have, by (5.7) and (2.11),

$$\begin{aligned}
&\|P(t)x - P_T x\|_H \\
&\quad \le \left\| \int_t^T U(r,t)^* M(r)\varphi(r,t)x \, dr \right\|_H \\
&\qquad + \|U(T,t)^* P_T^{1/2}[P_T^{1/2}\hat{y}(T,t;x) - P_T^{1/2}x]\|_H + \|[U(T,t)^* - 1_H]P_T x\|_H \\
&\qquad \text{(by (5.5), Hypotheses 1.3, 1.7, and Proposition 4.2(ii))} \\
&\quad \le c(T-t)^{2\alpha} + o(1) \qquad \text{as} \quad s \to T^-.
\end{aligned}$$

□

6. Differentiability Properties of $\varphi(t,s)$

From now on our technique differs from that of [LT3]. We start from the integral equation established in Remark 5.3:

$$\begin{aligned}
\varphi(t,s) &= U(t,s) \\
&\quad - \int_s^t U(t,\tau)A(\tau)G(\tau)N(\tau)^{-1}G(\tau)^* A(\tau)^* P(t)\varphi(\tau,s)\, d\tau. \qquad (6.1)
\end{aligned}$$

We state some results concerning the differentiability properties of the operator $\varphi(t,s)$ with respect to both t and s. Since the proofs are very long and technical, we omit them here: all details can be found in [AT5].

We start with the study of $t \to \varphi(t,s)$.

Proposition 6.1. *Under Hypotheses* 1.1–1.7 *let* $\varphi(t,s)$ *be the operator defined by* (5.1). *Then for* $0 \le s < t < T$ *we have*

$$\begin{aligned}
&\lim_{h\to 0} \left(\frac{\varphi(t+h,s) - \varphi(t,s)}{h} x \mid y \right)_H \\
&\quad = ([1_H - G(t)N(t)^{-1}G(t)^* A(t)^* P(t)]\varphi(t,s)x \mid A(t)^* y)_H, \\
&\quad \forall x \in H, \quad \forall y \in D_{A(t)^*}.
\end{aligned}$$

Proof. See [AT5]. □

Remark 6.2. In the autonomous case of [LT3], using the analyticity of $t \to \varphi(t,s)x$, it is shown that $\hat{y}(t,s;x) - G\hat{u}(t,s;x) \in D_A$, i.e., the range of the operator $[1_H - GN^{-1}G^* A^* P(t)]\varphi(t,s)$ is contained in D_A for each $t \in [s,T[$; this proves, in that situation, the strong differentiability of $t \to \varphi(t,s)$ and the formula

$$\frac{d}{dt}\varphi(t,s)x = A(t)[1_H - G(t)N(t)^{-1}G(t)^* A(t)^* P(t)]\varphi(t,s)x, \qquad t \in]s,T[\quad (6.2)$$

(compare with Lemma 4.4 of [LT3]). On the contrary, in our situation we cannot use analyticity and it is not clear whether or not

$$\begin{aligned}
&[1_H - G(t)N(t)^{-1}G(t)^* A(t)^* P(t)]\varphi(t,s)x \\
&\quad = \hat{y}(t,s;x) - G(t)\hat{u}(t,s;x) \in D_{A(t)}. \qquad (6.3)
\end{aligned}$$

However, this property holds true provided we impose some restrictions on the exponents δ, α in Hypotheses 1.4 and 1.5, as shown in Proposition 6.12 below. In concrete parabolic initial-boundary value problems, (6.3) and consequently (6.2) may be proved directly: indeed, for fixed t the vector in (6.3) solves an elliptic system with homogeneous boundary datum, so that the elliptic regularity theorems apply (see Remark 9.5 below).

We now consider the properties of $s \to \varphi(t,s)$. Fix a number $\varepsilon \in]0,T[$. For $0 \le q < T-\varepsilon$ we introduce the integral operator

$$[K_q g](t) := \int_q^t U(t,\tau)A(\tau)G(\tau)N(\tau)^{-1}G(\tau)^* A(\tau)^* P(\tau)g(\tau)\,d\tau,$$
$$t \in [q, T-\varepsilon], \tag{6.4}$$

whose kernel is

$$K(t,\tau) := U(t,\tau)A(\tau)G(\tau)N(\tau)^{-1}G(\tau)^* A(\tau)^* P(\tau),$$
$$0 \le \tau < t < T, \tag{6.5}$$

and satisfies, by Hypotheses 1.4–1.6 and (5.9),

$$\|K(t,\tau)\|_{\mathcal{L}(H)} \le c(t-\tau)^{\alpha-1}(T-\tau)^{\alpha-1} \qquad \text{for} \quad 0 \le \tau < t < T; \tag{6.6}$$

in particular,

$$\|K(t,\tau)\|_{\mathcal{L}(H)} \le c_\varepsilon (t-\tau)^{\alpha-1} \qquad \text{for} \quad 0 \le \tau < t \le T-\varepsilon. \tag{6.7}$$

It is shown in [AT5] that we have $(1+K_q)^{-1} \in \mathcal{L}(B_\gamma(]q, T-\varepsilon], \mathcal{L}(H)))$ for each $\gamma \in [0,1[$ and

$$\|(1+K_q)^{-1}\|_{\mathcal{L}(B_\gamma(]q,T-\varepsilon],H))} \le c_\varepsilon, \qquad \forall q \in [0, T-\varepsilon], \quad \forall \gamma \in [0,1[$$

(the space B_γ is introduced in Definition A.1 of Appendix A). Moreover this operator can be written as

$$[(1+K_q)^{-1}g](t) = g(t) + \int_q^t R(t,\sigma)g(\sigma)\,d\sigma,$$
$$\forall g \in B_\gamma(]q, T-\varepsilon], H), \quad \forall t \in [q, T-\varepsilon], \tag{6.8}$$

where the kernel $R(t,\sigma)$ is given by

$$R(t,\sigma) := \sum_{m=1}^{\infty} (-1)^m K_m(t,\sigma), \tag{6.9}$$

with

$$K_1(t,\sigma) := K(t,\sigma), \qquad K_{m+1}(t,\sigma) := \int_\sigma^t K_m(t,q)K(q,\sigma)\,dq, \qquad \forall m \in \mathbb{N}^+.$$

It also satisfies

$$\|R(t,\tau)\|_{\mathcal{L}(H)} \le c_\varepsilon (t-\tau)^{\alpha-1} \qquad \text{for} \quad 0 \le \tau < t \le T-\varepsilon. \tag{6.10}$$

All these facts are proved in [AT5]. Hence we can rewrite (6.1), for $0 \le s < t \le T - \varepsilon$, as

$$\varphi(t,s) = [(1+K_s)^{-1}(U(\cdot,s))](t) = U(t,s) + \int_s^t R(t,\sigma)U(\sigma,s)\,d\sigma, \quad t \in [s, T-\varepsilon]. \tag{6.11}$$

This formula is the starting point for the direct computation of the derivative $\varphi_s(t,s)$. The following preliminary lemmas are proved in [AT5].

Lemma 6.3. *Under Hypotheses* 1.1–1.7 *let* $\varphi(t,s)$ *be defined by* (5.1). *Then*

$$\|\varphi(t,\tau) - \varphi(t,s)\|_{\mathcal{L}(H)} \le c_\varepsilon(\tau - s)^\delta (t-\tau)^{-\delta} \qquad \textit{for} \quad 0 \le s \le \tau < t \le T - \varepsilon.$$

Lemma 6.4. *Under Hypotheses* 1.1–1.7 *let* $P(t)$ *be defined by* (5.7). *Then*

$$\|[-A(\tau)^*]^{1-\alpha}P(\tau) - [-A(s)^*]^{1-\alpha}P(s)\|_{\mathcal{L}(H)} \le c_\varepsilon(\tau - s)^\delta \quad \textit{for} \quad 0 \le s \le \tau \le T - \varepsilon.$$

Lemma 6.5. *Under Hypotheses* 1.1–1.7 *let* $K(t,\tau)$ *be defined by* (6.5). *Then*:

(i) $\|K(t,s) - K(\tau,s)\|_{\mathcal{L}(H)} \le c_\varepsilon(t-\tau)^\delta(\tau-s)^{\alpha-1-\delta}$ *for* $0 \le s < \tau \le t \le T - \varepsilon$;
(ii) $\|K(t,\tau) - K(t,s)\|_{\mathcal{L}(H)} \le c_\varepsilon(\tau-s)^\delta(t-\tau)^{\alpha-1-\delta}$ *for* $0 \le s \le \tau < t \le T - \varepsilon$;
(iii) $\|K(t,q) - K(\tau,q) - K(t,s) + K(\tau,s)\|_{\mathcal{L}(H)} \le c_\varepsilon(t-\tau)^\delta(q-s)^\delta(\tau-q)^{\alpha-1-2\delta}$ *for* $0 \le s \le q < \tau \le t \le T - \varepsilon$.

Remark 6.6. (i) As shown in [AT5], Lemma 6.5 tells us that the operators K_q and $(1+K_q)^{-1}$ belong to $\mathcal{L}(I_\gamma(]q, T-\varepsilon], H))$, $\gamma \in [1, 1+\delta[$, for each $q \in [0, T-\varepsilon[$, with norms bounded independently of q (the space I_γ is introduced in Definition A.6 of Appendix A).

(ii) The kernel $R(t,\sigma)$ introduced in (6.9) satisfies the same estimates as $K(t,\sigma)$ does, i.e., (6.10) and

$$\|R(t,s) - R(\tau,s)\|_{\mathcal{L}(H)} \le c_\varepsilon(t-\tau)^\delta(\tau-s)^{\alpha-1-\delta} \quad \text{for} \quad 0 \le s < \tau \le t \le T - \varepsilon, \tag{6.12}$$

$$\|R(t,\tau) - R(t,s)\|_{\mathcal{L}(H)} \le c_\varepsilon(\tau-s)^\delta(t-\tau)^{\alpha-1-\delta} \quad \text{for} \quad 0 \le s \le \tau < t \le T - \varepsilon, \tag{6.13}$$

$$\|R(t,q) - R(\tau,q) - R(t,s) + R(\tau,s)\|_{\mathcal{L}(H)} \le c_\varepsilon(t-\tau)^\delta(q-s)^\delta(\tau-q)^{\alpha-1-2\delta} \quad \text{for} \quad 0 \le s \le q < \tau \le t \le T - \varepsilon. \tag{6.14}$$

Indeed, it follows by induction (see [AT5]) that these estimates hold for each iterated kernel $K_m(t,\sigma)$ with constants c_m such that $c_m T^m \to 0$ as $m \to \infty$, so that they hold for $R(t,\sigma)$ too.

Lemma 6.7. *Under Hypotheses* 1.1–1.4 *there exists an operator* $V(t,s) \in \mathcal{L}(H)$, *continuous for* $0 \le s < t \le T$, *such that for* $0 \le s \le \sigma < t \le T$ *we have*:

(i) $(d/ds)U(t,s) = V(t,s)$, $V(t,s)x = -U(t,s)A(s)x$, $\forall x \in D_{A(s)}$;
(ii) $\|V(t,s)\|_{\mathcal{L}(H)} \le c(t-s)^{-1}$;
(iii) $\|V(t,s) + A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(H)} \le c(t-s)^{\delta-1}$;
(iv) $\|V(t,\sigma) - V(t,s)\|_{\mathcal{L}(H)} \le c_\eta(\sigma-s)^\eta(t-\sigma)^{-1-\eta}$, $\forall \eta \in]0,\delta[$;
(v) $\|V(t,\sigma) + A(\sigma)e^{(t-\sigma)A(\sigma)} - V(t,s) - A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(H)} \le c_\eta(\sigma-s)^\eta(t-\sigma)^{\delta-1-\eta}$, $\forall \eta \in]0,\delta[$;
(vi) $\|[V(t,\sigma)-V(t,s)]A(s)^{-1}\|_{\mathcal{L}(H)} \le c_\eta[(\sigma-s)^\eta(t-\sigma)^{-\eta}+(\sigma-s)^\mu(t-\sigma)^{\rho-1}]$, $\forall \eta \in]0,\delta[$.

Remark 6.8. Lemma 6.7 implies that for each $x \in H$ the function $V(\cdot,s)x$ belongs to the space $I_1(]s,T],H)$, since by (iii) we obtain, for each $t \in]s,T]$,

$$\exists \lim_{h\to 0^+} \int_{s+h}^t V(\tau,s)x\,d\tau = \int_s^t [V(\tau,s) + A(s)e^{(\tau-s)A(s)}]x\,d\tau - [e^{(t-s)A(s)}x - x] \qquad \text{in } H;$$

hence by (ii) and (iv) we also get, for each $\eta \in]0,\delta[$,

$$V(\cdot,s) \in Z_{1,\eta}(]s,T],\mathcal{L}(H)),$$
$$V(\cdot,s)x \in Z^*_{1,\eta}(]s,T],H), \qquad \forall x \in H. \tag{6.15}$$

Similarly, by (iii) and (v) we obtain, for each $\eta \in]0,\delta[$,

$$V(\cdot,s) + A(s)e^{(\cdot-s)A(s)} \in Z_{1-\delta,\eta}(]s,T],\mathcal{L}(H)). \tag{6.16}$$

Now we return to (6.11). For small h we easily obtain

$$\frac{\varphi(t,s+h)-\varphi(t,s)}{h} = \left[(1+K_{s+h})^{-1}\left(\frac{U(\cdot,s+h)-U(\cdot,s)}{h} + \frac{1}{h}\int_s^{s+h} K(\cdot,\tau)\varphi(\tau,s)\,d\tau\right)\right](t)$$
$$\text{for} \quad s < s+h \le t \le T-\varepsilon, \tag{6.17}$$

$$\frac{\varphi(t,s-h)-\varphi(t,s)}{-h} = \left[(1+K_s)^{-1}\left(\frac{U(\cdot,s-h)-U(\cdot,s)}{-h} + \frac{1}{h}\int_{s-h}^{s} K(\cdot,\tau)\varphi(\tau,s-h)\,d\tau\right)\right](t)$$
$$\text{for} \quad 0 \le s-h < s \le t \le T-\varepsilon; \tag{6.18}$$

by (6.8) we can rewrite them as

$$\frac{\varphi(t,s+h)-\varphi(t,s)}{h}$$

$$= \frac{U(t,s+h)-U(t,s)}{h} + \frac{1}{h}\int_s^{s+h} K(t,\tau)\varphi(\tau,s)\,d\tau$$
$$+ \int_{s+h}^t R(t,\sigma)\left(\frac{U(\sigma,s+h)-U(\sigma,s)}{h} + \frac{1}{h}\int_s^{s+h} K(\sigma,\tau)\varphi(\tau,s)\,d\tau\right) d\sigma,$$
$$t \in]s+h, T-\varepsilon], \tag{6.19}$$

$$\frac{\varphi(t,s-h)-\varphi(t,s)}{-h}$$
$$= \frac{U(t,s-h)-U(t,s)}{-h} + \frac{1}{h}\int_{s-h}^{s} K(t,\tau)\varphi(\tau,s-h)\,d\tau$$
$$+ \int_s^t R(t,\sigma)\left(\frac{U(\sigma,s-h)-U(\sigma,s)}{-h} + \frac{1}{h}\int_{s-h}^{s} K(\sigma,\tau)\varphi(\tau,s-h)\,d\tau\right) d\sigma,$$
$$t \in]s, T-\varepsilon]. \tag{6.20}$$

Proceeding formally, letting $h \to 0^+$ we find

$$\varphi_s(t,s) = V(t,s) + K(t,s) + \int_s^t R(t,\sigma)[V(\sigma,s)+K(\sigma,s)]\,d\sigma,$$

i.e.,

$$\varphi_s(t,s) = [(1+K_s)^{-1}[V(\cdot,s)+K(\cdot,s)]](t), \qquad t \in]s, T-\varepsilon]. \tag{6.21}$$

Notice that this formula is not meaningful in $\mathcal{L}(H)$, since the operator $(1+K_s)^{-1}$ acts in $I_1(]s, T-\varepsilon], \mathcal{L}(H))$ but does not operate in $Z_{1,\eta}(]s, T-\varepsilon], \mathcal{L}(H))$, whereas, by (6.21), $V(\cdot,s)$ is in the latter space but is not in the former one. However, for each $x \in X$ we have $V(\cdot,s)x \in I_1(]s, T-\varepsilon], H)$, so that instead of (6.20) we may write

$$\frac{d}{ds}[\varphi(t,s)x] = [(1+K_s)^{-1}[V(\cdot,s)x + K(\cdot,s)x]](t),$$
$$\forall t \in]s, T-\varepsilon], \quad \forall x \in H. \tag{6.22}$$

Nevertheless, it can be shown that $\varphi_s(t,s)$ exists in the sense of $\mathcal{L}(H)$; in fact we have:

Theorem 6.9. *Under Hypotheses* 1.1–1.7, *let* $\varphi(t,s)$ *be the operator defined by* (6.1). *Then for* $0 \le s < t < T$ *it holds, in the sense of* $\mathcal{L}(H)$, *that*

$$\frac{d}{ds}\varphi(t,s) = V(t,s) + \int_s^t R(t,\sigma)[V(\sigma,s) + A(s)e^{(\sigma-s)A(s)}]\,d\sigma$$
$$- \int_s^t [R(t,\sigma)-R(t,s)]A(s)e^{(\sigma-s)A(s)}\,d\sigma - R(t,s)e^{(t-s)A(s)},$$

where $V(t,s) = (d/ds)U(t,s)$ *and* $R(t,s)$ *is defined by* (6.9).

We remark that this formula reduces to (6.22) when applied to any $x \in H$.

Proof. See [AT5]. □

Corollary 6.10. *Under Hypotheses* 1.1–1.7, *let* $\varphi(t,s)$ *be the operator defined by* (6.1). *Then for* $0 \le s < t < T$ *we have, in the sense of* $\mathcal{L}(H)$,

$$\left[\frac{d}{ds}\varphi(t,s)\right] A(s)^{-1} = -U(t,s) - \int_s^t R(t,\sigma)U(\sigma,s)\, d\sigma - R(t,s)A(s)^{-1}.$$

Proof. It is an easy consequence of Theorem 6.9 and Lemma 6.7(i). □

Remark 6.11. The result of Theorem 6.9 guarantees that $\varphi_s(t,s)$ exists for $0 \le s < t < T$; in addition, for each $\eta > 0$ and $0 \le s < t < T$,

$$\|\varphi_s(t,s)\|_{\mathcal{L}(H)} \le c_{T-s}(T-t)^{2\alpha-1}(t-s)^{-1}, \tag{6.23}$$
$$\|\varphi_s(t,s)A(s)^{-1}\|_{\mathcal{L}(H)} \le c_{T-s}(T-t)^{2\alpha-1}(t-s)^{\alpha-1}. \tag{6.24}$$

Indeed, for fixed $s \in [0,T[$, if $t \le (T+s)/2$ we have, by (6.22), Lemma 6.7(ii), and (6.7),

$$\|\varphi_s(t,s)\|_{\mathcal{L}(H)} \le c_{(T-s)/2}(t-s)^{-1};$$

on the other hand, if $(T+s)/2 < t < T$ the identity (5.2) implies, using also (5.5), that

$$\begin{aligned}\|\varphi_s(t,s)\|_{\mathcal{L}(H)} &= \|\varphi(t,(T+s)/2)\varphi_s((T+s)/2,s)\|_{\mathcal{L}(H)} \\ &\le c(T-t)^{2\alpha-1}c_{(T-s)/2}(T-s)^{-1},\end{aligned}$$

so that (6.23) follows. Similarly, if $t \le (T+s)/2$ we have, by Corollary 6.10 and (6.10),

$$\|\varphi_s(t,s)A(s)^{-1}\|_{\mathcal{L}(H)} \le c_{(T-s)/2}(t-s)^{\alpha-1},$$

whereas if $t \le (T+s)/2$ we have, by (5.5),

$$\begin{aligned}\|\varphi_s(t,s)A(s)^{-1}\|_{\mathcal{L}(H)} &= \|\varphi(t,(T+s)/2)\varphi_s((T+s)/2,s)A(s)^{-1}\|_{\mathcal{L}(H)} \\ &\le c(T-t)^{2\alpha-1}c_{(T-s)/2}(t-s)^{\alpha-1},\end{aligned}$$

and (6.24) also follows.

We also remark that by (5.6) we deduce that $P_T^{1/2}\varphi_s(t,s)$ exists even when $t=T$, and for each $s \in [0,T[$ we obtain

$$P_T^{1/2}\varphi_s(T,s) = P_T^{1/2}\varphi\left(T,\frac{T+s}{2}\right)\varphi_s\left(\frac{T+s}{2},s\right), \tag{6.25}$$
$$\|P_T^{1/2}\varphi_s(T,s)\|_{\mathcal{L}(H)} \le c_{T-s}. \tag{6.26}$$

We end this section with a result concerning the differentiability of $\varphi(t,s)$ with respect to t, under some restrictions on the exponents δ and α.

Proposition 6.12. *Under Hypotheses* 1.1–1.7, *let* $(\hat{y}(t,s;x), \hat{u}(t,s;x))$ *be the optimal pair for the control problem* (2.9)–(2.10). *We have*

$$\hat{y}(t,s;x) - G(t)\hat{u}(t,s;x) \in D_{A(t)}, \qquad \forall t \in [s, T[,$$

provided the numbers δ *and* α *satisfy* $\alpha + \delta > 1$.

Proof. Assume $\alpha + \delta > 1$. By (2.10) and (2.3) we have for $0 \le s < t < T$, using Lemma 6.7(i),

$$\begin{aligned}
\hat{y}(t,s;x) &= U(t,s)x - \int_s^t U(t,q)A(q)G(q)\hat{u}(q,s;x)\,dq \\
&= U(t,s)x + \int_s^t \left[[-A(q)^*]^{1-\alpha}U(t,q)^*\right]^* \\
&\quad \times [-A(q)]^\alpha G(q)[\hat{u}(q,s;x) - \hat{u}(t,s;x)]\,dq \\
&\quad + \int_s^t \left[[-A(q)^*]^{1-\alpha}U(t,q)^*\right]^* \\
&\quad \times \left[[-A(q)]^\alpha G(q) - [-A(t)]^\alpha G(t)\right]\hat{u}(t,s;x)\,dq \\
&\quad - \int_s^t \left[[-A(q)^*]U(t,q)^*\right]^* \\
&\quad \times \left[[-A(q)]^{-\alpha} - [-A(t)]^{-\alpha}\right][-A(t)]^\alpha G(t)\hat{u}(t,s;x)\,dq \\
&\quad + G(t)\hat{u}(t,s;x) - U(t,s)G(t)\hat{u}(t,s;x);
\end{aligned}$$

hence, denoting by T_i, $i = 1, 2, 3$, the integral terms in the last member, we get

$$\hat{y}(t,s;x) - G(t)\hat{u}(t,s;x) = U(t,s)[x - G(t)\hat{u}(t,s;x)] + \sum_{i=1}^3 T_i, \tag{6.27}$$

and the first term on the right-hand side belongs to $D_{A(t)}$. We now show that $A(t)T_i \in H$ for $i = 1, 2, 3$: by Hypotheses 1.3 and 1.5 and Theorem 4.3(ii) we have

$$\begin{aligned}
&\|A(t)T_1\|_H \\
&\quad = \left\| \int_s^t \left[[-A(q)^*]^{1-\alpha}U(t,q)^*A(t)^*\right]^* \right. \\
&\qquad \left. \times [-A(q)]^\alpha G(q)[\hat{u}(q,s;x) - \hat{u}(t,s;x)]\,dq \right\|_H \\
&\quad \le c\int_s^t (t-q)^{\alpha-2+\delta}(T-t)^{\alpha-1+\delta}dq \le c(t-s)^{\alpha+\delta-1}(T-t)^{\alpha+\delta-1},
\end{aligned} \tag{6.28}$$

and similarly

$$\begin{aligned}
\|A(t)T_2\|_H &= \left\| \int_s^t \left[[-A(q)^*]^{1-\alpha}U(t,q)^*A(t)^*\right]^* \right. \\
&\qquad \left. \times \left[[-A(q)]^\alpha G(q) - [-A(t)]^\alpha G(t)\right]\hat{u}(t,s;x)\,dq \right\|_H \\
&\le c\int_s^t (t-q)^{\alpha-2+\delta}(T-t)^{\alpha-1}dq \le c(t-s)^{\alpha+\delta-1}(T-t)^{\alpha-1}.
\end{aligned} \tag{6.29}$$

Now we remark that

$$\|[-A(q)]^{-\alpha} - [-A(t)]^{-\alpha}\|_{\mathcal{L}(H)} \le c(t-q)^{\mu} \qquad \text{for} \quad 0 \le q \le t \le T; \tag{6.30}$$

indeed, as $\alpha + \rho \ge \alpha + \delta > 1$, this estimate follows, proceeding as in Lemma 2.7(i) of [AFT]. By (6.30) we deduce

$$\begin{aligned}\|A(t)T_3\|_H &= \Big\| \int_s^t \big[[-A(q)^*]U(t,q)^*A(t)^*\big]^* \big[[-A(q)]^{-\alpha} - [-A(t)]^{-\alpha}\big] \\ &\qquad \times [-A(t)]^{\alpha} G(t)\hat{u}(t,s;x)\, dq \Big\|_H \\ &\le c\int_s^t (t-q)^{\alpha-2+\mu}(T-t)^{\alpha-1}\, dq \\ &\le c(t-s)^{\alpha+\mu-1}(T-t)^{\alpha-1}. \end{aligned} \tag{6.31}$$

By (6.28), (6.29), and (6.31) we see that the right member of (6.27) belongs to $D_{A(t)}$. The proof is complete. □

7. The Operators $\Lambda(t)$

Under Hypotheses 1.1 and 1.2, a precise definition of the linear unbounded operator $P \to A(t)^*P + PA(t)$, appearing in (0.4), can be given in the following way (compare with [D]). Fix P in the Banach space $(\Sigma(H), \|\cdot\|_{\mathcal{L}(H)})$ and consider the sesquilinear form defined on $D_{A(t)} \times D_{A(t)}$ by

$$\varphi_P(t;x,y) := (A(t)x, Py)_H + (Px, A(t)y)_H\,, \qquad x,y \in D_{A(t)}. \tag{7.1}$$

We set

$$\begin{aligned} D_{\Lambda(T)} := \big\{P \in \Sigma(H) : \exists c(t;P) > 0 : |\varphi_P(t;x,y)| \\ \le c(t;P)\|x\|_H\|y\|_H, \forall x,y \in D_{A(t)}\big\}. \end{aligned} \tag{7.2}$$

If $P \in D_{\Lambda(t)}$, then $\varphi_P(t;\cdot,\cdot)$ has a unique extension $\bar{\varphi}_P(t;\cdot,\cdot)$ to $H \times H$ such that

$$\begin{cases} \bar{\varphi}_P(t;x,y) = \varphi_P(t;x,y), & \forall x,y \in D_{A(t)}, \\ |\bar{\varphi}_P(t;x,y)| \le c(t;P)\|x\|_H\|y\|_H, & \forall x,y \in H; \end{cases} \tag{7.3}$$

hence by Riesz' Representation Theorem there exists an operator $Q_P(t) \in \mathcal{L}(H)$ such that

$$\bar{\varphi}_P(t;x,y) = (Q_P(t)x \mid y)_H\,, \qquad \forall x,y \in H. \tag{7.4}$$

Now we define

$$\Lambda(t)P := Q_P(t), \qquad \forall P \in D_{\Lambda(t)}, \tag{7.5}$$

i.e.,

$$(\Lambda(t)Px \mid y)_H = \bar{\varphi}_P(t;x,y), \qquad \forall x,y \in H. \tag{7.6}$$

We remark that if $P \in D_{\Lambda(t)}$ and $x \in D_{A(t)}$, then in particular

$$|(Px \mid A(t)y)_H| = |\bar{\varphi}_P(t; x, y) - (A(t)x \mid Py)_H| \\ \leq [c(t; P)\|x\|_H + \|A(t)x\|_H]\,\|y\|_H;$$

this means $Px \in D_{A(t)^*}$ and

$$\Lambda(t)Px = A(t)^*Px + PA(t)x, \qquad \forall x \in D_{A(t)}, \quad \forall P \in D_{\Lambda(t)}, \tag{7.7}$$

i.e., (0.4) holds when evaluated at any $x \in D_{A(t)}$. In particular, by (7.4), (7.3), (7.1), and (7.7) it follows easily that

$$(Q_P(t)x \mid y)_H = (x \mid Q_P(t)y)_H\,, \qquad \forall x, y \in D_{A(t)},$$

and therefore $\Lambda(t)P \equiv Q_P(t) \in \Sigma(H)$ for each $P \in D_{\Lambda(t)}$.

The properties of the family $\{\Lambda(t), t \in [0, T]\}$ are summarized in the following statement.

Proposition 7.1. *Under Hypotheses* 1.1 *and* 1.2, *the family* $\{\Lambda(t)\}$ *satisfies*:

(i) *For each* $t \in [0, T]$, $\Lambda(t)$ *generates in* $\Sigma(H)$ *the analytic semigroup* $e^{\xi\Lambda(t)}$ *given by*

$$e^{\xi\Lambda(t)}P = e^{\xi A(t)^*}Pe^{\xi A(t)}, \qquad P \in \Sigma(H), \tag{7.8}$$

and in particular Hypothesis 1.1 *holds for* $\{\Lambda(t)\}$ *in* $\Sigma(H)$ *for each* $\theta_0 \in]\pi/2, \theta[$ *with a suitable constant* $M_0 := M(\theta_0) \geq M$, *i.e., we have*

$$\left\|[\lambda - \Lambda(t)]^{-1}\right\|_{\mathcal{L}(\Sigma(H))} \leq M_0\,[1 + |\lambda|]^{-1}\,, \\ \forall \lambda \in \overline{S(\theta_0)}, \quad \forall t \in [0, T]. \tag{7.9}$$

(ii) *Hypothesis* 1.2 *holds for* $\{\Lambda(t)\}$ *in* $\Sigma(H)$ *with the same* μ *and for each* $\rho_0 \in]1 - \mu, \rho[$, *with a suitable constant* $N_0 := N(\rho_0) \geq N$, *i.e., we have*

$$\left\|\Lambda(t)[\lambda - \Lambda(t)]^{-1}[\Lambda(t)^{-1} - \Lambda(s)^{-1}]\right\|_{\mathcal{L}(\Sigma(H))} \\ \leq N_0|t - s|^{\mu}[1 + |\lambda|]^{-\rho_0}, \qquad \forall \lambda \in \overline{S(\theta_0)}, \quad \forall t, s \in [0, T]. \tag{7.10}$$

(iii) *The evolution operator* $E(t, s)$ *of the family* $\{\Lambda(T - t)\}$ *is given by*

$$E(t, s)P = U(T - s, T - t)^*PU(T - s, T - t), \\ 0 \leq s \leq t \leq T, \quad P \in \Sigma(H). \tag{7.11}$$

Proof. See Section 2 of [A2]. □

Remark 7.2. (i) According to the remarks in Section 1, it can be shown that $E(t, s)$ satisfies Hypotheses 1.3 and 1.4 in $\Sigma(H)$, with $A(t)$ replaced by $\Lambda(T - t)$ and δ replaced by $\delta_0 := \rho_0 + \mu - 1$. In particular we have

$$E(T - t, 0)P = U(T, t)^*PU(T, t), \qquad \forall t \in [0, T], \quad \forall P \in \Sigma(H). \tag{7.12}$$

(ii) The domains $D_{\Lambda(t)}$ are in general not dense in $\mathcal{L}(H)$, as the following characterization shows (compare with Remark 8.3 below). Thus the analytic semigroups generated by each $\Lambda(t)$ in $\Sigma(H)$ are not continuous at 0. A detailed study of such kinds of semigroups can be found in [Si] or in the book [Lu].

Proposition 7.3. *Under Hypotheses* 1.1–1.4 *we have*

$$\begin{aligned}\overline{D_{\Lambda(\tau)}} &= \left\{P \in \Sigma(H) : \lim_{s\to\tau} \left\|U(\tau, s)^* P - P\right\|_{\mathcal{L}(H)} = 0\right\} \\ &= \left\{P \in \Sigma(H) : \lim_{t\to\tau} \left\|U(t, \tau)^* P - P\right\|_{\mathcal{L}(H)} = 0\right\}.\end{aligned}$$

Proof. We just prove the result when $\tau = T$, since the case $\tau \in [0, T[$ is quite similar. If $P \in \overline{D_{\Lambda(T)}}$, fix $\varepsilon > 0$ and choose $Q \in D_{\Lambda(T)}$ such that $\|P - Q\|_{\mathcal{L}(H)} < \varepsilon$. Then, recalling that

$$\frac{d}{ds}U(T, s)^* = A(s)^* U(T, s)^*, \qquad \forall s \in [0, T[$$

(compare with Lemma 6.7(i)), we get

$$\begin{aligned}&\left\|U(T, s)^* P - P\right\|_{\mathcal{L}(H)} \\ &\quad\le \left\|\left[U(T, s)^* - 1_H\right](P - Q)\right\|_{\mathcal{L}(H)} + \left\|U(T, s)^* Q - Q\right\|_{\mathcal{L}(H)} \\ &\quad\le c\|P - Q\|_{\mathcal{L}(H)} + \left\|\int_s^T A(\tau)^* U(T, \tau)^* Q \, d\tau\right\|_{\mathcal{L}(H)}. \end{aligned} \tag{7.13}$$

Now by the representation formula in Proposition 3.1 of [A2] is is easy to see that

$$D_{\Lambda(T)} \subseteq \left\{Q \in \Sigma(H) : [-A(T)^*]^\eta Q \in \mathcal{L}(H)\right\}, \qquad \forall \eta \in [0, 1[;$$

thus by (7.13) and Hypothesis 1.3 it follows that

$$\begin{aligned}&\|U(T, s)^* P - P\|_{\mathcal{L}(H)} \\ &\quad\le c\|P - Q\|_{\mathcal{L}(H)} \\ &\qquad + \left\|\int_s^T \left[[-A(\tau)^*] U(T, \tau)^* [-A(T)^*]^{-\eta}\right] [-A(T)^*]^\eta Q \, d\tau\right\|_{\mathcal{L}(H)} \\ &\quad\le c\|P - Q\|_{\mathcal{L}(H)} + c\int_s^T (T - \tau)^{\eta-1} \|[-A(T)^*]^\eta Q\|_{\mathcal{L}(H)} \, d\tau \\ &\quad\le c\varepsilon + c\|[-A(T)^*]^\eta Q\|_{\mathcal{L}(H)} (T - s)^\eta,\end{aligned}$$

so that $U(T, s)^* P \to P$ in $\mathcal{L}(H)$ as $s \to T^-$. Conversely, assume that $U(T, s)^* P \to P$ in $\mathcal{L}(H)$ as $s \to T^-$; then we also have

$$\|PU(T, s) - P\|_{\mathcal{L}(H)} = \left\|\left[U(T, s)^* P - P\right]^*\right\|_{\mathcal{L}(H)} \to 0 \qquad \text{as} \quad s \to T^-.$$

Hence, by (7.12),

$$\begin{aligned}&\|E(T - s, 0)P - P\|_{\mathcal{L}(H)} \\ &\quad= \|U(T, s)^* PU(T, s) - P\|_{\mathcal{L}(H)}\end{aligned}$$

$$\leq \left\| \left[U(T,s)^* - 1_H \right] PU(T,s) \right\|_{\mathcal{L}(H)} + \| P\,[U(T,s) - 1_H] \|_{\mathcal{L}(H)}$$
$$\leq c \left\| \left[U(T,s)^* P - P \right\|_{\mathcal{L}(H)} + \| PU(T,s) - P \|_{\mathcal{L}(H)} \to 0 \qquad \text{as} \quad s \to T^-;$$

this shows that $P \in \overline{D_{\Lambda(T)}}$. □

The following proposition is a consequence of the results of Proposition B.1 in Appendix B below.

Proposition 7.4. *Under Hypotheses* 1.1 *and* 1.2, *consider the linear backward Cauchy problem*:

$$\begin{cases} Q'(t) = -\Lambda(t)Q(t) - F(t), & t \in [0,T[, \\ Q(T) = Q_T. \end{cases}$$

(i) *If* $Q_T \in \overline{D_{\Lambda(T)}}$ *and* $F \in L^1(0,T;\Sigma(H)) \cap Z_{1,\eta}([0,T[,\Sigma(H))$, $\eta \in]0,\delta[$, *then there exists a unique classical solution* Q, *such that moreover* Q' *and* $\Lambda(\cdot)Q(\cdot)$ *belong to* $Z^*_{1,\eta}([0,T[,\Sigma(H))$.

(ii) *If* $Q_T \in D_{\Lambda(T)}$ *and* $F \in B(0,T;\Sigma(H)) \cap Z_{0,\eta}([0,T[,\Sigma(H))$, *then the classical solution* Q *satisfies* $Q', \Lambda(\cdot)Q(\cdot) \in B(0,T;\Sigma(H)) \cap Z_{0,\eta}([0,T[,\Sigma(H))$.

(iii) *If* $Q_T \in D_{\Lambda(T)}$, $F \in C([0,T],\Sigma(H)) \cap Z_{0,\eta}([0,T[,\Sigma(H))$, *and* $\Lambda(T)Q_T + F(T) \in \overline{D_{\Lambda(T)}}$, *then there exists a unique strict solution* Q, *such that moreover* $Q', \Lambda(\cdot)Q(\cdot) \in Z_{0,\eta}([0,T[,\Sigma(H))$.

(iv) *In all cases* Q *is given by*

$$Q(t) = E(T-t,0)Q_T + \int_t^T E(T-t,T-\sigma)F(\sigma)\,d\sigma, \qquad t \in [0,T],$$

where $E(\tau,s)$ *is the evolution operator associated to* $\{\Lambda(T-\cdot)\}$.

(*The spaces* $Z_{\gamma,\eta}$ *and* $Z^*_{\gamma,\eta}$ *are defined in Definitions A.*1 *and A.*6 *of Appendix A below.*

Proof. The function $R(t) := Q(T-t)$, i.e.,

$$R(t) = E(t,0)Q_T + \int_0^t E(t,s)F(T-s)\,ds, \qquad t \in [0,T],$$

is, by Proposition B.1 of Appendix B, the classical or strict solution of the problem

$$\begin{cases} R'(t) = \Lambda(T-t)R(t) + F(T-t), & t \in]0,T], \\ R(0) = Q_T; \end{cases}$$

hence all statements follow by the corresponding ones in Proposition B.1. □

We now return to the Riccati operator $P(t)$ introduced in (5.7). In order to prove that it is a classical solution of Riccati equation (0.4), we show that $P(t)$ belongs to $D_{\Lambda(t)}$ for each $t \in [0,T[$. More precisely:

Theorem 7.5. *Under Hypotheses* 1.1–1.7, *let* $P(t)$ *be the operator defined in* (5.7). *Then we have* $P(t) \in D_{\Lambda(t)}$ *for each* $t \in [0, T[$ *and*

$$(\Lambda(t)P(t)x \mid y)_H$$
$$= \int_t^T \left\{(M(\tau)\varphi(\tau,t)x \mid V(\tau,t)y)_H + (V(\tau,t)x \mid M(\tau)\varphi(\tau,t)y)_H\right\} d\tau$$
$$+ (P_T\varphi(T,t)x \mid V(T,t)y)_H + (V(T,t)x \mid P_T\varphi(T,t)y)_H, \qquad \forall t \in [0,T[,$$

where $\varphi(\tau,t)$ *is defined by* (5.1) *and* $V(\tau,t) = (d/dt)U(\tau,t)$.

Proof. For $x, y \in D_{A(t)}$ we write, by (7.1) and (5.7),

$$\varphi_{P(t)}(x,y) = (P(t)x \mid A(t)y)_H + (A(t)x \mid P(t)y)_H$$
$$= \int_t^T \left\{\big(M(\tau)\hat{y}(\tau,t;x) \mid U(\tau,t)A(t)y\big)_H + \big(U(\tau,t)A(t)x \mid M(\tau)\hat{y}(\tau,t;y)\big)_H\right\} d\tau$$
$$+ \big(P_T\hat{y}(T,t;x) \mid U(T,t)A(t)y\big)_H + \big(U(T,t)A(t)x \mid P_T\hat{y}(T,t;y)\big)_H. \tag{7.14}$$

By (2.11) we obtain

$$\varphi_{P(t)}(x,y)$$
$$= \int_t^T \left\{(M(\tau)U(\tau,t)x \mid U(\tau,t)A(t)y)_H + (U(\tau,t)A(t)x \mid M(\tau)U(\tau,t)y)_H\right\} d\tau$$
$$- \int_t^T \left\{ \big(M(\tau)L_t[\hat{u}(\cdot,t;x)](\tau) \mid U(\tau,t)A(t)y\big)_H + \big(U(\tau,t)A(t)x \mid M(\tau)L_t[\hat{u}(\cdot,t;y)](\tau)\big)_H \right\} d\tau$$
$$+ \left\{(P_TU(T,t)x \mid U(T,t)A(t)y)_H + (U(T,t)A(t)x \mid P_TU(T,t)y)_H\right\}$$
$$- \left\{\big(P_TL_{tT}[\hat{u}(\cdot,t;x)] \mid U(T,t)A(t)y\big)_H - \big(U(T,t)A(t)x \mid P_TL_{tT}[\hat{u}(\cdot,t;y)]\big)_H\right\}$$
$$=: \sum_{i=1}^4 I_i.$$

Let us estimate the terms I_i. By Hypothesis 1.6, Theorem 4.3(i), (4.3), and Lemma 6.7(i)–(ii),

$$|I_2| \le c\int_t^T (\tau-t)^{\alpha-1}(T-\tau)^{\alpha-1}\, d\tau \|x\|_H\|y\|_H$$
$$\le c(T-t)^{2\alpha-1}\|x\|_H\|y\|_H, \tag{7.15}$$

whereas by (3.2), Proposition 4.1(i), and Lemma 6.7(i)–(ii),

$$|I_4| \le c\left\|P_T^{1/2}\right\|_{\mathcal{L}(H)} (T-t)^{-1}\left\{\|\hat{u}(\cdot,t;x)\|_{X_t}\|y\|_H + \|\hat{u}(\cdot,t;y)\|_{X_t}\|x\|_H\right\}$$
$$\le c(T-t)^{-1}\|x\|_H\|y\|_H. \tag{7.16}$$

Next, by (7.12) and Proposition 7.1,

$$\begin{aligned} |I_3| &= |(A(t)^* U(T,t)^* P_T U(T,t)x \mid y)_H + (U(T,t)^* P_T U(T,t)A(t)x \mid y)_H \\ &= |(\Lambda(t)E(T-t,0)P_T x \mid y)_H| \le c(T-t)^{-1}\|x\|_H\|y\|_H. \end{aligned} \tag{7.17}$$

Finally, concerning the term I_1, we remark that by Proposition 7.4 the function $t \to \int_t^T E(T-t, T-\tau)M(\tau)\,d\tau$ is the classical solution of

$$\begin{cases} R'(t) = -\Lambda(t)R(t) - M(t), & t \in [0,T[, \\ R(T) = 0, \end{cases}$$

with $M \in C^\delta([0,T], \Sigma(H))$ by Hypothesis 1.6, and obviously $0 \in D_{\Lambda(T)}$; hence it follows that

$$t \to \Lambda(t)\int_t^T E(T-t,T-\tau)M(\tau)\,d\tau \in B(0,T;\Sigma(H)) \cap Z_{0,\eta}([0,T[,\Sigma(H)), \quad \forall \eta \in]0,\delta[.$$

Hence we have, by (7.12),

$$\begin{aligned} |I_1| &= \Bigg|\left(A(t)^* \int_t^T U(\tau,t)^* M(\tau)U(\tau,t)\,d\tau\; x \mid y\right)_H \\ &\qquad + \left(\int_t^T U(\tau,t)^* M(\tau)U(\tau,t)\,d\tau A(t)x \mid y\right)_H\Bigg| \\ &= \left|\left(\Lambda(t)\int_t^T E(T-t,T-\tau)M(\tau)\,d\tau\; x \mid y\right)_H\right| \le c\|x\|_H\|y\|_H. \end{aligned} \tag{7.18}$$

By (7.15)–(7.18) we conclude that

$$|\varphi_{P(t)}(x,y)| \le c(T-t)^{-1}\|x\|_H\|y\|_H,$$

which shows, by definition (see (7.2)), that $P(t) \in D_{\Lambda(t)}$. In addition, by (7.14), (5.1), and Lemma 6.7(i), we obtain the desired expression for the operator $\Lambda(t)P(t)$. □

8. The Riccati Equation

We are now ready to show that the operator $P(t)$ defined by (5.7) solves the Riccati differential equation (0.4). We start with the following result, which is more general than Theorem 1.1(viii) of [LT3] even in the autonomous case:

Theorem 8.1. *Under Hypotheses* 1.1–1.7, *let* $P(t)$ *be defined by* (5.7). *Then for each* $x, y \in H$ *we have* $(P(\cdot)x \mid y)_H \in C^1([0,T[)$ *and*

$$\begin{aligned} \frac{d}{dt}(P(t)x \mid y)_H &= -(M(t)x \mid y)_H - (\Lambda(t)P(t)x \mid y)_H \\ &\qquad + \left(P(t)A(t)G(t)N(t)^{-1}G(t)^* A(t)^* P(t)x \mid y\right)_H. \end{aligned} \tag{8.1}$$

Proof. Assume first $x, y \in D_{A(t)}$, with fixed $t \in [0, T[$. Then, starting from (5.7), for $0 < h < (T-t)/2$ we can write

$$\begin{aligned}
&\left(\frac{P(t+h)-P(t)}{h}x \mid y\right)_H \\
&\quad= \left(\int_{t+h}^{T} \frac{U(\sigma,t+h)^* - U(\sigma,t)^*}{h} M(\sigma)\varphi(\sigma,t+h)x\, d\sigma \mid y\right)_H \\
&\qquad + \left(\int_{t+h}^{T} U(\sigma,t)^* M(\sigma)\frac{\varphi(\sigma,t+h)-\varphi(\sigma,t)}{h} x\, d\sigma \mid y\right)_H \\
&\qquad - \left(\frac{1}{h}\int_{t}^{t+h} U(\sigma,t)^* M(\sigma)\varphi(\sigma,t)x\, d\sigma \mid y\right)_H \\
&\qquad + \left(\frac{U(T,t+h)^* - U(T,t)^*}{h} P_T\varphi(T,t+h)x \mid y\right)_H \\
&\qquad + \left(U(T,t)^* P_T \frac{\varphi(T,t+h)-\varphi(T,t)}{h} x \mid y\right)_H =: \sum_{i=1}^{5} I_i .
\end{aligned}$$

Now by lemma 6.7 we have

$$\begin{aligned}
I_1 &= \int_{t+h}^{T} \left(M(\sigma)\varphi(\sigma,t+h)x \mid \frac{1}{h}\int_t^{t+h} [V(\sigma,q)-V(\sigma,t)]y\, dq\right)_H d\sigma \\
&\quad + \int_{t+h}^{T} (M(\sigma)\varphi(\sigma,t+h)x \mid V(\sigma,t)y)_H\, d\sigma \\
&\quad \text{(by (5.5) and Lemma 6.7(vi)–(i))} \\
&= -\int_t^T (M(\sigma)\varphi(\sigma,t)x \mid U(\sigma,t)A(t)y)_H\, d\sigma + O(h^{\delta/2}) + O(h^{\mu}), \\
&\quad \text{as} \quad h \to 0^+.
\end{aligned}$$

Next,

$$\begin{aligned}
I_2 &= \int_{t+h}^{T} \left(M(\sigma)\frac{1}{h}\int_t^{t+h} [\varphi_q(\sigma,q)-\varphi_t(\sigma,t)]x\, dq \mid U(\sigma,t)y\right)_H d\sigma \\
&\quad + \int_{t+h}^{T} (M(\sigma)\varphi_t(\sigma,t)x \mid U(\sigma,t)y)_H\, d\sigma \\
&\quad \text{(by (6.22))} \\
&= \int_{t+h}^{T} \left(M(\sigma)\frac{1}{h}\int_t^{t+h} [V(\sigma,q)-V(\sigma,t)]x\, dq \mid U(\sigma,t)y\right)_H d\sigma \\
&\quad + \int_{t+h}^{T} \left(M(\sigma)\frac{1}{h}\int_t^{t+h} [K(\sigma,q)-K(\sigma,t)]x\, dq \mid U(\sigma,t)y\right)_H d\sigma \\
&\quad + \int_{t+h}^{T} \left(M(\sigma)\frac{1}{h}\int_t^{t+h}\int_q^{\sigma} R(\sigma,p)\right. \\
&\qquad\qquad \left. \times [V(p,q)x - V(p,t)x]\, dp\, dq \mid U(\sigma,t)y\right)_H d\sigma
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t+h}^{T}\left(M(\sigma)\frac{1}{h}\int_t^{t+h}\int_q^{\sigma} R(\sigma,p)\right.\\
&\qquad\left.\times\,[K(p,q)x-K(p,t)x]\,dp\,dq \mid U(\sigma,t)y\right)_H\,d\sigma\\
&-\int_{t+h}^{T}\left(M(\sigma)\frac{1}{h}\int_t^{t+h}\int_t^{q} R(\sigma,p)\right.\\
&\qquad\left.\times\,[-U(p,t)A(t)x+K(p,t)x]\,dp\,dq \mid U(\sigma,t)y\right)_H\,d\sigma\\
&+\int_{t+h}^{T}(M(\sigma)\varphi_t(\sigma,t)x \mid U(\sigma,t)y)_H\,d\sigma
\end{aligned}
$$

(by Lemma 6.7(vi), Lemma 6.5(ii), (6.10), and Lemma 6.7(i))

$$
=\int_t^T (M(\sigma)\varphi_t(\sigma,t)x \mid U(\sigma,t)y)_H\,d\sigma + O(h^{\delta/2})+O(h^{\delta})+O(h^{\alpha})
$$

as $h\to 0^+$.

We remark that in view of (6.24) the limit of I_2 is meaningful, i.e., the integral converges.

Concerning I_3, I_4, and I_5, by Proposition 5.1 we have

$$
\begin{aligned}
I_3 &= -\left(\frac{1}{h}\int_t^{t+h} U(\sigma,t)^*M(\sigma)\varphi(\sigma,t)x\,d\sigma \mid y\right)_H\\
&= -(M(t)x \mid y)_H + o(1) \qquad \text{as} \quad h\to 0^+;
\end{aligned}
$$

by Proposition 5.1 and Lemma 6.7(i) we get

$$
\begin{aligned}
I_4 &= \left(P_T\varphi(T,t+h)x \mid \frac{U(T,t+h)-U(T,t)}{h}y\right)_H\\
&= -(P_T\varphi(T,t)x \mid U(T,t)A(t)y)_H + o(1) \qquad \text{as} \quad h\to 0^+;
\end{aligned}
$$

finally by (6.25) we obtain

$$
\begin{aligned}
I_5 &= \left(P_T\frac{\varphi(T,t+h)-\varphi(T,t)}{h}x \mid U(T,t)y\right)_H\\
&= (P_T\varphi_t(T,t)x \mid U(T,t)y)_H + o(1) \qquad \text{as} \quad h\to 0^+.
\end{aligned}
$$

Summing up, we have

$$
\begin{aligned}
&\lim_{h\to 0^+}\left(\frac{P(t+h)-P(t)}{h}x \mid y\right)_H\\
&\quad = -\int_t^T (M(\sigma)\varphi(\sigma,t)x \mid U(\sigma,t)A(t)y)_H\,d\sigma\\
&\qquad + \int_t^T (M(\sigma)\varphi_t(\sigma,t)x \mid U(\sigma,t)y)_H\,d\sigma - (M(t)x \mid y)_H\\
&\qquad - (P_T\varphi(T,t)x \mid U(T,t)A(t)y)_H + (P_T\varphi_t(T,t)x \mid U(T,t)y)_H\,,\\
&\quad \forall x, y\in D_{A(t)}.
\end{aligned}
$$

A quite similar calculation shows that the same limit is obtained as $h \to 0^-$. This shows that for each $t \in [0, T[$ and $x, y \in D_{A(t)}$ the derivative of $(P(\tau)x \mid y)_H$ exists at the point $\tau = t$ and

$$\begin{aligned}
&\left[\frac{d}{d\tau}(P(\tau)x \mid y)_H\right]_{\tau=t} \\
&\quad = -\int_t^T (M(\sigma)\varphi(\sigma,t)x \mid U(\sigma,t)A(t)y)_H \, d\sigma \\
&\qquad + \int_t^T (M(\sigma)\varphi_t(\sigma,t)x \mid U(\sigma,t)y)_H \, d\sigma - (M(t)x \mid y)_H \\
&\qquad - (P_T\varphi(T,t)x \mid U(T,t)A(t)y)_H + (P_T\varphi_t(T,t)x \mid U(T,t)y)_H \,. \qquad (8.2)
\end{aligned}$$

We show not that (8.1) holds. Denote by T_i, $i = 1, 2, 3, 4, 5$, the terms on the right-hand side of (8.2): then by (5.7) we see that

$$\begin{aligned}
&T_1 + T_4 = -(P(t)x \mid A(t)y)_H, \\
&T_3 = -(M(t)x \mid y)_H, \\
&T_2 + T_5 = \left(\left[\int_t^T U(\sigma,t)^* M(\sigma)\varphi_t(\sigma,t)\, d\sigma + U(T,t)^* P_T\varphi_t(T,t)\right] x \mid y\right)_H .
\end{aligned}$$

Now, choosing any $r \in]t, T[$ and using (5.7) and (5.2), we can rewrite $T_2 + T_5$ as

$$T_2 + T_5 = (U(r,t)^* P(r)\varphi_t(r,t)x \mid y)_H + \left(\int_t^r U(\sigma,t)^* M(\sigma)\varphi_t(\sigma,t)\, d\sigma \; x \mid y\right)_H ;$$

now the second term tends to 0 as $r \to t^+$, since, by (6.24),

$$\left|\left(\int_t^r U(r,\sigma)^* M(\sigma)\varphi_t(\sigma,t)\, d\sigma \; x \mid y\right)_H\right| \le c_{(T-t)/2}\|A(t)x\|_H\|y\|_H(r-t)^\alpha,$$

whereas the first term transforms in the following manner: using (6.22),

$$\begin{aligned}
&\big(U(r,t)^* P(r)\varphi_t(r,t)x \mid y\big)_H \\
&\quad = \big(U(r,t)^* P(r)V(r,t)x \mid y\big)_H + (U(r,t)^* P(r)K(r,t)x \mid y)_H \\
&\qquad - \left(U(r,t)^* P(r)\int_t^r K(r,\sigma)\varphi_t(\sigma,t)x\, d\sigma \mid y\right)_H \\
&\qquad \text{(by Lemma 6.7(i) and (6.5))} \\
&\quad = -(U(r,t)^* P(r)U(r,t)A(t)x \mid y)_H \\
&\qquad + (U(r,t)^* P(r)U(r,t)A(t)G(t)N(t)^{-1}G(t)^* A(t)^* P(t)x \mid y)_H \\
&\qquad - (U(r,t)^* P(r)\int_t^r K(r,\sigma)\varphi_t(\sigma,t)x\, d\sigma \mid y)_H;
\end{aligned}$$

thus as $r \to t^+$ we obtain that

$$-(U(r,t)^* P(r)U(r,t)A(t)x \mid y)_H \to -(P(t)A(t)x \mid y)_H = -(A(t)x \mid P(t)y)_H,$$

$$\begin{aligned}
&(U(r,t)^*P(r)U(r,t)A(t)G(t)N(t)^{-1}G(t)^*A(t)^*P(t)x \mid y)_H \\
&\quad = -([-A(t)]^\alpha G(t)N(t)^{-1}G(t)^*A(t)^*P(t)x \mid [-A(t)^*]^{1-\alpha}U(r,t)^*P(r)y)_H \\
&\qquad \text{(since } P(t) \in \mathcal{L}(H, D([-A(t)^*]^\eta)) \text{ for each } \eta \in]0,1[) \\
&\quad \to -([-A(t)]^\alpha G(t)N(t)^{-1}G(t)^*A(t)^*P(t)x \mid [-A(t)^*]^{1-\alpha}P(t)y)_H \\
&\quad = (P(t)A(t)G(t)N(t)^{-1}G(t)^*A(t)^*P(t)x \mid y)_H,
\end{aligned}$$

and the last term tends to 0 as $r \to t^+$, since, by (6.7) and (6.24),

$$\left| \left(U(r,t)^*P(r)\int_t^r K(r,\sigma)\varphi_t(\sigma,t)x\,d\sigma \mid y \right)_H \right| \le c_{(T-t)/2}\|A(t)x\|_H\|y\|_H(r-t)^{2\alpha}.$$

Summing up, we have obtained

$$\begin{aligned}
\left[\frac{d}{d\tau}(P(\tau)x \mid y)_H\right]_{\tau=t} &= \sum_{i=1}^5 T_i \\
&= -(P(t)x \mid A(t)y)_H - (M(t)x \mid y)_H - (A(t)x \mid P(t)y)_H \\
&\quad + (P(t)A(t)G(t)N(t)^{-1}G(t)^*A(t)^*P(t)x \mid y)_H;
\end{aligned}$$

hence by Theorem 7.5

$$\begin{aligned}
\left[\frac{d}{d\tau}(P(\tau)x \mid y)_H\right]_{\tau=t} &= -(\Lambda(t)P(t)x \mid y)_H - (M(t)x \mid y)_H \\
&\quad + (P(t)A(t)G(t)N(t)^{-1}G(t)^*A(t)^*P(t)x \mid y)_H, \\
&\forall x, y \in D_{A(t)}.
\end{aligned}$$

Finally we observe that the right member of the above equality is a bounded linear operator in H for each $t \in [0,T[$; thus, since $D_{A(t)}$ is dense in H, we obtain in a standard way that the equality holds for each $x, y \in H$. The proof of Theorem 8.1 is complete. □

We remark that this result, even when $A(t) \equiv A$, improves Theorem 4.5 of [LT3]. However, in fact, we have more: the next result in the autonomous case was known only under additional assumptions on P_T (see Corollary 1.7 of [LT4]).

Theorem 8.2. *Under Hypotheses* 1.1–1.7, *let* $P(t)$ *be defined by* (5.7). *Then* $P \in C^1([0,T[, \Sigma(H))$, $\Lambda(\cdot)P(\cdot) \in C([0,T[, \Sigma(H))$, *and*

$$P'(t) + \Lambda(t)P(t) = -M(t) + P(t)A(t)G(t)N(t)^{-1}G(t)^*A(t)^*P(t), \quad \forall t \in [0,T[. \tag{8.3}$$

Proof. Let $P(t)$ be given by (5.7) and fix $\varepsilon \in]0,T[$. Then $P(T-\varepsilon) \in D(\Lambda(T-\varepsilon))$ by Theorem 7.5, and in addition the function

$$t \to -M(t) + P(t)A(t)G(t)N(t)^{-1}G(t)^*A(t)^*P(t)$$

belongs to $C^{\delta}([0, T - \varepsilon], \Sigma(H))$ by Hypotheses 1.5 and 1.6 and Lemma 6.4; hence if we consider the linear problem

$$\begin{cases} Q'(t) + \Lambda(t)Q(t) = -M(t) + P(t)A(t)G(t)N(t)^{-1}G(t)^* A(t)^* P(t), \\ \qquad t \in [0, T - \varepsilon], \\ Q(T - \varepsilon) = P(T - \varepsilon), \end{cases}$$

by Proposition 7.4 we obtain for this problem a unique classical solution $Q \in C([0, T - \varepsilon], \Sigma(H)) \cap C^1([0, T - \varepsilon[, \Sigma(H))$ with $\Lambda(\cdot)Q(\cdot) \in C([0, T - \varepsilon[, \Sigma(H))$.

We want to show that $Q(t) = P(t)$ for each $t \in [0, T - \varepsilon]$. In order to do this, we repeat the usual argument (see also the proof of Proposition B.1 in Appendix B): set $R = Q - P$ and fix $t \in]0, T - \varepsilon]$, $x, y \in H$; thus $R(\sigma) \in D_{\Lambda(\sigma)}$ for each $\sigma \in [t, T - \varepsilon[$ and, by Theorem 8.1,

$$\frac{d}{d\sigma}(R(\sigma)x \mid y)_H = -(\Lambda(\sigma)R(\sigma)x \mid y)_H, \qquad \forall x, y \in H, \quad \forall \sigma \in [t, T - \varepsilon[.$$

Consider the function

$$z(\sigma) = (E(T - t, T - \sigma)R(\sigma)x \mid y)_H, \qquad \sigma \in [t, T - \varepsilon];$$

we have, by (7.11),

$$z(\sigma) = (R(\sigma)U(\sigma, t)x \mid U(\sigma, t)y)_H,$$

so that we find that z is differentiable in $]t, T - \varepsilon[$ and, by (7.7),

$$\begin{aligned} z'(\sigma) = {} & \left[\frac{d}{d\sigma}(R(\sigma)\xi \mid \zeta)_H\right]_{\xi = U(\sigma,t)x,\, \zeta = U(\sigma,t)y} \\ & + (R(\sigma)A(\sigma)U(\sigma, t)x \mid U(\sigma, t)y)_H + (R(\sigma)U(\sigma, t)x \mid A(\sigma)U(\sigma, t)y)_H \\ = {} & -(\Lambda(\sigma)R(\sigma)U(\sigma, t)x \mid U(\sigma, t)y)_H \\ & + \big([R(\sigma)A(\sigma) + A(\sigma)^* R(\sigma)]U(\sigma, t)x \mid U(\sigma, t)y\big)_H = 0, \\ & \forall \sigma \in]t, T - \varepsilon[. \end{aligned}$$

Hence z is constant in $[t, T - \varepsilon]$, i.e.,

$$\begin{aligned} (R(t)x \mid y)_H &= z(t) = z(T - \varepsilon) \\ &= (E(T - t, \varepsilon)R(T - \varepsilon)x \mid y)_H = 0, \qquad \forall x, y \in H; \end{aligned}$$

this implies

$$R(t) = 0, \qquad \forall t \in [0, T - \varepsilon],$$

so that $P \equiv Q$ in $[0, T - \varepsilon]$; in particular, $P', \Lambda(\cdot)P(\cdot) \in C([0, T - \varepsilon], \Sigma(H))$ and (8.3) holds in $[0, T - \varepsilon]$. By the arbitrariness of ε, we get the desired result. □

Remark 8.3. We know by Theorem 5.5 that $P(t)x \to P_T x$ for each $x \in H$ as $t \to T^-$; in general, however, $P(\cdot)$ does not belong to $C([0,T], \Sigma(H))$. In fact we have

$$P(\cdot) \in C([0,T], \Sigma(H)) \quad \Leftrightarrow \quad P_T \in \overline{D_{\Lambda(T)}}. \tag{8.4}$$

This in turn is equivalent to saying that $P(\cdot)$ is a classical solution in $\Sigma(H)$ of the Riccati equation (8.3) , in the sense of Appendix B below.

Let us prove property (8.4): if $P_T \in \overline{D_{\Lambda(T)}}$, then we can refine the proof of Theorem 5.5 using Proposition 7.3, obtaining that $\|P(t) - P_T\|_{\mathcal{L}(H)} \to 0$ as $t \to T^-$. Conversely, if $P(\cdot) \in C([0,T], \Sigma(H))$, then obviously $P(t) \to P_T$ in $\mathcal{L}(H)$ as $t \to T^-$, so that by (5.7) and Proposition 7.3 we readily get $P_T \in \overline{D_{\Lambda(T)}}$.

Notice that the identity operator 1_H does not belong to $\overline{D_{\Lambda(T)}}$ unless $A(T)$ is bounded, and hence in that case $P(t) \to 1_H$ just strongly, as $t \to T^-$, although the optimal control of the associated problem (0.1)–(0.2) exists. On the other hand, any compact operator $Q \in \Sigma(H)$, and in particular every $Q \in \Sigma(H)$ with finite dimensional range, belongs to $\overline{D_{\Lambda(T)}}$. Hence such operators originate classical solutions $P(t)$ of the differential Riccati equations such that $P(t) \to P_T$ in $\mathcal{L}(H)$ as $t \to T^-$, provided they satisfy Hypothesis 1.7; this assumption is basic for the existence of an optimal control, as the counterexample in [F3] and the argument in Section 7.1 of [LT2] show.

9. Examples

We consider in this section two control problems whose state equations are nonautonomous parabolic systems with Dirichlet and Neumann boundary conditions, respectively. We think of them as prototypes of the class of problems which are covered by the abstract assumptions of Section 1.

Following [AFT], let Ω be a bounded open set of $\mathbb{R}^n$, with boundary $\partial\Omega$ of class C^2. Let $\{A_{sj}(t,x)\}_{s,j=1,\dots,n}$ be a set of $N \times N$ complex-valued matrices defined in $[0,T] \times \bar{\Omega}$, fulfilling the following hypotheses:

$$\begin{cases} A_{sj} \in C^{\gamma+1/2}\left([0,T], [C^0(\bar{\Omega})]^{N^2}\right) \cap C^{\gamma}\left([0,T], [C^1(\bar{\Omega})]^{N^2}\right), \\ \mathcal{R}e \displaystyle\sum_{s,j=1}^{n} (A_{sj}(t,x) \cdot \xi_j \mid \xi_s)_{\mathbb{C}^N} \ge \nu_0 \sum_{s=1}^{n} |\xi_s|^2_{\mathbb{C}^N}, \\ \qquad \forall \xi_1, \dots, \xi_n \in \mathbb{C}^N, \quad \forall (t,x) \in [0,T] \times \bar{\Omega}, \end{cases} \tag{9.1}$$

where $\gamma \in]0,1[$, $\nu_0 > 0$. We consider the following problems:

$$\begin{cases} D_t y(t,x) - \displaystyle\sum_{s,j=1}^{n} D_s[A_{sj}(t,x) \cdot D_j y(t,x)] & \\ \quad + y(t,x) = 0 & \text{in} \quad [0,T] \times \bar{\Omega}, \\ y(t,x) = u(t,x) & \text{in} \quad [0,T] \times \partial\Omega, \\ y(0,x) = y_0(x) & \text{in} \quad \bar{\Omega}, \end{cases} \tag{9.2}$$

$$\begin{cases} D_t y(t,x) - \displaystyle\sum_{s,j=1}^{n} D_s[A_{sj}(t,x) \cdot D_j y(t,x)] & \\ \quad + y(t,x) = 0 & \text{in} \quad [0,T] \times \bar{\Omega}, \\ \displaystyle\sum_{s,j=1}^{n} A_{sj}(t,x) \cdot D_j y(t,x) \nu_s(x) = u(t,x) & \text{in} \quad [0,T] \times \partial\Omega, \\ y(0,x) = y_0(x) & \text{in} \quad \bar{\Omega}, \end{cases} \tag{9.3}$$

where y_0, u are prescribed data on the parabolic boundary of $[0,T]\times\bar{\Omega}$. Here $\nu(x)$ is the unit outward normal vector at $x \in \partial\Omega$.

The corresponding control problems are the following:

$$\begin{cases} \text{minimize} \\ \quad J(u) := \displaystyle\int_0^T \int_\Omega (m(t,x)y(t,x) \mid y(t,x))_{\mathbb{C}^N}\, dx\, dt \\ \qquad\qquad + \displaystyle\int_0^T \int_{\partial\Omega} (n(t,x)u(t,x) \mid u(t,x))_{\mathbb{C}^N}\, d\sigma_x\, dt + \int_\Omega |y(T,x)|^2_{\mathbb{C}^N}\, dx \\ \text{over all controls } u \in L^2(]0,T[\times\partial\Omega, \mathbb{C}^N) \text{ subject to the state} \\ \quad \text{equation (9.2) or (9.3);} \end{cases} \tag{9.4}$$

here the matrices m, n satisfy $m \in C^\gamma([0,T],[L^\infty(\Omega)]^{N^2})$, $n \in C^\gamma([0,T],[L^\infty(\partial\Omega)]^{N^2})$, with $m(t,x)$ and $n(t,x) - \nu I_N$ $(\nu > 0)$ positive definite; hence Hypothesis 1.6 is satisfied in $H := [L^2(\Omega)]^N$ and $U := [L^2(\partial\Omega)]^N$ with $\delta = \gamma$. Moreover, Hypothesis 1.7 is certainly fulfilled since P_T is just the identity operator on H.

Introducing, for each $t \in [0,T]$, the differential operators

$$\mathcal{A}(t,x,D)v := \sum_{s,j=1}^n D_s[A_{sj}(t,x)\cdot D_j v] - v, \qquad x \in \bar{\Omega}, \tag{9.5}$$

$$\mathcal{B}_0 v := v|_{\partial\Omega}, \tag{9.6}$$

$$\mathcal{B}_1(t,x,D)v := \sum_{s,j=1}^n A_{sj}(t,x)\cdot D_j v \nu_s(x), \qquad x \in \partial\Omega, \tag{9.7}$$

we can define for $t \in [0,T]$ the following abstract operators on H:

$$\begin{cases} D_{A_0(t)} := \left[W_0^{2,2}(\Omega) \cap W^{1,2}(\Omega)\right]^N, \\ A_0(t)v := \mathcal{A}(t,\cdot,D)v, \end{cases} \tag{9.8}$$

$$\begin{cases} D_{A_1(t)} := \left\{v \in [W^{2,2}(\Omega)]^N : \mathcal{B}_1(t,\cdot,D)v = 0\right\}, \\ A_1(t)v := \mathcal{A}(t,\cdot,D)v. \end{cases} \tag{9.9}$$

The adjoint operators $A_r(t)^*$ of $A_r(t)$ $(r = 0,1)$ are defined by

$$\begin{cases} D_{A_0(t)^*} := \left[W^{1,2}(\Omega) \cap W_0^{1,2}(\Omega)\right]^N, \\ A_0(t)^* y := \overline{\mathcal{A}(t,\cdot,D)}y = \displaystyle\sum_{s,j=1}^n D_j\left[\overline{{}^tA_{sj}(t,\cdot)}\cdot D_s y\right] - y, \end{cases} \tag{9.10}$$

$$\begin{cases} D_{A_1(t)^*} := \left\{ y \in [W^{2,2}(\Omega)]^N : \overline{\mathcal{B}_1(t,\cdot,D)}y \right. \\ \qquad\qquad \left. = \displaystyle\sum_{s,j=1}^n \overline{{}^tA_{sj}(t,\cdot)}\cdot D_s y \nu_j = 0 \right\}, \\ A_1(t)^* y := \overline{\mathcal{A}(t,\cdot,D)}y, \end{cases} \tag{9.11}$$

where $\overline{{}^tA_{sj}}$ is the matrix whose elements are the conjugates of the elements of the transposed ${}^tA_{sj}$ of A_{sj}.

The main properties of the operators $A_r(t)$, $A_r(t)^*$ in the Hilbert space $H = [L^2(\Omega)]^N$ are listed in the following statement:

Proposition 9.1. *Under assumption* (9.1) *we have, for* $r = 0, 1$:

(i) *for each* $t \in [0, T]$, *the operator* $A_r(t)$ *is the infinitesimal generator of an analytic semigroup in* H, *with dense domain* $D_{A_r(t)}$;
(ii) *the family* $\{A_r(t)\}$ *satisfies Hypothesis* 1.2 *in* H, *with* $\mu = \gamma + \frac{1}{2}$, $\rho = \frac{1}{2}$, *so that we have* $\delta := \gamma$.

Proof. See Propositions 2.4 and 2.6 of [AFT]. □

By Proposition 9.1 we can associate to the family $\{A_r(t)\}$, $r = 0, 1$, an evolution operator $U_r(t, s) \in \mathcal{L}(H)$. Indeed, we have:

Proposition 9.2. *Under assumption* (9.1), *for* $r = 0, 1$ *and* $0 \le s < t \le T$, *the evolution operator* $U_r(t, s)$, *associated to* $\{A_r(t)\}$, *exists and satisfies all its usual properties*; *in particular it fulfills Hypotheses* 1.3 *and* 1.4.

Proof. See Propositions 2.8 and 2.9 and Corollaries 2.10 and 2.11 of [AFT]; see also Remark 1.8. □

We now define the operator $G(t)$ of Hypothesis 1.5. We introduce the Dirichlet and Neumann maps $G_0(t)$, $G_1(t)$ from $U = [L^2(\partial\Omega)]^N$ to $H = [L^2(\Omega)]^N$, relative to the operators (9.8) and (9.9), in the following way:

$$u := G_0(t)g \quad \Leftrightarrow \quad \begin{cases} \mathcal{A}(t, \cdot, D)u = 0 & \text{in } \Omega, \\ \mathcal{B}_0 u = g & \text{on } \partial\Omega, \end{cases} \tag{9.12}$$

$$u := G_1(t)g \quad \Leftrightarrow \quad \begin{cases} \mathcal{A}(t, \cdot, D)u = 0 & \text{in } \Omega, \\ \mathcal{B}_1(t, \cdot, D)u = g & \text{on } \partial\Omega. \end{cases} \tag{9.13}$$

Theorem 9.3. *Under assumption* (9.1), *we have, for* $r = 0, 1$:

(i) *the operator* $G_r(t)$ *is well defined from* $[L^2(\partial\Omega)]^N$ *into the domain of* $[-A_r(t)]^\alpha$ *for each* $\alpha \in]0, \alpha_r[$, *where* $\alpha_0 := \frac{1}{4}$ *and* $\alpha_1 := \frac{3}{4}$;
(ii) $\{G_r(t)\}$ *satisfies Hypothesis* 1.5 *in* $U = [L^2(\partial\Omega)]^N$ *and* $H = [L^2(\Omega)]^N$ *with any* $\alpha \in]0, \alpha_r[$, *provided we have* $\gamma[1 - 2(\alpha_r - \alpha)] < \alpha_r - \alpha$.

Proof. Part (i) is proved in (2.71) of [AFT].

In order to prove part (ii) we need the following:

Lemma 9.4. *Under assumption* (9.1), *we have, for* $\tau, t \in [0, T]$:

(i) $\|G_0(t) - G_0(\tau)\|_{\mathcal{L}([L^2(\partial\Omega)]^N, W^{1/2,2}(\Omega)]^N} \le c|t - \tau|^{\gamma+1/4}$;
(ii) $\|G_1(t) - G_1(\tau)\|_{\mathcal{L}([L^2(\partial\Omega)]^N, W^{3/2,2}(\Omega)]^N} \le c|t - \tau|^{\gamma+1/4}$.

Proof. Fix $g \in [C^\infty(\partial\Omega)]^N \subseteq U$ and set $v := G_r(t)g$, $w := G_r(\tau)g$, and $u := v - w = [G_r(t) - G_r(\tau)]g$. Then the function u solves

$$\begin{cases} \mathcal{A}(t, x, D)u = -[\mathcal{A}(t, x, D) - \mathcal{A}(\tau, x, D)]w =: f & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \qquad \text{if} \quad r = 0, \tag{9.14}$$

$$\begin{cases} \mathcal{A}(t,x,D)u = -[\mathcal{A}(t,x,D) - \mathcal{A}(\tau,x,D)]w =: f & \text{in } \Omega \\ \mathcal{B}_1(t,x,D)u = -[\mathcal{B}_1(t,x,D) - \mathcal{B}_1(\tau,x,D)]w =: \varphi & \text{on } \partial\Omega \end{cases} \quad \text{if} \quad r = 1. \tag{9.15}$$

We first consider the case $r = 0$: then (9.14) means

$$\begin{cases} \sum_{s,j=1}^n D_s[A_{sj}(t,x) \cdot D_j u] - u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{9.16}$$

Multiplying by $\bar{u}$ and integrating by parts we easily get, by (9.2),

$$\nu_0 \|Du\|^2_{[L^2(\Omega)]^{Nn}} + \|u\|^2_{[L^2(\Omega)]^N} \le \left| (f \mid u)_{[L^2(\Omega)]^N} \right|;$$

recalling the definition of f and integrating by parts again we obtain, by (9.1),

$$\begin{aligned} &\nu_0 \|Du\|^2_{[L^2(\Omega)]^{Nn}} + \|u\|^2_{[L^2(\Omega)]^N} \\ &\quad \le \left| \sum_{s,j=1}^n \left(\left[A_{sj}(\tau,\cdot) - A_{sj}(t,\cdot) \right] \cdot D_j w \mid D_s u \right)_{[L^2(\Omega)]^N} \right| \\ &\quad \le c|t-\tau|^{\gamma+1/2} \|Dw\|_{[L^2(\Omega)]^{Nn}} \|Du\|^2_{[L^2(\Omega)]^{Nn}}, \end{aligned} \tag{9.17}$$

which implies

$$\|u\|_{[W^{1,2}(\Omega)]^N} \le c|t-\tau|^{\gamma+1/2} \|G_0(\tau)g\|_{[W^{1,2}(\Omega)]^N},$$

and hence, by (2.52) of [AFT],

$$\|u\|_{[W^{1,2}(\Omega)]^N} \le c|t-\tau|^{\gamma+1/2} \|g\|_{[W^{1/2,2}(\partial\Omega)]^N}. \tag{9.18}$$

On the other hand, let $\psi \in [W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)]^N$ be the solution of the problem

$$\begin{cases} \overline{\mathcal{A}(t,\cdot,D)}\varphi = u & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega; \end{cases}$$

we remark that, by the classical estimates of [ADN],

$$\|w\|_{[W^{2,2}(\Omega)]^N} \le c\|u\|_{[L^2(\partial\Omega)]^N}. \tag{9.19}$$

Multiplying (9.14) by $\bar{\psi}$ and integrating twice by parts we get, after some manipulations,

$$(f \mid \psi)_{[L^2(\Omega)]^N} = (\mathcal{A}(t,\cdot,D)u \mid \psi)_{[L^2(\Omega)]^N} = \|u\|^2_{[L^2(\Omega)]^N};$$

hence, using the definition of f, two more integrations by parts yield

$$\begin{aligned} \|u\|^2_{[L^2(\Omega)]^N} = \Bigg| &\left(g \mid \sum_{s,j=1}^n [{}^t\overline{A_{sj}}(\tau,\cdot) - {}^t\overline{A_{sj}}(t,\cdot)] \cdot D_s \psi \nu_j \right)_{[L^2(\partial\Omega)]^N} \\ &- \left(w \mid \sum_{s,j=1}^n D_j([{}^t\overline{A_{sj}}(\tau,\cdot) - {}^t\overline{A_{sj}}(t,\cdot)] \cdot D_s \psi) \right)_{[L^2(\Omega)]^N} \Bigg| \end{aligned}$$

$$\leq \|g\|_{[W^{-1/2,2}(\partial\Omega)]^N} \left\| \sum_{s,j=1}^{n} [{}^t\overline{A_{sj}}(\tau,\cdot) - {}^t\overline{A_{sj}}(t,\cdot)] \cdot D_s\psi\nu_j \right\|_{[W^{1/2,2}(\partial\Omega)]^N}$$

$$+ \|w\|_{[L^2(\Omega)]^N} \left\| \sum_{s,j=1}^{n} D_j([{}^t\overline{A_{sj}}(\tau,\cdot) - {}^t\overline{A_{sj}}(t,\cdot)] \cdot D_s\psi) \right\|_{[L^2(\Omega)]^N}.$$

By (2.58) of [AFT], (9.1), and (9.19) we deduce that

$$\begin{aligned} \|u\|^2_{[L^2(\Omega)]^N} &\leq c\|g\|_{[W^{-1/2,2}(\partial\Omega)]^N} \\ &\quad \times \left\{ |t-\tau|^\gamma \|D\psi\|_{[L^2(\Omega)]^{Nn}} + |t-\tau|^{\gamma+1/2} \|D^2\psi\|_{[L^2(\Omega)]^{Nn^2}} \right\} \\ &\leq c|t-\tau|^\gamma \|\psi\|_{[W^{2,2}(\Omega)]^N} \|g\|_{[W^{-1/2,2}(\partial\Omega)]^N} \\ &\leq c|t-\tau|^\gamma \|u\|_{[L^2(\Omega)]^N} \|g\|_{[W^{-1/2,2}(\partial\Omega)]^N}; \end{aligned}$$

thus we conclude that

$$\|u\|_{[L^2(\Omega)]^N} \leq c|t-\tau|^\gamma \|g\|_{[W^{-1/2,2}(\partial\Omega)]^N}. \tag{9.20}$$

Now by interpolating between (9.18) and (9.20) we simply obtain

$$\|u\|_{[W^{2\theta,2}(\Omega)]^N} \leq c|t-\tau|^{\gamma+\theta} \|g\|_{[W^{2\theta-1/2,2}(\partial\Omega)]^N}, \qquad \forall\theta \in [0, \tfrac{1}{2}],$$

and, in particular, when $\theta = \frac{1}{4}$ we obtain (i).

We now consider the case $r = 1$. Problem (9.15) becomes

$$\begin{cases} \displaystyle\sum_{s,j=1}^{n} D_s[A_{sj}(t,x) \cdot D_j u] - u = f & \text{in } \Omega, \\ \displaystyle\sum_{s,j=1}^{n} A_{sj}(t,x) \cdot D_j u \nu_s = \varphi & \text{on } \partial\Omega. \end{cases} \tag{9.21}$$

Multiplying by $\bar{u}$ and integrating by parts, as before we arrive at (9.17) and hence

$$\|u\|_{[W^{1,2}(\Omega)]^N} \leq c|t-\tau|^{\gamma+1/2} \|G_1(\tau)g\|_{[W^{1,2}(\Omega)]^N};$$

thus by (2.63) of [AFT]

$$\|u\|_{[W^{1,2}(\Omega)]^N} \leq c|t-\tau|^{\gamma+1/2} \|g\|_{[W^{-1/2,2}(\partial\Omega)]^N}. \tag{9.22}$$

On the other hand, by the classical estimates of [ADN], we have

$$\|u\|_{[W^{2,2}(\Omega)]^N} \leq c\left[\|f\|_{L^2(\Omega)]^N} + \|\varphi\|_{[W^{1/2,2}(\partial\Omega)]^N}\right],$$

so that by definition of f, φ we easily get

$$\|u\|_{[W^{2,2}(\Omega)]^N} \leq c\left[|t-\tau|^{\gamma+1/2} \|w\|_{[W^{2,2}(\Omega)]^N} + |t-\tau|^\gamma \|w\|_{[W^{1,2}(\Omega)]^N}\right],$$

and finally, by (2.62) of [AFT],

$$\|u\|_{[W^{2,2}(\Omega)]^N} \leq c|t-\tau|^\gamma \|g\|_{[W^{1/2,2}(\partial\Omega)]^N}. \tag{9.23}$$

Now by interpolating between (9.22) and (9.23) we obtain

$$\|u\|_{[W^{2\theta,2}(\Omega)]^N} \le c|t-\tau|^{\gamma-\theta+1}\|g\|_{[W^{2\theta-3/2,2}(\partial\Omega)]^N}, \qquad \forall\theta \in [\tfrac{1}{2}, 1],$$

and in particular, choosing $\theta = \frac{3}{4}$, we obtain (ii). The proof of Lemma 9.4 is complete. □

We return to the proof of Theorem 9.3(ii). We recall that

$$D([-A_0(t)]^\alpha) = [W^{2\alpha,2}(\Omega)]^N, \qquad \forall\alpha \in]0, \tfrac{1}{4}[, \tag{9.24}$$

$$D([-A_1(t)]^\alpha) = [W^{2\alpha,2}(\Omega)]^N, \qquad \forall\alpha \in]0, \tfrac{3}{4}[- \{\tfrac{1}{2}\}. \tag{9.25}$$

Now fix $\alpha \in]0, \alpha_r[$, $\beta \in]\alpha, \alpha_r[$, and $\tau, t \in [0, T]$ with $0 \le \tau < t \le T$: then, writing for the sake of simplicity $G(t)$ in place of $G_r(t)$ we have

$$\begin{aligned}
&[-A(t)]^\alpha G(t) - [-A(\tau)]^\alpha G(\tau)\\
&\quad = [-A(t)]^\alpha[G(t) - G(\tau)] + [-A(t)]^\alpha[1 - e^{(t-\tau)A(t)}]G(\tau)\\
&\qquad + \left[[-A(t)]^\alpha e^{(t-\tau)A(t)} - [-A(\tau)]^\alpha e^{(t-\tau)A(\tau)}\right] G(\tau)\\
&\qquad + [-A(\tau)]^\alpha[e^{(t-\tau)A(\tau)} - 1]G(\tau) =: \sum_{i=1}^4 I_i.
\end{aligned} \tag{9.26}$$

By (9.24), (9.25), and Lemma 9.4 we get

$$\|I_1\|_{\mathcal{L}(U,H)} \le c(t-\tau)^{\gamma+1/4}, \tag{9.27}$$

whereas by representing the semigroup $e^{\sigma A(t)}$ as a Dunford integral we easily deduce, using the boundedness of $\|[-A(\tau)]^\beta G(\tau)\|_{\mathcal{L}(U,H)}$ and Proposition 9.1(ii), that

$$\begin{aligned}
&\|I_3\|_{\mathcal{L}(U,H)}\\
&\quad \le c_\beta \left\| \left[[-A(t)]^\alpha e^{(t-\tau)A(t)} - [-A(\tau)]^\alpha e^{(t-\tau)A(\tau)}\right] [-A(\tau)]^{-\beta} \right\|_{\mathcal{L}(H)}\\
&\quad = c_\beta \left\| (2\pi i)^{-1} \int_\Gamma (-\lambda)^\alpha e^{(t-\tau)\lambda} \right.\\
&\qquad\qquad \left. \times \left[[\lambda - A(t)]^{-1} - [\lambda - A(\tau)]^{-1}\right] [-A(\tau)]^{-\beta}\ d\lambda \right\|_{\mathcal{L}(H)}\\
&\quad \le c_\beta (t-\tau)^{\gamma+\beta-\alpha}.
\end{aligned} \tag{9.28}$$

Finally, using again the boundedness of $\|[-A(t)]^\beta G(\tau)\|_{\mathcal{L}(U,H)}$ we obtain, for each $\eta \in]0, 2\beta - 2\alpha[$,

$$\begin{aligned}
&\|I_2 + I_4\|_{\mathcal{L}(U,H)}\\
&\le c_\beta \int_0^{t-\tau} \left\| \left[[-A(t)]^{1+\alpha} e^{\sigma A(t)} - [A(\tau)]^{1+\alpha} e^{\sigma A(\tau)}\right] [-A(\tau)]^{-\beta} \right\|_{\mathcal{L}(H)}^\eta\\
&\quad \times \left[\left\| [-A(t)]^{1+\alpha-\beta} e^{\sigma A(t)} \right\|_{\mathcal{L}(H)} + \left\| [-A(\tau)]^{1+\alpha-\beta} e^{\sigma A(\tau)} \right\|_{\mathcal{L}(H)} \right]^{1-\eta} d\sigma\\
&\le c_\beta \int_0^{t-\tau} \left[(t-\tau)^{\alpha+1/2} \sigma^{\beta-\alpha-3/2}\right]^\eta \left[\sigma^{\beta-\alpha-1}\right]^{1-\eta}\ d\sigma \le c_\beta (t-\tau)^{\gamma\eta+\beta-\alpha}.
\end{aligned} \tag{9.29}$$

By (9.26)–(9.29) we obtain that the Hölder exponent of $[-A(\cdot)]^{\alpha}G(\cdot)$ is any number in the set $]0, (2\gamma + 1)(\alpha_r - \alpha)[\cap]0, \gamma + \frac{1}{4}]$, since η and β are arbitrarily close to $2(\alpha_r - \alpha)$ and α_r, respectively. In order that the number γ belongs to this set, we must have $\gamma < (2\gamma + 1)(\alpha_r - \alpha)$, which is equivalent to $\gamma[1 - 2(\alpha_r - \alpha)] < \alpha_r - \alpha$. The proof of Theorem 9.3 is complete. □

Remark 9.5. By Proposition 2.13 of [AFT] it follows that formula (2.2) in its correct form, i.e.,

$$y(t) = U_r(t, 0)x + \int_0^t \left[[-A_r(q)^*]^{1-\alpha} U_r(t, q)^*\right]^* [-A_r(q)]^{\alpha} G_r(q)u(q)\, dq,$$
$$t \in [0, T[, \tag{9.30}$$

where $\alpha \in]0, \alpha_r[$, is meaningful for each $x \in [L^2(\Omega)]^N$ and $u \in [L^2(]0, T[\times\partial\Omega)]^N$ and defines a function $y \in [L^2(]0, T[\times\Omega)]^N$; if in addition x and u are sufficiently smooth (see [AFT] for details), then (9.30) is the solution of problem (9.2) or (9.3), with $y_t, \mathcal{A}(\cdot, \cdot, D)y \in [L^2(]0, T[\times\Omega)]^N$. Hence (9.30) is a reformulation of the state equation for the control problem (9.4). Moreover, when one has $y_t \in [L^2(]0, T[\times\Omega)]^N$, then it also holds that

$$y(t, \cdot) - G_r(t)u(t, \cdot) \in D_{A_r(t)}, \qquad \forall t \in [0, T[.$$

Indeed, for fixed t the function $z := y(t, \cdot) - G_r(t)u(t, \cdot)$ solves the elliptic problem

$$\begin{cases} \mathcal{A}(t, x, D)z = y_t(t, x), & x \in \Omega \\ z|_{\partial\Omega} = 0 \quad \text{or} \quad \mathcal{B}_1(t, x, D)z = 0, & x \in \partial\Omega, \end{cases}$$

so that, by classical results, $z \in D_{A(t)}$ (compare with Remark 6.2).

By the results of this section we see that the control problem (9.2)–(9.4) (resp. (9.3)–(9.4)) satisfies Hypotheses 1.1–1.7 of Section 1 with $\delta = \mu + \rho - 1 = \gamma$ and any $\alpha \in]0, \alpha_r]$, provided we verify the required relationship among the numbers α and γ, i.e., $0 < \gamma < \alpha$ and, as required in Theorem 9.3, $\gamma[1 - 2(\alpha_r - \alpha)] < \alpha_r - \alpha$.

Now in the Dirichlet case ($r = 0$) we have $\alpha_0 = \frac{1}{4}$ and it follows that γ and α must satisfy

$$0 < \alpha < \tfrac{1}{4}, \qquad 0 < \gamma < \frac{1 - 4\alpha}{2 + 8\alpha} \wedge \alpha; \tag{9.31}$$

in the Neumann case ($r = 1$) we have $\alpha_1 = \frac{3}{4}$ and we find that γ and α must satisfy

$$0 < \alpha < \tfrac{3}{4}, \qquad \begin{cases} 0 < \gamma < \alpha & \text{if} \quad \alpha \in]0, \frac{1}{4}], \\ 0 < \gamma < \dfrac{3 - 4\alpha}{8\alpha - 2} \wedge \alpha & \text{if} \quad \alpha \in]\frac{1}{4}, \frac{3}{4}[. \end{cases} \tag{9.32}$$

Appendix A. Some Spaces of Singular Functions

We collect here the definitions of some useful spaces of functions defined in $[a, b] \subset \mathbb{R}$ with values in a Banach space X, and some related properties which are often used in this paper.

Definition A.1.

(i) If $\gamma \geq 0$, $B_\gamma([a, b[, X)$ (resp. $B_\gamma(]a, b], X)$) is the Banach space of Bochner measurable functions $u\colon [a, b[\to X$ (resp. $u\colon]a, b] \to X$) such that $\|u\|_\gamma < \infty$, where

$$\|u\|_\gamma := \sum_{s\in[a,b[} (b-s)^\gamma \|u(s)\|_X \qquad \left(\text{resp. } \sup_{s\in]a,b]} (s-a)^\gamma \|u(s)\|_X\right).$$

(ii) If $\gamma \geq 0$, $C_\gamma([a, b[, X)$ (resp. $C_\gamma(]a, b], X)$) is the space of continuous functions belonging to $B_\gamma([a, b[, X)$ (resp. $B_\gamma(]a, b], X)$), endowed by the same norm.

(iii) If $\eta \in]0, 1]$ and $\gamma \geq 0$, $Z_{\gamma,\eta}([a, b[, X)$ (resp. $Z_{\gamma,\eta}(]a, b], X)$) is the space of functions $u \in C_\gamma([a, b[, X)$ (resp. $C_\gamma(]a, b], X)$) such that $[u]_{\gamma,\eta} < \infty$, where

$$[u]_{\gamma,\eta} := \sup_{s\in[a,b[} \left\{(b-s)^{\gamma+\eta} \sup_{s\leq p<q\leq(s+b)/2} (q-p)^{-\eta}\|u(q)-u(p)\|_X\right\}$$
$$\left(\text{resp. } \sup_{s\in]a,b]} \left\{(s-a)^{\gamma+\eta} \sup_{(s+a)/2\leq p<q\leq s} (q-p)^{-\eta}\|u(q)-u(p)\|_X\right\}\right).$$

(iv) If $\eta \in]0, 1]$ and $\gamma \in [-\eta, 0[$, $Z_{\gamma,\eta}([a, b[, X)$ (resp. $Z_{\gamma,\eta}(]a, b], X)$) is the space of functions $u \in C^{|\gamma|}([a, b], X)$ such that $[u]_{\gamma,\eta} < \infty$, where $[u]_{\gamma,\eta}$ is defined as before.

The spaces $Z_{\gamma,\eta}$ are Banach spaces with their obvious norms, i.e.,

$$\|u\|_{Z_{\gamma,\eta}} := \begin{cases} \|u\|_\gamma + [u]_{\gamma,\eta} & \text{if} \quad \gamma \geq 0, \\ \|u\|_\infty + [u]_{|\gamma|} + [u]_{\gamma,\eta} & \text{if} \quad \gamma \in [-\nu, 0[; \end{cases}$$

they are useful in treating Hölder continuous functions which blow up at an endpoint of their interval of definition. These spaces were introduced in [AT1] (with blow up at $a = 0$) and used in various situations, but an earlier use of them can be found in [So].

The following characterization of the spaces $Z_{\gamma,\eta}$ is useful:

Proposition A.2. *If $\eta \in]0, 1]$ and $\gamma \geq -\eta$, $\gamma \neq 0$, then we have $w \in Z_{\gamma,\eta}([a, b[, X)$ if and only if $w\colon [a, b[\to X$ fulfills*

$$\|w(p) - w(q)\|_X \leq c(p-q)^\eta (b-p)^{-\gamma-\eta} \qquad \textit{for} \quad a \leq q \leq p < b. \tag{A.1}$$

Proof. See [AT5]. □

Remark A.3. When $\gamma = 0$ the above proposition fails, since an element of the space $Z_{0,\eta}([a,b[,X)$ may be unbounded at the point b; however, one can show that $Z_{0,\eta}([a,b[,X) \subseteq BMO(a,b;X)$ (the space of functions of bounded mean oscillation).

Here are some properties of the spaces $Z_{\gamma,\eta}$, whose proof is very easy; we state them in the case of singularities at the second endpoint, but the same statements hold of course in the opposite situation.

Proposition A.4.

(i) *If* $\eta \in]0,1]$ *and* $-\eta \le \gamma \le \nu$, *then* $Z_{\gamma,\eta}([a,b[,X) \subseteq Z_{\nu,\eta}([a,b[,X)$.
(ii) *If* $0 < \lambda < \eta \le 1$ *and* $\gamma \ge -\lambda$, *then* $Z_{\gamma,\eta}([a,b[,X) \subseteq Z_{\gamma,\lambda}([a,b[,X)$.
(iii) *If* $0 < \varepsilon < \eta \le 1$ *and* $\gamma \ge \varepsilon - \eta$, *then* $Z_{\gamma,\eta}([a,b[,X) \subseteq Z_{\gamma-\varepsilon,\eta-\varepsilon}([a,b[,X)$.

Proof. Easy consequences of Definition A.1. □

Remark A.5. In view of Definition A.1, Hypotheses 1.3 and 1.4 just say that, for each $\gamma, \eta \ge 0$,

$$\begin{cases} t \to [-A(t)]^\eta U(t,s)[-A(s)]^{-\gamma} \in Z_{(\eta-\gamma)\vee 0,\delta}(]s,T],\mathcal{L}(H)), \\ s \to [-A(s)^*]^\eta U(t,s)^*[-A(t)^*]^{-\gamma} \in Z_{(\eta-\gamma)\vee 0,\delta}([0,t[,\mathcal{L}(H)). \end{cases} \tag{A.2}$$

In case of nonintegrable singularities, it is useful to introduce another class of more suitable function spaces.

Definition A.6.

(i) If $\gamma \ge 1$, $I_\gamma([a,b[,X)$ (resp. $I_\gamma(]a,b],X)$) is the space of functions $u \in C_\gamma([a,b[,X)$ (resp. $u \in C_\gamma(]a,b],X)$) such that the limit

$$\lim_{h\to 0^+} \int_a^{b-h} u(t)\,dt \qquad \left(\text{resp. } \lim_{h\to 0^+} \int_{a+h}^{b} u(t)\,dt\right)$$

exists in the norm of X.

(ii) If $\gamma \ge 1$ and $\eta \in]0,1]$, we set

$$Z^*_{\gamma,\eta}([a,b[,X) := Z_{\gamma,\eta}([a,b[,X) \cap I_\gamma([a,b[,X),$$
$$Z^*_{\gamma,\eta}(]a,b],X) := Z_{\gamma,\eta}(]a,b],X) \cap I_\gamma(]a,b],X).$$

The spaces $I_\gamma([a,b[,X)$ (resp. $I_\gamma(]a,b],X)$) are Banach spaces with the norm

$$\|u\|_{I_\gamma} := \|u\|_\gamma + \|u\|_*,$$

where

$$\|u\|_* := \sup\left\{ \left\| \int_c^d u(t)\,dt \right\|_X : a \le c \le d < b \right\}$$
$$\left(\text{resp. } \|u\|_* := \sup\left\{ \left\| \int_c^d u(t)\,dt \right\|_X : a < c \le d \le b \right\}\right);$$

for a proof see Lemma 1.7 of [AT2]. The spaces $Z^*_{\gamma,\eta}([a,b[,X)$, $Z^*_{\gamma,\eta}(]a,b],X)$ are Banach spaces with the norm

$$\|u\|^*_{Z_{\gamma,\eta}} := \|u\|_\gamma + [u]_{\gamma,\eta} + \|u\|_*.$$

Appendix B. Linear Nonautonomous Parabolic Equations

We quote here some of the results contained in [AT2] about existence and regularity of strict and classical solutions of the Cauchy problem

$$\begin{cases} u'(t) = A(t)u(t) + f(t), & t \in [0,T], \\ u(0) = u_0 \end{cases} \tag{B.1}$$

in a Banach space E, under Hypotheses 1.1 and 1.2.

A strict solution is a function $u \in C^1([0,T],E)$ such that $A(\cdot)u(\cdot) \in C([0,T],E)$ and such that (B.1) holds in $[0,T]$; a classical solution is a function $u \in C^1(]0,T],E) \cap C([0,T],E)$ such that $A(\cdot)u(\cdot) \in C(]0,T],E)$ and (B.1) holds only in $]0,T]$.

Proposition B.1. *Under Hypotheses* 1.1–1.2*, we have*:

(i) *If* $u_0 \in \overline{D_{A(0)}}$ *and* $f \in L^1(0,T;E) \cap Z_{1,\eta}(]0,T],E)$, $\eta \in]0,\delta]$, *then there exists a unique classical solution* u, *such that moreover* u', $A(\cdot)u(\cdot) \in Z^*_{1,\eta}(]0,T],E)$.

(ii) *If* $u_0 \in D_{A(0)}$ *and* $f \in Z_{0,\eta}(]0,T],E)$, $\eta \in]0,\delta]$, *then the classical solution* u *satisfies moreover* u', $A(\cdot)u(\cdot) \in Z_{0,\eta}(]0,T],E)$.

(iii) *If* $u_0 \in D_{A(0)}$, $f \in C([0,T],E) \cap Z_{0,\eta}(]0,T],E)$, $\eta \in]0,\delta]$, *and in addition* $A(0)u_0 + f(0) \in \overline{D_{A(0)}}$, *then there exists a unique strict solution* u, *such that moreover* u', $A(\cdot)u(\cdot) \in Z_{0,\eta}(]0,T],E)$.

(The spaces $Z_{\delta,\eta}$ and $Z^*_{\delta,\eta}$ are defined in Appendix A, Definitions A.1 and A.6.)

Proof. All statements are proved in Theorems 6.1 and 6.5 of [AT2], except for the uniqueness of the classical solution. This property follows by a standard argument: if u, v are two classical solutions of problem (B.1) with $x \in \overline{D_{A(0)}}$ and $f \in C(]0,T],E)$, set $w = u - v$ and fix $t \in]0,T]$: then the function

$$z(s) = U(t,s)w(s), \qquad s \in [0,t],$$

is differentiable in $]0,t[$ and, by Lemma 6.7(i),

$$z'(s) = V(t,s)w(s) + U(t,s)A(s)w(s) = 0, \qquad \forall s \in]0,t[;$$

hence $z(s)$ is constant in $[0,t]$, i.e.,

$$w(t) = z(t) = U(t,0)w(0) = 0,$$

so that $w(t) = 0$ for each $t \in [0,T]$. □

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