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Abstract

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BOUNDARY CONTROL PROBLEMS FOR NON-AUTONOMOUS PARABOLIC SYSTEMS

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0. Goal

This paper is concerned with boundary control, over finite time horizon, of some linear non-autonomous systems of parabolic type. We continue the analysis of [1] where we studied, from an abstract point of view, a class of parabolic systems obeying the abstract assumptions of [3,4] which, essentially, force the system to have a variational structure. We consider here a different class of not necessarily variational systems, whose "parabolicity" is somewhat less stringent (for instance, the order of some boundary operators may reduce by one, or more, for some values of t), provided we pay a price in terms of higher time regularity assumptions. The abstract hypotheses corresponding to this class are those of [8,2], which are in fact independent of those of [3,4], as remarked in [3, §7].

1. The initial-boundary value problem

Let Ω be a bounded open set of $\mathbb{R}^n$, $n\ge1$, with sufficiently smooth $\partial\Omega$. Consider a pair $\big(\mathcal{A}(t,x,D),\mathcal{B}(t,x,D)\big)$ of N-ples of differential operators, acting on functions $u:\Omega\to\mathbb{C}^N$, $N\ge1$, defined respectively in $[0,T]\times\Omega$ and in $[0,T]\times\partial\Omega$. We assume that $(\mathcal{A},\mathcal{B})$ is an elliptic system of order 2m, $m\ge1$, in the sense of [7, §5], and that all related properties hold uniformly with respect to $t\in[0,T]$; moreover we require that all coefficients of $(\mathcal{A},\mathcal{B})$ are $C^{1+\alpha}$ in t for some $\alpha\in]0,1[$.

Here is the initial-boundary value problem we are going to control:

$$\begin{cases} y_t(t,x)-\mathcal{A}(t,x,D)y = f(t,x), & (t,x)\in[0,T]\times\Omega, \\ y(0,x) = y_0(x), & x\in\Omega, \\ \mathcal{B}(t,x,D)y = u(t,x), & (t,x)\in[0,T]\times\partial\Omega; \end{cases} \tag{1.1}$$

as we are interested to boundary control, we will take $f\equiv0$.

2. The abstract initial-boundary value problem

Set $H:=[L^2(\Omega)]^N$, and define the linear unbounded operator $A(t):D_{A(t)}\subseteq H\to H$ by:

$$\begin{cases} D_{A(t)}:=\{u\in[W^{2,2}(\Omega)]^N:\ \mathcal{B}(t,\cdot,D)u=0 \text{ on } \partial\Omega\} \\ A(t)u:=\mathcal{A}(t,\cdot,D)u. \end{cases} \tag{2.1}$$

It is known that $\{A(t)\}_{t\in[0,T]}$ is a family of closed, densely defined generators of analytic semigroups in H; in particular it is not restrictive to assume:

$$\begin{cases} \text{there exists } \omega\in]\pi/2,\pi[\text{ such that the resolvent set of} \\ A(t) \text{ contains the sector } S_\omega:=\{0\}\cup\{z\in\mathbb{C}:\ |\arg z|<\omega\}, \text{ and} \\ \|[\lambda-A(t)]^{-1}\|_{\mathcal{L}(H)}\le \dfrac{M}{1+|\lambda|}\quad \forall\lambda\in S_\omega,\ \forall t\in[0,T]. \end{cases} \tag{2.2}$$

Next, arguing as in [10, § 5.3], it is easy to obtain:

$$\begin{cases} \text{there exists } \alpha\in]0,1[\text{ such that } t\to[\lambda-A(t)]^{-1}\in C^{1+\alpha}([0,T],\mathcal{L}(H)) \\ \text{for each } \lambda\in S_\omega,\ \text{and} \\ \left\|\dfrac{d}{dt}[\lambda-A(t)]^{-1}\right\|_{\mathcal{L}(H)}\le \dfrac{K}{1+|\lambda|^\alpha}\quad \forall\lambda\in S_\omega,\ \forall t\in[0,T], \\ \left\|\dfrac{d}{dt}A(t)^{-1}-\dfrac{d}{ds}A(s)^{-1}\right\|_{\mathcal{L}(H)}\le N\ |t-s|^\alpha\quad \forall s,t\in[0,T]. \end{cases} \tag{2.3}$$

Conditions (2.2) and (2.3) are the assumptions of [8]. The abstract Cauchy problem

$$\begin{cases} y'(t)-A(t)y(t) = f(t), & t\in[0,T] \\ y(0) = y_0 \end{cases} \tag{2.4}$$

corresponds to the concrete problem (1.1) with $u\equiv0$.

To give an abstract version of the non-homogeneous boundary conditions of (1.1) we introduce the elliptic "Green map" G(t), defined by:

$$v:=G(t)g \Leftrightarrow \begin{cases} \mathcal{A}(t,\cdot,D)v = 0 & \text{in } \Omega, \\ \mathcal{B}(t,\cdot,D)v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

The existence of $v=G(t)g$, for smooth g, is guaranteed by the results of [7]. However we need a stronger property: namely, setting $U:=[L^2(\partial\Omega)]^N$, we assume:

$$\begin{cases} \text{there exists } \vartheta\in]0,1[\text{ such that } G(t)\in\mathcal{L}(U,D_{[-A(t)]^\vartheta}) \text{ for} \\ \text{each } t\in[0,T] \text{ and } t\to[-A(t)]^\vartheta G(t)\in C([0,T],\mathcal{L}(U,H)). \end{cases} \tag{2.6}$$

This requirement is natural in variational problems, but in our situation it is not easy to get it; we will see later some examples where (2.6) is satisfied.

Using the operator G(t), we may rewrite problem (1.1) as:

$$\begin{cases} y'(t)-A(t)[y(t)-G(t)u(t)] = 0, & t\in[0,T] \\ y(0) = y_0\ . \end{cases} \tag{2.7}$$

3. The evolution operator

The results of [8] and [2] show that we can construct the evolution operator $\{U(t,s)\}_{0\le s<t\le T}$ associated to $\{A(t)\}_{t\in[0,T]}$; thus if $y_0\in H$ and $f\in L^2(0,T;H)$ the solution of problem (2.4) is

$$y(t) = U(t,0)y_0+\int_0^t U(t,s)f(s)ds,\quad t\in[0,T].$$

Consequently, at least formally, if $u\in L^2(0,T;U)$ the solution of problem (2.7) is

$$y(t) = U(t,0)y_0+\int_0^t U(t,s)A(s)G(s)u(s)ds,\quad t\in[0,T]; \tag{3.1}$$

but (3.1) is not meaningful because the range of G(s) is never contained in $D_{A(s)}$. In order to give sense to (3.1) let us recall some properties of $\{U(t,s)\}_{0\le s<t\le T}$, proved in [5]:

PROPOSITION 3.1 *Under assumptions* (2.1), (2.2), (2.3) *let*

$\{U(t,s)\}$ *be the evolution operator associated to* $\{A(t)\}$. *Then:*

(i) $\|[-A(t)]^{\gamma}U(t,s)[-A(s)]^{\beta}\|_{\mathcal{L}(H)} \le M_{\gamma\beta}[1+(t-s)^{\beta-\gamma}]$

$$\forall 0\le s<t\le T,\ \forall\gamma,\beta\in[0,1];$$

(ii) *for each* $0\le s<t\le T$ *and* $x\in H$ *there exists* $\frac{d}{ds}U(t,s)=:V(t,s)$, *and*

$$\|V(t,s)\|_{\mathcal{L}(D_{[-A(s)]^{\vartheta}},H)} \le B(t-s)^{\vartheta-1} \quad \forall 0\le s<t\le T,\ \forall\vartheta\in[0,1],$$

$$\|V(t,s)+A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(D_{[-A(s)]^{\vartheta}},H)} \le B(t-s)^{\vartheta+\alpha-1}$$

$$\forall 0\le s<t\le T,\ \forall\vartheta\in[0,1];$$

(iii) $[-A(t)]^{-\beta}U(t,s)[-A(s)]^{\gamma}$ *has an extension to* $\mathcal{L}(H)$ *bounded by* $M_{\gamma\beta}[1+(t-s)^{\beta-\gamma}]$ $\forall 0\le s<t\le T$, $\forall\gamma,\beta\in[0,1]$. □

Using Proposition 3.1(ii) and the obvious fact that $\frac{d}{ds}U(t,s)x = -U(t,s)A(s)x\ \forall x\in D_{A(s)}$, we may rewrite (3.1) as:

$$y(t) = U(t,0)y_0-\int_0^t \frac{d}{ds}U(t,s)G(s)u(s)ds, \quad t\in[0,T], \tag{3.2}$$

and this expression is obviously meaningful in view of (2.6).

4. The control problem

Equation (3.2) will be considered as the state equation of the following L.-Q.-R. problem:

$$\begin{cases} \text{minimize} \\ J(u):=\int_0^T [(M(t)y(t)|y(t))_H+(N(t)u(t)|u(t))_U]dt + \\ \qquad +(P_Ty(T)|y(T))_H \\ \text{over all controls } u\in L^2(0,T;U), \text{ where } y \text{ is the solution of} \\ \text{problem (2.7), i.e. } y \text{ is given by (3.2).} \end{cases} \tag{4.1}$$

Here we assume (besides (2.1), (2.2), (2.3), (2.5) and (2.6)):

$$M\in L^{\infty}(0,T;\Sigma^+(H)), \tag{4.2}$$

$$N\in C_s([0,T],\Sigma^+(U)) \quad \text{and} \quad N(t)\ge\nu>0 \quad \forall t\in[0,T], \tag{4.3}$$

$$\begin{cases} P_T\in\Sigma^+(H) \text{ and } [-A(T)^*]^{2\beta}P_T \text{ can be continuously extended} \\ \text{to } \mathcal{L}(H) \text{ for some } \beta\in]\frac{1}{2}-\alpha,\frac{1}{2}]\cap[0,\frac{1}{2}]. \end{cases} \tag{4.4}$$

We remark that by (4.4) the term $(P_Ty(T)|y(T))_H$ is well defined and in addition, thinking of it as a function of u (via (3.2)), it turns out that such a function is continuous on $L^2(0,T;U)$ [1, Lemma 3.5].

We solve the control problem exactly as in [1], by a dynamic programming technique.

5. The Riccati equation

The Riccati equation associated to the control problem (4.1) is, formally,

$$\begin{aligned} P(t) &= U(T,t)^*P_TU(T,t) + \\ &+\int_t^T U(s,t)^*[M(s)-P(s)A(s)G(s)N(s)^{-1}G(s)^*A(s)^*P(s)]U(s,t)ds, \end{aligned} \tag{5.1}$$

but it is not meaningful since the range of G(s) is not contained in $D_{A(s)}$. However, recalling (2.6), we rewrite (5.1) as

$$\begin{aligned} P(t) &= U(T,t)^*P_TU(T,t) + \\ &+\int_t^T U(s,t)^*\left\{M(s)-[[-A(s)^*]^{1-\vartheta}P(s)]^*K(s)[-A(s)^*]^{1-\vartheta}P(s)\right\}U(s,t)ds \end{aligned} \tag{5.2}$$

where

$$K(s):=[-A(s)^{\vartheta}]G(s)N(s)^{-1}[[-A(s)]^{\vartheta}G(s)]^*; \tag{5.3}$$

by (2.6) and (4.3) we have $K\in C([0,T],\Sigma^+(H))$.

The same argument of [1] leads to:

PROPOSITION 5.1 *Equation* (5.2) *has a unique solution* P *in* [0,T]. *Moreover:*

(i) $P(t)\ge 0\quad\forall t\in[0,T]$;

(ii) P *satisfies the integral equation*

$$P(t) = U(T,t)^*P_T\Phi(T,t) + \int_t^T U(\sigma,t)^*M(\sigma)\Phi(\sigma,t)d\sigma, \quad t\in[0,T], \qquad (5.4)$$

where $\{\Phi(t,s)\}_{0\le s\le t\le T}$ *is an evolution operator which is the unique solution of this further integral equation:*

$$\Phi(t,s)x = U(t,s)x +$$
$$+ \int_s^t [[-A(r)^*]^{1-\vartheta}U(t,r)^*]^*K(r)[[-A(r)^*]^{1-\vartheta}P(r)]^*\Phi(r,s)xdr, \qquad (5.5)$$
$$t\in[0,T], \ x\in H;$$

(iii) *for each* $\eta\in[0,1]$ *(resp.* $\eta\in[0,\vartheta]$*),* $t\to[-A(t)^*]^{\eta}P(t)\in C([0,T[, \mathcal{L}(H))$ *(resp.* $C([0,T],\mathcal{L}(H))$*).* □

6. Synthesis

Following again [1] we get:

THEOREM 6.1 *Let* $y_0\in H$ *be given. Then:*

(i) *there exists a unique optimal control* $u_*\in L^2(0,T;U)$ *for problem* (4.1);

(ii) *the optimal cost* $J(u_*)$ *is given by*

$$J(u_*) = (P(0)y_0|y_0)_H ,$$

where P is the solution of the Riccati equation (5.2);

(iii) *the optimal state* $y_*\in L^2(0,T;H)$ *is given by*

$$y_*(t) = \Phi(t,0)y_0 , \quad t\in[0,T],$$

where Φ *is the solution of* (5.5);

(iv) *the optimal control* u_* *is given by the feedback formula*

$$u_*(t) = - N(t)^{-1}G(t)^*A(t)^*P(t)y_*(t), \quad t\in[0,T[,$$

where we have written $G(t)^*A(t)^*$ *instead of* $-[[-A(t)]^{\vartheta}G(t)]^*[-A(t)^*]^{1-\vartheta}$;

(v) *the optimal pair* (u_*,y_*) *is characterized by the optimality system*

$$\begin{cases} y_*(t) = U(t,0)y_0 - \int_0^t \frac{d}{ds}U(t,s)G(s)u(s)ds, \\ u_*(t) = - N(t)^{-1}G(t)^*A(t)^*p(t), & t\in[0,T[. \\ p(t) := U(T,t)^*P_Ty_*(T) + \int_t^T U(s,t)^*M(s)y_*(s)ds, \end{cases}$$
□

7. Examples

EXAMPLE 7.1 Take N=1, $\partial\Omega\in C^3$, $\lambda_0>0$ and

$$\mathcal{A}(t,x,D)u := \sum_{s,j=1}^n D_s(A_{sj}(t,x)D_ju) + \lambda_0 u , \quad (t,x)\in[0,T]\times\overline{\Omega}, \qquad (7.1)$$

$$\mathcal{B}(t,x,D)u := \sum_{s=1}^n \beta_s(t,x)D_su + \alpha(t,x)u , \quad (t,x)\in[0,T]\times\partial\Omega, \qquad (7.2)$$

where

$$A_{sj}, \beta_s, \alpha \text{ are } C^2 \text{ in } x \text{ and } C^{1+\alpha} \text{ in } t; \qquad (7.3)$$

for each $x\in\partial\Omega$ and $t\in[0,T]$ the polynomial

$$\sum_{s,j=1}^n A_{sj}(t,x)(\xi_j+\tau\nu_j(x))(\xi_s+\tau\nu_s(x)) \qquad (7.4)$$

has 1 root $\tau_+(t,x;\xi)$ with positive imaginary part for each $\xi\in\mathbb{R}^n$ tangent to $\partial\Omega$ at x;

$$\sum_{s=1}^n \beta_s(t,x)(\xi_s+\tau_+(t,x;\xi)\nu_s(x))\neq 0 \quad \forall\xi\in\mathbb{R}^n \text{ tangent to } \partial\Omega \text{ at } x \cdot \qquad (7.5)$$

(in particular, β may be a real, non-tangential vector).

It is easy to see that the abstract assumptions (2.2) and (2.3) hold for the operator A(t) given by (2.1), provided λ_0 is sufficiently large. Let us verify that G(t), given by (2.5), fulfills (2.6). First of all we have Green's formula:

$$\int_\Omega [\mathcal{A}(t,x,D)u\,\overline{v} - u\,\overline{\mathcal{A}^*(t,x,D)v}]dx = \int_{\partial\Omega}\left[\frac{\partial u}{\partial\nu_A}\overline{v} - u\,\overline{\frac{\partial v}{\partial\nu_{A^*}}}\right]d\sigma \qquad (7.6)$$
$$\forall u,v\in W^{2,2}(\Omega),$$

where

$$\mathcal{A}^*(t,x,D)v := \sum_{s,j=1}^{n} D_s\left(\overline{{}^tA_{sj}(t,x)D_j v}\right) + \lambda_0 v\ , \qquad (t,x)\in[0,T]\times\overline{\Omega}$$

and $\frac{\partial}{\partial\nu_A}$, $\frac{\partial}{\partial\nu_{A^*}}$ are the conormal derivatives relative to $\mathcal{A}$, $\mathcal{A}^*$.

Let us denote by $\tau^1(x)$, ..., $\tau^{n-1}(x)$ an orthonormal system of vectors tangent to $\partial\Omega$ at x; then

$$\frac{\partial u}{\partial\nu_A} = \sum_{j=1}^{n}\left\{-(\nu_A\cdot\nu)(\beta\cdot\nu)^{-1}(\beta\cdot\tau^j)+(\nu_A\cdot\tau^j)\right\}\frac{\partial u}{\partial\tau^j} + $$
$$+ (\nu_A\cdot\nu)(\beta\cdot\nu)^{-1}\mathcal{B}(t,\cdot,D)u - \alpha(\nu_A\cdot\nu)(\beta\cdot\nu)^{-1}u,$$

so that by (7.6) it is easy to get (compare with [9, Ch.2, Th.2.1])

$$\int_\Omega [\mathcal{A}(t,x,D)u\ \overline{v} - \overline{\mathcal{A}^*(t,x,D)v}]dx = \tag{7.7}$$
$$= \int_{\partial\Omega} [\mathcal{B}(t,x,D)u\ \ (\nu_A\cdot\nu)(\beta\cdot\nu)^{-1}\overline{v} - u\ \overline{\mathcal{C}(t,x,D)v}]d\sigma \quad \forall u,v\in W^{2,2}(\Omega),$$

where

$$\mathcal{C}(t,x,D)v := -\frac{\partial v}{\partial\nu_{A^*}} + \sum_{j=1}^{n-1}\overline{[(\nu_A\cdot\nu)(\beta\cdot\nu)^{-1}(\beta\cdot\tau^j)-(\nu_A\cdot\tau^j)]}\frac{\partial v}{\partial\tau^j} +$$
$$+ \left\{\sum_{j=1}^{n-1}\frac{\partial}{\partial\tau^j}\overline{[(\nu_A\cdot\nu)(\beta\cdot\nu)^{-1}(\beta\cdot\tau^j)-(\nu_A\cdot\tau^j)]} - \alpha(\nu_A\cdot\nu)(\beta\cdot\nu)^{-1}\right\}v; \tag{7.8}$$

in particular we have

$$\begin{cases} D_{A(t)^*} := \{v\in[W^{2,2}(\Omega)]^N:\ \mathcal{C}(t,\cdot,D)v=0 \text{ on } \partial\Omega\} \\ A(t)^*v := \mathcal{A}^*(t,\cdot,D)v. \end{cases} \tag{7.9}$$

Now let $g\in W^{2,2}(\partial\Omega)$ and set $u:=G(t)g$, i.e., by (2.5),

$$\begin{cases} \mathcal{A}(t,\cdot,D)u = 0 & \text{in } \Omega, \\ \mathcal{B}(t,\cdot,D)u = g & \text{on } \partial\Omega. \end{cases}$$

By classical results [7,9]

$$\|u\|_{W^{2,2}(\Omega)} \le c\ \|g\|_{W^{1/2,2}(\partial\Omega)} \qquad \forall g\in W^{1/2,2}(\partial\Omega). \tag{7.10}$$

On the other hand, let $\psi := [A(t)^*]^{-1}u$, i.e.

$$\begin{cases} \mathcal{A}^*(t,\cdot,D)\psi = u & \text{in } \Omega, \\ \mathcal{C}(t,\cdot,D)\psi = 0 & \text{on } \partial\Omega; \end{cases} \tag{7.11}$$

as $(\mathcal{A}^*,\mathcal{C})$ satisfies the assumptions of [7], again by classical results we have

$$\|\psi\|_{W^{2,2}(\Omega)} \le c\ \|u\|_{L^2(\Omega)}. \tag{7.12}$$

If we multiply the equation $\mathcal{A}(t,x,D)u=0$ by ψ, and integrate by parts in Ω, by (7.7) we get:

$$\|u\|^2_{L^2(\Omega)} \le -\int_{\partial\Omega} g\ (\nu_A\cdot\nu)(\beta\cdot\nu)^{-1}\overline{\psi}\ d\sigma =$$
$$\le \left|\overline{\left\langle g, (\nu_A\cdot\nu)(\beta\cdot\nu)^{-1}\psi\right\rangle}_{W^{-3/2,2}(\partial\Omega),W^{3/2,2}(\partial\Omega)}\right| \le$$
$$\le c\ \|g\|_{W^{-3/2,2}(\partial\Omega)}\ \|\psi\|_{W^{3/2,2}(\partial\Omega)},$$

and by (7.12) we obtain

$$\|u\|_{L^2(\Omega)} \le c\ \|g\|_{W^{-3/2,2}(\partial\Omega)} \qquad \forall g\in W^{1/2,2}(\partial\Omega). \tag{7.13}$$

Finally, interpolating between (7.10) and (7.13) we deduce in a standard way

$$\|u\|_{W^{3/2,2}(\Omega)} \le c\ \|g\|_{L^2(\partial\Omega)} \qquad \forall g\in W^{1/2,2}(\partial\Omega) \tag{7.14}$$

This estimate shows that G(t) has a bounded extension from $L^2(\partial\Omega)$ into $W^{3/2,2}(\Omega)$, and consequently we get assumption (2.6) for each $\vartheta\in]0,3/4[$.

REMARK 7.2 By (7.7) it follows that

$$(u|G(t)^*A(t)^*v)_H = \int_{\partial\Omega} u\ (\nu_A\cdot\nu)(\beta\cdot\nu)^{-1}\overline{v}\ d\sigma \quad \forall u\in W^{1/2,2}(\partial\Omega),\ \forall v\in D_{A(t)^*},$$

i.e.

$$G(t)^*A(t)^*v = \overline{(\nu_A\cdot\nu)(\beta\cdot\nu)^{-1}}\, v_{|\partial\Omega} \quad \forall v\in D_{A(t)^*}. \tag{7.15}$$

If, in particular, $\beta=\nu_A$ (the case of $A_1(t)$ in [1]), we get $G(t)^*A(t)^*v = v_{|\partial\Omega}\ \forall v\in D_{A(t)^*}$.

REMARK 7.3 If $\beta\equiv 0$ and $\alpha\equiv 1$ in (7.2), then we have, as in [1], that (2.6) holds for each $\vartheta\in]0,1/4[$ and that $G(t)^*A(t)^*v = \partial v/\partial\nu_A$ $\forall v\in D_{A(t)^*}$.

REMARK 7.4 In Example 7.1 the assumptions of [1] are also fulfilled, even assuming only C^α regularity in t (but in this case, of course, (2.3) is no longer true).

EXAMPLE 7.5 Take n=1, $\Omega=]0,1[$, $\lambda_0>0$ and

$$\mathcal{A}(t,x,D):= A(t,x)\cdot u'' + B(t,x)\cdot u' + C(t,x)\cdot u + \lambda_0 u, \tag{7.16}$$

$$\mathcal{B}_j(t,D):= \beta_j(t)\cdot u'(j) + \alpha_j(t)\cdot u(j), \quad j=0,1, \tag{7.17}$$

where $A(t,x)$, $B(t,x)$, $C(t,x)$, $\beta_j(t)$, $\alpha_j(t)$ are NxN matrices whose coefficients are continuous in x and $C^{1+\alpha}$ in t. We assume

$$\det A(t,x) \neq 0 \quad \forall (t,x)\in[0,T]\times[0,1]. \tag{7.18}$$

Consider the boundary value problem (for fixed t)

$$\begin{cases} \mathcal{A}(t,\cdot,D)u = f\in L^2(0,1;\mathbb{C}^N) \\ \mathcal{B}_j(t,D)u = z_j\in\mathbb{C}^N, \quad j=0,1; \end{cases} \tag{7.19}$$

setting $v:=u'$, by (7.18) we may rewrite (7.19) as a first order system of 2N equations:

$$\begin{cases} \begin{pmatrix}u\\v\end{pmatrix}' = \begin{pmatrix}0 & I\\ Q(t,x) & R(t,x)\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} + \begin{pmatrix}0\\F(t,x)\end{pmatrix}, \quad x\in]0,1[, \\ \alpha_j(t)\cdot u(j) + \beta_j(t)\cdot v(j) = z_j, \quad j=0,1, \end{cases} \tag{7.20}$$

where $Q:=-A^{-1}B$, $R:=-A^{-1}(C+\lambda_0 I)$, $F:=-A^{-1}f$. An easy check shows that (7.20) is uniquely solvable if and only if

$$\det\begin{pmatrix}\alpha_0(t)\cdot U(0) & \beta_0(t)\cdot V(0)\\ \alpha_1(t)\cdot U(1) & \beta_1(t)\cdot V(1)\end{pmatrix} \neq 0, \tag{7.21}$$

where the 2Nx2N matrix

$$\begin{pmatrix}U(x)\\V(x)\end{pmatrix}, \quad x\in[0,1]$$

is any Wronskian relative to the homogeneous system associated to (7.20). Thus the columns of U(x) give 2N linearly independent solutions in $C^1([0,1],\mathbb{C}^N)$ of the homogeneous system $\mathcal{A}(t,\cdot,D)u=0$, and the columns of V(x) give the derivatives of such solutions.

Under assumptions (7.18) and (7.21) (which is obviously intrinsic, i.e. does not depend on the particular Wronskian) it is now easy to verify that the abstract assumptions (2.2), (2.3) and (2.6) are fulfilled.

EXAMPLE 7.6 Our abstract theory also applies to certain dynamic systems acting on non-cylindrical domains, which are studied in [6].

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