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EVOLUTION OPERATORS AND STRONG SOLUTIONS OF ABSTRACT LINEAR PARABOLIC EQUATIONS

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Abstract. We consider the linear non-autonomous Cauchy problem of parabolic type in a Banach space E, under general assumptions which allow the domains of the operators to be non-constant in t and not dense in E. We study the regularity properties of the evolution operator, and prove existence, uniqueness and sharp regularity results for strong solutions. Applications to parabolic partial differential equations are also given.

0. Introduction. Let E be a Banach space. We are concerned with the linear parabolic non-autonomous Cauchy problem

$$\begin{cases} u'(t) - A(t)u(t) = f(t), & t \in [s,T] \\ u(s) = x. \end{cases} \tag{0.1}$$

Here, $T > 0$, $s \in [0,T]$ and $x \in E$, $f : [s,T] \to E$ are prescribed data, whereas $\{A(t)\}$ is a family of closed linear operators in E, which are generators of analytic semigroups and whose domains $D_{A(t)}$ may change with t and be not dense in E. In [3], we studied existence, uniqueness, and maximal regularity of strict and classical (i.e., continuously differentiable) solutions of (0.1), and in [4] we constructed the evolution operator $U(t,s)$ for problem (0.1). In both cases, the initial point was $s = 0$, but the general situation $s \in [0,T]$ requires no substantial changes. Here, under the same assumptions of those papers, we consider the variation of parameters formula

$$u(t) := U(t,s)x + \int_s^t U(t,r)f(r)\,dr, \quad t \in [s,T], \tag{0.2}$$

and show that u is the unique strong solution (see Definition 1.6 (c) below) of (0.1), if and only if $x \in \overline{D_{A(s)}}$ and $f \in C([s,T],E)$. Furthermore, we prove very precise regularity results, both in time and in space, for the function (0.2); such results generalize those of [7], [14], and [2], and are sharper than the similar ones obtained in [1] under hypotheses which are independent of ours (see Remark 4.5 below). In addition, we shortly consider classical solutions, showing that formula (0.2) holds for them too, under very general conditions on the data (compare with [4, Remark 2.3]). We note that, as in our previous papers, [3]

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and [4], our assumptions are generally weaker than those known in the literature; detailed comparisons and references can be found in [3, Section 7].

Let us describe the subject of the next sections. Section 1 contains some preliminaries. Section 2 is devoted to a careful analysis of the properties of the evolution operator $U(t,s)$. In Section 3, we prove existence and uniqueness of the strong solution of (0.1), whereas in Section 4, we study its time and space regularity. Section 5 is concerned with the representation of classical solutions by formula (0.2), and finally, in Section 6, we discuss the validity of Hypotheses I and II (see Section 1 below) in some concrete examples.

1. Notations, Assumptions, and Preliminary Results. Let Y be a Banach space, and let a, $b \in \mathbf{R}$ with $a < b$. If $\mu \in [0, \infty[$, we consider the spaces

$$B_\mu(a,b,Y) := \{f :]a,b] \to Y \mid \sup_{a<t\le b} (t-a)^\mu \|f(t)\|_Y < \infty\},$$
$$C_\mu(]a,b],Y) := B_\mu(a,b,Y) \cap C(]a,b],Y),$$

$$C_\mu([a,b],Y) := \{f \in C_\mu(]a,b],Y) \mid t \to (t-a)^\mu f(t) \in C([a,b],Y)\},$$

which are Banach spaces with the obvious norm

$$\|f\|_{B_\mu(a,b,Y)} := \sup_{a<t\le b} (t-a)^\mu \|f(t)\|_Y$$

(of course, $C_0([a,b],Y)$ means $C([a,b],Y)$ whereas $C_0(]a,b],Y)$ is strictly contained in $C(]a,b],Y)$; we will similarly denote $B_0(a,b,Y)$ by $B(a,b,Y)$). Next, for $\alpha \in]0,1[$, we will use the Hölder spaces

$$C^\alpha([a,b],Y) := \left\{f \in C([a,b],Y) \mid [f]_{C^\alpha([a,b],Y)} := \sup_{a\le s<t\le b} \frac{\|f(t)-f(s)\|_Y}{(t-s)^\alpha} < \infty\right\},$$

$$h^\alpha([a,b],Y) := \left\{f \in C^\alpha([a,b],Y) \mid \lim_{r\downarrow 0} \sup_{a\le c\le b-r} [f]_{C^\alpha([c,c+r],Y)} = 0\right\},$$

endowed with their usual norm. For $\alpha = 1$, we will use both the Lipschitz space

$$\mathrm{Lip}([a,b],Y) := \left\{f \in C([a,b],Y) \mid [f]_{\mathrm{Lip}([a,b],Y)} := \sup_{a\le s<t\le b} \frac{\|f(t)-f(s)\|_Y}{t-s} < \infty\right\}$$

with its usual norm, and the Zygmund classes

$$C^{*,1}([a,b],Y) := \Big\{f \in C([a,b],Y) \mid [f]_{C^{*,1}([a,b],Y)}$$
$$:= \sup_{a\le s<t\le b} \frac{\|f(t) - 2f((t+s)/2) + f(s)\|_Y}{t-s} < \infty\Big\},$$

$$h^{*,1}([a,b],Y) := \left\{f \in C^{*,1}([a,b],Y) \mid \lim_{r\downarrow 0} \sup_{a\le c\le b-r} [f]_{C^{*,1}([c,c+r],Y)} = 0\right\}$$

with the obvious norm

$$\|f\|_{C^{*,1}([a,b],Y)} := \|f\|_{C([a,b],Y)} + [f]_{C^{*,1}([a,b],Y)}\,.$$

We will also work with the usual Lebesgue spaces $L^p(a,b,Y)$, $1 \le p \le \infty$. Finally, $C^1([a,b],Y)$ (resp. $C^1(]a,b],Y)$) is the class of functions $f \in C([a,b],Y)$ (resp. $f \in C(]a,b],Y)$), which are continuously differentiable in $[a,b]$ (resp. in $]a,b]$) with respect to the Y-norm. The Banach space $C^1([a,b],Y)$ is endowed with its natural norm.

Let E be a Banach space and let $A : D_A \subseteq E \to E$ be a closed linear operator, generating an analytic semigroup $\{e^{sA}\}_{s\ge 0}$. Then, this semigroup is strongly continuous at $s=0$, if and only if D_A is dense in E (see [14]). By endowing D_A and D_{A^2} with the graph norm, they become Banach spaces continuously imbedded into E; hence, we can consider the real interpolation spaces between D_{A^m}, $m=1,2$, and E, introduced in [13] and [8] (see also [9]).

Definition 1.1. For $\beta \in]0,1[$ and $m=1,2$, we set

$$D_{A^m}(\beta,\infty) := (D_{A^m},E)_{1-\beta,\infty}\,, \tag{1.1}$$

$$D_{A^m}(\beta) = (D_{A^m},E)_{1-\beta} \qquad \text{(see [8, Def. 2.2])}. \tag{1.2}$$

The following characterizations hold (see [14, [9], and [11]):

$$D_{A^m}(\beta,\infty) = \left\{x \in E \,:\, [x]^{(1)}_{m,\beta} := \sup_{s>0} s^{-m\beta}\|(e^{SA}-1)^m x\|_E < \infty\right\}, \qquad (1.3)_1$$

$$D_{A^m}(\beta,\infty) = \left\{x \in E \,:\, [x]^{(2)}_{m,\beta} := \sup_{s>0} s^{m(1-\beta)}\|A^m e^{sA}x\|_E < \infty\right\}, \qquad (1.3)_2$$

$$D_{A^m}(\beta,\infty) = \left\{x \in E \,:\, [x]^{(3)}_{m,\beta} := \sup_{\lambda\in\rho(A)} |\lambda|^{m\beta}\|(AR(\lambda,A))^m x\|_E < \infty\right\}, \qquad (1.3)_3$$

$$D_{A^m}(\beta) = \left\{x \in D_{A^m}(\beta,\infty) \,:\, \lim_{s\downarrow 0} s^{-m\beta}\|(e^{sA}-1)^m x\|_E = 0\right\}, \qquad (1.4)_1$$

$$D_{A^m}(\beta) = \left\{x \in D_{A^m}(\beta,\infty) \,:\, \lim_{s\downarrow 0} s^{m(1-\beta)}\|A^m e^{sA}x\|_E = 0\right\}, \qquad (1.4)_2$$

$$D_{A^m}(\beta) = \left\{x \in D_{A^m}(\beta,\infty) \,:\, \lim_{|\lambda|\to\infty,\lambda\in\rho(A)} |\lambda|^{m\beta}\|(AR(\lambda,A))^m x\|_E = 0\right\}. \qquad (1.4)_3$$

Moreover, the corresponding norms

$$\|x\|^{(i)}_{m,\beta} := \|x\|_E + [x]^{(i)}_{m,\beta}\,, \quad i=1,2,3 \qquad (1.5)_i$$

are all equivalent to the norm of the interpolation space $(D_{A^m},E)_{1-\beta,\infty}$. Thus, we will denote by $[x]_{D_{A^m}(\beta,\infty)}$, $\|x\|_{D_{A^m}(\beta,\infty)}$, any of the seminorms $(1.3)_i$ and of the norms $(1.5)_i$. When $\beta=0$, the sets $(1.3)_i$ reduce to the whole E and the sets $(1.4)_1$, $(1.4)_3$ reduce to $\overline{D_{A^m}}(\equiv \overline{D_A})$, but the set $(1.4)_2$ becomes a space containing $\overline{D}_A$. When $\beta=1$, the sets $(1.4)_i$ reduce to $\{0\}$, but the sets $(1.3)_i$ become strictly larger, in general, than D_{A^m} (see [6, Corollary 3.18]). Thus, we are led to the following definition.

Definition 1.2. We set

$$D_{A^m}(0) := \overline{D_A},\;\; D_{A^m}(0,\infty) := E,\;\; D_{A^m}(1) := \{0\}\,,$$

$$D_{A^m}(1,\infty) := \left\{x \in E \,|\, [x]^{(1)}_{m,1} := \sup_{s>0} s^{-m}\|(e^{sA}-1)^m x\|_E < \infty\right\}.$$

By [6, Theorems 2.5.4 and 3.5.3] we have also

$$\begin{aligned} D_{A^m}(1,\infty) &= \left\{x \in E \,|\, [x]^{(2)}_{m,1} := \sup_{s>0} \|A^m e^{sA}x\|_E < \infty\right\} \\ &= \left\{x \in E \,|\, [x]^{(3)}_{m,1} := \sup_{\lambda\in\rho(A)} |\lambda|^m \|(AR(\lambda,A))^m x\|_E < \infty\right\}. \end{aligned}$$

As before, we denote by $[x]_{D_{A^m}(1,\infty)}$ any of the seminorms $[x]^{(i)}_{m,1}$, and set

$$\|x\|_{D_{A^m}(1,\infty)} := \|x\|_E + [x]_{D_{A^m}(1,\infty)}\,.$$

Remark 1.3. (a). If $\beta \in [0,1[$, $D_{A^m}(\beta)$ coincides with the closure of D_A in the norm of $D_{A^m}(\beta,\infty)$ (see [6, Prop. 3.16]).

(b). We have $D_{A^2}(\beta,\infty) = D_A(2\beta,\infty)$, $D_{A^2}(\beta) = D_A(2\beta)$ for each $\beta \in [0,1/2[$, and $D_{A^2}(\beta,\infty) = \{x \in D_A \,|\, Ax \in D_A(2\beta-1,\infty)\}$, $D_{A^2}(\beta) = \{x \in D_A \,|\, Ax \in D_A(2\beta-1)\}$ for each $\beta \in]1/2,1]$ (see [6, Theorem 3.4.6]). Thus, $D_{A^2}(\beta,\infty)$ and $D_{A^2}(\beta)$ are relevant just when $\beta = 1/2$.

(c). The following continuous inclusions hold $(0 < \beta < \sigma < 1)$:

$$\{x \in D_A \,:\, Ax \in \overline{D}_A\} \hookrightarrow \left\langle \begin{array}{c} D_A \hookrightarrow D_A(1,\infty) \\ \\ D_{A^2}(\tfrac{1}{2}) \end{array} \right\rangle \hookrightarrow D_{A^2}(\frac{1}{2},\infty) \hookrightarrow D_A(\sigma)$$
$$\hookrightarrow D_A(\sigma,\infty) \hookrightarrow D_A(\beta) \hookrightarrow \overline{D}_A \hookrightarrow E,$$

without equality in general; however, if E is reflexive, then $D_A = D_A(1,\infty)$ and $\overline{D}_A = E$ (see [6, Corollary 2.2.15] and [12]). It is also easily seen that (compare with [6, Corollary 3.1.8])

$$\{x \in D_A \,:\, Ax \in \overline{D}_A\} = \big\{x \in D_A(1,\infty) \,:\, \lim_{t\downarrow 0} \|e^{tA}x - x\|_{D_A(1,\infty)} = 0\big\}\,.$$

Let us list now our assumptions, which are the same as in [3], and [4]. We fix a Banach space E and a positive number T.

Hypothesis I. *For each $t \in [0,T]$, $A(t) : D_{A(t)} \subseteq E \to E$ is a closed linear operator and there exist $M > 0$ and $\theta_0 \in]\pi/2,\pi[$ such that*

(i) $$\rho(A(t)) \supseteq S_{\theta_0} := \big\{z \in \mathbf{C} : |\arg z| \le \theta_0\big\} \cup \{0\} \quad \forall t \in [0,T],$$

(ii) $$\|R(\lambda, A(t))\|_{\mathcal{L}(E)} \le \frac{M}{1+|\lambda|} \quad \forall \lambda \in S_{\theta_0},\ \forall t \in [0,T].$$

Hypothesis II. *There exist $B > 0$, $k \in \mathbf{N}^+$, $\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k$ with $0 \le \beta_i < \alpha_i \le 2$, such that*

$$\|A(t)R(\lambda,A(t))\,\big[A(t)^{-1} - A(s)^{-1}\big]\,\|_{\mathcal{L}(E)} \le B\sum_{i=1}^{k}(t-s)^{\alpha_i}|\lambda|^{\beta_i-1}$$

$\forall \lambda \in S_{\theta_0} - \{0\}$, $\forall 0 \le s \le t \le T$. *We also assume (which is not restrictive)*

$$\delta := \min_{1\le i\le k}(\alpha_i - \beta_i) \in]0,1[. \tag{1.6}$$

Remark 1.4. (a). Hypotheses I and II will be always assumed throughout this paper. Comments and comparisons with other kinds of assumptions can be found in [3, Section 7].

(b). An immediate consequence of Hypothesis I is the estimate

$$\|A(t)^m e^{\xi A(t)}\|_{L(E)} \le c(m)\xi^{-m} \quad \forall\, t \in [0,T],\ \forall \xi > 0,\ \forall m \in \mathbf{N}, \tag{1.7}$$

which follows by the usual representation of $e^{\xi A(t)}$ by a Dunford integral (see [3, formula (1.17)]).

In the next sections, we shall handle functions $t \mapsto g(t)$ such that, at each time t, $g(t)$ belongs to $D_{A(t)}$, or $D_{A(t)}(\beta,\infty)$, or $\overline{D}_{A(t)}$, and so on. Thus, we are led to define, with slight abuse of notations, the following spaces ($\mu \in [0,\infty[$, $m = 1,2$, $0 \le a < b \le T$) :

$$B_\mu(a,b,D_{A^m}) := \{g \in B_\mu(a,b,E) \,|\, t \to A(t)^m g(t) \in B_\mu(a,b,E)\}, \tag{1.8}$$

and $C_\mu(]a,b],D_{A^m})$, $C_\mu([a,b],D_{A^m})$ which are defined similarly; they are Banach spaces with norm

$$\|g\|_{B_\mu(a,b,D_{A^m})} := \sup_{a<t\le b} (t-a)^\mu \|g(t)\|_{D_{A(t)^m}} . \tag{1.9}$$

Similarly, we also define the space $C(]a,b],D_{A^m})$. Next, we set for $\beta \in [0,1]$:

$$B(a,b,D_{A^m}(\beta,\infty)) := \{g \in B(a,b,E) \,|\, [g]_{B(a,b,D_{A^m}(\beta,\infty))} := \sup_{a\le t\le b} [g(t)]_{D_{A(t)^m}(\beta,\infty)} < \infty\}, \tag{1.10}$$

which is a Banach space with norm

$$\|g\|_{B(a,b,D_{A^m}(\beta,\infty))} := \|g\|_{B(a,b,E)} + [g]_{B(a,b,D_{A^m}(\beta,\infty))} . \tag{1.11}$$

Finally, for $\beta \in [0,1[$, we will consider the closed subspace of $B(a,b,D_{A^m}(\beta,\infty))$ defined by:

$$C([a,b],D_{A^m}(\beta)) := \text{closure of } C([a,b],D_{A^m}) \text{ in } B(a,b,D_{A^m}(\beta,\infty)) . \tag{1.12}$$

The space $C([a,b],D_{A^m}(\beta))$ can be characterized as follows:

$$C([a,b],D_{A^m}(\beta)) = \{g \in C([a,b],E) : \lim_{s\downarrow 0} \sup_{a\le t\le b} s^{m(q-\beta)} \|A(t)^{mq} e^{sA(t)} g(t)\|_E = 0\} \ \forall q \in \mathbf{N}^+ ; \tag{1.13$_1$}$$

$$C([a,b],D_{A^m}(\beta)) = \{g \in C([a,b],E) \,|\, \lim_{s\downarrow 0} \sup_{a\le t\le b} s^{-m\beta} \|(e^{sA(t)}-1)^m g(t)\|_E = 0\} ; \tag{1.13$_2$}$$

$$C([a,b],D_{A^m}(\beta)) = \{g \in C([a,b],E) \,|\, g(t) \in D_{A(t)^m}(\beta) \ \forall t \in [0,T] \text{ and } \lim_{h\downarrow 0} \sup_{a\le t\le b-h} \sup_{s>0} s^{m(1-\beta)} \|A(t+h)^m e^{sA(t+h)} g(t+h) - A(t)^m e^{sA(t)} g(t)\|_E = 0\} . \tag{1.13$_3$}$$

The proof of $(1.13)_i$, $i = 1,2,3$, is rather involved and will be omitted.

Remark 1.5. In the autonomous case, $A(t) \equiv A$, the space $C([a,b],D_{A^m}(\beta))$ has an obvious intrinsic meaning; however, it coincides, as $(1.13)_2$ easily shows, with the definition (1.12). Similarly, under the assumptions of [2], we have $D_{A(t)} \equiv D_{A(0)}$, so that for $m = 1$ we also get $D_{A(t)}(\beta) \equiv D_{A(0)}(\beta)$; hence, again, the space $C([a,b],D_A(\beta))$ has an intrinsic meaning. But also in this case, using $(1.13)_2$ and a refinement of the argument of [2, Prop. 2.3], it can be shown that such space agrees with (1.12).

We conclude this section by defining our solutions.

Definition 1.6.

(a) *A strict solution of (0.1) is a function* $u \in C^1([s,T],E) \cap C([s,T],D_A)$ *such that* $u(s) = x$, $u' - A(\cdot)u(\cdot) = f$ *in* $[s,T]$.

(b) *A classical solution of (0.1) is a function* $u \in C([s,T],E) \cap C^1(]s,T],E) \cap C(]s,T],D_A)$ *such that* $u(s) = x$, $u' - A(\cdot)u(\cdot) = f$ *in* $]s,T]$.

(c) *A strong solution of (0.1) is a function* $u \in C([s,T],E)$ *such that there exists a sequence* $\{u_n\} \subset C^1([s,T],E) \cap C([s,T],D_A)$ *satisfying as* $n \to \infty$:

$$u_n \to u \ \text{ in } \ C([s,T],E), \quad u_n(0) \to x \ \text{ in } \ E, \quad u_n' - A(\cdot)u_n(\cdot) \to f \ \text{ in } \ C([s,T],E).$$

2. The Fundamental solution. In [4], an explicit representation of the evolution operator for problem (0.1) is given in the case $s = 0$, but only notational changes are needed for general $s \in [0,T[$. Namely, denote by Q_s the integral operator

$$(Q_s g)(t) = \int_s^t Q(t,r)g(r)\,dr, \quad 0 \le s \le t \le T, \tag{2.1}$$

where

$$Q(t,s) = A(t)^2 e^{(t-s)A(t)} \left[A(t)^{-1} - A(s)^{-1}\right], \quad 0 \le s < t \le T; \tag{2.2}$$

then by [3, Lemma 2.3 (i)], it follows that

$$\|Q(t,s)\|_{\mathcal{L}(E)} \le k(t-s)^{\delta-1}, \quad \forall 0 \le s < t \le T, \tag{2.3}$$

where δ is defined in (1.6). Hence, $Q_s \in \mathcal{L}(L^p(s,T,E))\ \forall p \in [1,\infty]$; moreover, the Neumann series $\sum_{n=0}^{\infty} Q_s^n$ converges in $\mathcal{L}(L^p(s,T,E))$, so that $(1-Q_s)^{-1}$ is well defined and

$$\left[(1-Q_s)^{-1}g\right](t) = g(t) + \sum_{n=1}^{\infty} \int_s^t Q_n(t,r)g(r)\,dr, \quad \forall g \in L^p(s,T,E), \tag{2.4}$$

$Q_n(t,s)$, being defined inductively by

$$Q_1(t,s) := Q(t,s), \quad Q_n(t,s) := \int_s^t Q_{n-1}(t,r)Q(r,s)\,dr, \tag{2.5}$$

(see [4, Lemma 1.2]). Following [4], we can define the evolution operator $U(t,s)$ of problem (0.1) by

$$U(t,s) := e^{(t-s)A(s)} + \int_s^t Z(r,s)\,dr, \quad 0 \le s \le t \le T, \tag{2.6}$$

where

$$\begin{aligned} Z(t,s) := {} & \left\{(1-Q_s)^{-1}\left[A(\cdot)e^{(\cdot-s)A(\cdot)} - A(s)e^{(\cdot-s)A(s)}\right]\right\}(t) \\ & + \sum_{n=1}^{\infty} \int_s^t \left[Q_n(t,q) - Q_n(t,s)\right] A(s)e^{(q-s)A(s)}\,dq \\ & + \sum_{n=1}^{\infty} Q_n(t,s)\left[e^{(t-s)A(s)} - 1\right], \quad 0 \le s < t \le T. \end{aligned} \tag{2.7}$$

In the next propositions, we list the main properties of $Z(t,s)$ and $U(t,s)$. First of all, however, we need the following lemma concerning the kernels $Q_n(t,s)$.

Lemma 2.1. *For $0 \le s < r < t \le T$ we have:*

(i) $\sum_{n=1}^{\infty} \|Q_n(t,s)\|_{\mathcal{L}(E)} \le c(t-s)^{\delta-1}$,

(ii) $\sum_{n=1}^{\infty} \|Q_n(t,s) - Q_n(r,s)\|_{\mathcal{L}(E)} \le c(\eta)(t-r)^{\eta}(r-s)^{\delta-\eta-1}$, $\forall \eta \in]0,\delta[$,

(iii) $\sum_{n=1}^{\infty} \|Q_n(t,r) - Q_n(t,s)\|_{\mathcal{L}(E)} \le c(\eta)(r-s)^{\eta}(t-r)^{\delta-\eta-1}$, $\forall \eta \in]0,\delta[$.

Proof: (i) It is [4, Lemma 1.2 (i)]. (ii) For $n = 1$, the estimate follows by [3, Lemma 2.3 (ii)]; for $n > 1$ we write

$$Q_n(t,s) - Q_n(r,s) = \int_r^t Q(t,q)Q_{n-1}(q,s)\,dq + \int_s^r [Q(t,q) - Q(r,q)]Q_{n-1}(q,s)\,dq\,,$$

and using (i), we easily obtain the result. The proof of (iii) is similar to (ii).

Lemma 2.2. *For $0 \le s < q < r \le T$ we have:*

(i) $\|Z(r,s)\|_{\mathcal{L}(D_{A(s)}(\beta,\infty),E)} \le c(\beta)(r-s)^{\delta+\beta-1}$, $\forall \beta \in [0,1]$,

(ii) $\|Z(r,s) - Z(q,s)\|_{\mathcal{L}(D_{A(s)}(\beta,\infty),E)} \le c(\beta,\eta)(r-q)^{\eta}(q-s)^{\beta+\delta-1-\eta}$, $\forall \beta \in [0, 1-\delta]$, $\forall \eta \in [0,\delta[$,

(iii) $\|Z(r,s) - Z(q,s)\|_{\mathcal{L}(D_{A(s)}(\beta,\infty),E)} \le c(\beta,\eta)(r-q)^{\eta}(q-s)^{\beta+\delta-1-\eta}$, $\forall \beta \in]1-\delta, 1]$, $\forall \eta \in [\beta+\delta-1, \delta[$,

(iv) $\|Z(r,s) - Z(q,s)\|_{\mathcal{L}(D_{A(s)}(1,\infty),E)} \le c(\eta)(r-q)^{\eta}$, $\forall \eta \in]0,\delta[$,

(v) $\|Z(r,q) - Z(r,s)\|_{\mathcal{L}(E)} \le c(\eta)(q-s)^{\eta}(r-q)^{\delta-1-\eta}$, $\forall \eta \in]0,\delta[$.

Proof: (i) The result follows by using [3, Lemma 1.10 (i) and Prop. 2.6 (i)] and [4, Lemma 1.2 (i)-(iii)].
(ii)-(iii)-(iv) By (2.7) we can write:

$$\begin{aligned}
Z(r,s) - Z(q,s) = {} & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.8)\\
& \left[A(r)e^{(r-s)A(r)} - A(q)e^{(r-s)A(q)}\right] + \int_q^r \left[A(q)^2 e^{(p-s)A(q)} - A(s)^2 e^{(p-s)A(s)}\right] dp \\
& + \int_q^r \sum_{n=1}^{\infty} Q_n(r,p)\left[A(p)e^{(p-s)A(p)} - A(s)e^{(p-s)A(s)}\right] dp \\
& + \int_s^q \sum_{n=1}^{\infty} [Q_n(r,p) - Q_n(q,p)]\left[A(p)e^{(p-s)A(p)} - A(s)e^{(p-s)A(s)}\right] dp \\
& + \int_q^r \sum_{n=1}^{\infty} [Q_n(r,p) - Q_n(r,s)]A(s)e^{(p-s)A(s)}\,dp \\
& + \int_s^q \sum_{n=1}^{\infty} [Q_n(r,p) - Q_n(r,s) - Q_n(q,p) + Q_n(q,s)]\,A(s)e^{(p-s)A(s)}\,dp \\
& + \sum_{n=1}^{\infty} Q_n(r,s)\int_q^r A(s)e^{(p-s)A(s)}\,dp + \sum_{n=1}^{\infty}[Q_n(r,s) - Q_n(q,s)]\left[e^{(q-s)A(s)} - 1\right].
\end{aligned}$$

The results follow in a tedious but standard way, using [3, Lemma 1.10 (i)] and Lemma 2.1 above.

(v) We write:

$$
\begin{aligned}
Z(r,q) - Z(r,s) = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.9)\\
& -\int_{r-q}^{r-s} \left[A(r)^2 e^{pA(r)} - A(q)^2 e^{pA(q)}\right] dp - \left[A(q)e^{(r-s)A(q)} - A(s)e^{(r-s)A(s)}\right] \\
& -\int_q^r \sum_{n=1}^{\infty} Q_n(r,p) \int_{p-q}^{p-s} \left[A(p)^2 e^{uA(p)} - A(q)^2 e^{uA(q)}\right] du\, dp \\
& -\int_q^r \sum_{n=1}^{\infty} Q_n(r,p) \left[A(q)e^{(p-s)A(q)} - A(s)e^{(p-s)A(s)}\right] dp \\
& -\int_s^q \sum_{n=1}^{\infty} Q_n(r,p) \left[A(p)e^{(p-s)A(p)} - A(s)e^{(p-s)A(s)}\right] dp \\
& -\int_q^r \sum_{n=1}^{\infty} [Q_n(r,p) - Q_n(r,q)] \int_{p-q}^{p-s} A(q)^2 e^{uA(q)}\, du\, dp \\
& +\int_q^r \sum_{n=1}^{\infty} [Q_n(r,p) - Q_n(r,q)] \left[A(q)e^{(p-s)A(q)} - A(s)e^{(p-s)A(s)}\right] dp \\
& -\sum_{n=1}^{\infty} [Q_n(r,q) - Q_n(r,s)] \int_q^r A(s)e^{(p-s)A(s)}\, dp \\
& -\sum_{n=1}^{\infty} \int_s^q [Q_n(r,p) - Q_n(r,s)]\, A(s)e^{(p-s)A(s)}\, dp - \sum_{n=1}^{\infty} Q_n(r,q) \int_{r-q}^{r-s} A(q)e^{pA(q)}\, dp \\
& +\sum_{n=1}^{\infty} Q_n(r,q) \left[e^{(r-s)A(q)} - e^{(r-s)A(s)}\right] + \sum_{n=1}^{\infty} [Q_n(r,q) - Q_n(r,s)] \left[e^{(r-s)A(s)} - 1\right];
\end{aligned}
$$

again a straightforward computation using [3, Lemma 1.10 (i)] and Lemma 2.1, leads to the result.

We can now list the main regularity properties of the evolution operators $U(t,s)$, defined in (2.6).

Theorem 2.3. *Set $\Delta := \{(t,s) \in [0,T]^2 : t > s\}$. We have:*

(i) $(t,s) \to U(t,s) \in B(\overline{\Delta}, \mathcal{L}(E)) \cap C(\Delta, \mathcal{L}(E))$ *and*

$$U(t,t) = 1, \quad U(t,r)U(r,s) = U(t,s), \quad \forall 0 \le s \le r \le t \le T;$$

(ii) $\lim_{s \uparrow t} \|U(t,s)x - x\|_E = 0$ *if and only if* $x \in \overline{D_{A(t)}}$, *and*

$$\lim_{t \downarrow s} \|U(t,s)x - x\|_E = 0 \quad \textit{if and only if} \quad x \in \overline{D_{A(s)}};$$

(iii) $\|U(t,s) - U(r,s)\|_{\mathcal{L}(E)} \le c\left\{\log\left(1 + \frac{t-r}{r-s}\right) + (t-r)^\delta\right\}, \ \forall 0 \le s < r \le t \le T;$

(iv) $\|U(t,r) - U(t,s)\|_{\mathcal{L}(E)} \le c\left\{\log\left(1 + \frac{r-s}{t-r}\right) + (r-s)^\delta\right\}, \ \forall 0 \le s \le r < t \le T;$

(v) $t \to U(t,s) \in C^1(]s,T], \mathcal{L}(E)) \cap C((]s,T], \mathcal{L}(E, D_A))$ and $\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s)$; *moreover,*

$$\|A(t)U(t,s)\|_{\mathcal{L}(E)} \leq c(t-s)^{-1}, \quad \forall 0 \leq s < t \leq T;$$

(vi) $t \to A(t)U(t,s)A(s)^{-1} \in B(\overline{\Delta}, \mathcal{L}(E)) \cap C(\Delta, \mathcal{L}(E))$;

(vii) $\lim_{s \uparrow t} \|A(t)U(t,s)A(s)^{-1}x - x\|_E = 0$ *if and only if* $x \in \overline{D_{A(t)}}$, *and* $\lim_{t \downarrow s} \|A(t)U(t,s)A(s)^{-1}x - x\|_E = 0$ *if and only if* $x \in \overline{D_{A(s)}}$;

(viii) $\|A(t)U(t,s)A(s)^{-1} - U(t,s)\|_{\mathcal{L}(E)} \leq c(t-s)^{\delta}, \quad \forall 0 \leq s \leq t \leq T$;

(ix) *if* $x \in \overline{D_{A(s)}}$, *then*

$$E - \lim_{h \downarrow 0} h^{-1}[U(t,s+h) - U(t,s)]A(s)^{-1}x = -U(t,s)x \quad \forall 0 \leq s < t \leq T,$$

and

$$E - \lim_{h \uparrow 0} h^{-1}[U(t,s+h) - U(t,s)]A(s+h)^{-1}x = -U(t,s)x, \ \forall 0 < s \leq t \leq T.$$

Proof: (i) The first part follows by [3, Lemma 1.10] and Lemma 2.2 (ii)-(v)-(i); next, obviously, $U(t,t) = 1$. The last assertion will be proved after part (v).
(ii) We have

$$U(t,s)x - x = \left[e^{(t-s)A(s)} - e^{(t-s)A(t)}\right]x + \left[e^{(t-s)A(t)} - 1\right]x + \int_s^t Z(r,s)x\,dr,$$

so that by [3, Lemma 1.10 (i)] and Lemma 2.2 (i)

$$\|U(t,s)x - x\|_E = O((t-s)^{\delta}) + \|e^{(t-s)A(t)} - 1)x\|_E \ \text{ as } \ s \uparrow t,$$

and the first part follows by [14, Prop. 1.2 (i)]. The proof of the second part is even simpler.
(iii) We write

$$U(t,s) - U(r,s) = \int_r^t A(s)e^{(q-s)A(s)}\,dp + \int_r^t Z(q,s)\,dq,$$

and the result follows by (1.7) and Lemma 2.2 (i).
(v) Formula (2.6) clearly implies that $U(\cdot,s) \in C^1(]s,T], \mathcal{L}(E))$ and

$$\frac{\partial}{\partial t}U(t,s) = A(s)e^{(t-s)A(s)} + Z(t,s), \quad 0 \leq s < t \leq T. \tag{2.10}$$

Consider for each $m \in \mathbb{N}^+$, the problem

$$\begin{cases} u_m'(t) - A_m(t)u_m(t) = 0, & t \in [s,T] \\ u_m(s) = x \in E, \end{cases} \tag{2.11}$$

where $A_m(t) := mA(t)R(m, A(t)) \in \mathcal{L}(E)$ is the Yosida approximation of $A(t)$. It is easy to show (see [3, Prop. 4.5]) that problem (2.11) has a unique solution $u_m \in C^1([s,T],E)$. Moreover, we can introduce the kernels $Q_m(t,s)$, $Q_{n,m}(t,s)$ as well as the integral operator $Q_{s,m}$ simply by replacing $A(t)$ with $A_m(t)$ in the corresponding expressions, defining $Q(t,s)$, $Q_n(t,s)$ and Q_s; i.e., in (2.2), (2.5) and (2.1). Then we can define

$$U_m(t,s) := e^{(t-s)A_m(s)} + \int_s^t Z_m(r,s)\,dr\,, \tag{2.12}$$

where again, $Z_m(t,s)$ is defined by replacing Q_s, $A(t)$, $Q_n(t,s)$ with $Q_{s,m}$, $A_m(t)$, $Q_{n,m}(t,s)$ in the definition (2.7) of $Z(r,s)$. Now it is easy to see, following [4], that the solution of (2.11) must be given by

$$u_m(t) = U_m(t,s)x, \quad T \in [s,T],$$

so that, by (2.12)

$$A_m(t)U_m(t,s) = \frac{\partial}{\partial t}U_m(t,s) = A_m(t)e^{(t-s)A_m(t)} + Z_m(t,s), \ \ t \in [s,T]. \tag{2.13}$$

Now, by (2.12), (2.13), (2.10) and the results of [3, Sections 4-5], it is readily shown that if $0 \le s < t \le T$, we have:

$$U_m(t,s) \to U(t,s) \quad \text{in} \quad \mathcal{L}(E) \quad \text{as} \quad m \to \infty,$$

$$A_m(t)U_m(t,s) \to \frac{\partial}{\partial t}U(t,s) \quad \text{in} \quad \mathcal{L}(E) \quad \text{as} \quad m \to \infty.$$

This easily implies that $U(t,s) \in \mathcal{L}(E, D_{A(t)})$ and $A(t)U(t,s) = \frac{\partial}{\partial t}U(t,s)$. Finally, by (2.10), (1.7) and Lemma 2.2 (i), we get

$$\left\| \frac{\partial}{\partial t}U(t,s) \right\|_{\mathcal{L}(E)} = \|A(t)U(t,s)\|_{\mathcal{L}(E)} \le c(t-s)^{-1},$$

and (v) is proved.

We prove now the last part of (i). For $r > s$, we have $U(r,s)x \in D_{A(r)}$ $\forall x \in E$, so that by [3, Theorem 6.3], the problem

$$\begin{cases} u'(t) - A(t)u(t) = 0, & t \in]r,T] \\ u(r) = U(r,s)x\,, \end{cases} \tag{2.14}$$

has a unique classical solution. But by (v), the functions $v_1(t) := U(t,s)x$ and $v_2(t) := U(t,r)U(r,s)x$ both solve (2.14), and therefore $v_1 \equiv v_2$; this proves the identity $U(t,s) = U(t,r)U(r,s)$. Note that, in fact, v_1 is a strict solution of (2.14), since by (v), it belongs to $C^1([r,T],E) \cap C([r,T],D_A)$; this implies, by [3, Prop. 3.7 (i)] and by the arbitrariness of $r > s$, that

$$A(t)U(t,s) \in \overline{D_{A(t)}} \ \ \forall 0 \le s < t \le T. \tag{2.15}$$

(iv) Suppose first $t - r \le r - s$. Then, we write:

$$U(t,r) - U(t,s) = -\int_{t-r}^{t-s} A(r)e^{qA(r)}\,dq + \left[e^{(t-s)A(r)} - e^{(t-s)A(s)}\right] + \int_r^t [Z(q,r) - Z(q,s)]\,dq - \int_r^s Z(q,s)\,dq\,.$$

By (1.7), [3, Lemma 1.10 (i)] and Lemma 2.2 (v)-(i), we get for any $\eta \in]0,\delta[$

$$\|U(t,r)-U(t,s)\|_{\mathcal{L}(E)} \le c(\eta)\left\{\log\left(1+\frac{r-s}{t-r}\right)+(r-s)^\delta+(r-s)^\eta(t-r)^{\delta-\eta}\right\};$$

as $t-r\le r-s$, we get the result in this case. On the other hand, if $t-r>r-s$, we use the last assertion of (i), writing

$$U(t,r)-U(t,s)=U(t,2r-s)[U(2r-s,r)-U(2r-s,s)],$$

and the desired estimate follows.
(vi) For each $x\in E$, $t\to U(t,s)A(s)^{-1}x$ is the classical solution of

$$\begin{cases} u'(t)-A(t)u(t)=0, & t\in]s,T] \\ u(s)=A(s)^{-1}x, \end{cases} \tag{2.16}$$

and by [3, Theorem 6.3], we have $t\to A(t)U(t,s)A(s)^{-1}x\in C(]s,T],E)$ and

$$\|A(t)U(t,s)A(s)^{-1}x\|_E\le c\|x\|_E \;\; \forall 0\le s\le t\le T, \tag{2.17}$$

so that $t\to A(t)U(t,s)A(s)^{-1}\in C(]s,T],\mathcal{L}(E))$ $\forall s\in[0,T[$. On the other hand, if $0\le r< s<t$, we have, choosing any $q\in]s,t[$,

$$\begin{aligned} A(t)U(t,s)A(s)^{-1}x-A(t)U(t,r)A(r)^{-1}x &= A(t)U(t,s)\left[A(s)^{-1}-A(r)^{-1}\right]x \\ &\quad + A(t)U(t,q)[U(q,s)-U(q,r)]A(r)^{-1}x, \end{aligned}$$

so that by (v), Hypothesis II and (iv), we get

$$\begin{aligned} &\left\|A(t)U(t,s)A(s)^{-1}x-A(t)U(t,r)A(r)^{-1}x\right\|_E \\ &\le c\left[(t-s)^{-1}(s-r)^\delta+(t-q)^{-1}\log\left(1+\frac{s-r}{q-s}\right)+(t-q)^{-1}(s-r)^\delta\right]\|x\|_E \\ &= o(1)\cdot\|x\|_E \;\text{ as }\; s-r\downarrow 0. \end{aligned}$$

This proves that $s\to A(t)U(t,s)A(s)^{-1}\in C([0,t[,\mathcal{L}(E))$ $\forall t\in]0,T]$. The proof of (vi) is complete.
(vii) The function $t\to U(t,s)A(s)^{-1}x$ is a strict solution of (2.16) if and only if $x\in \overline{D_{A(s)}}$ ([3, Theorem 6.1 and Prop. 3.1 (ii)]); this proves the second part. In addition, $t\to U(t,s)A(s)^{-1}x$ solves the integral equation

$$\begin{aligned} &A(t)U(t,s)A(s)^{-1}x= \\ &\quad Q_s(A(\cdot)U(\cdot,s)A(s)^{-1}x)(t)+A(t)e^{(t-s)A(t)}A(s)^{-1}x,\;\; s\in[0,T[,\;\; t\in[s,T], \end{aligned} \tag{2.18}$$

([3, Theorem 6.3 (i)]). Since (2.17) holds, we have by (2.3) and (2.18), as $s\uparrow t$,

$$\begin{aligned} &A(t)U(t,s)A(s)^{-1}x-x \\ &= O((t-s)^\delta)+A(t)e^{(t-s)A(t)}\left[A(s)^{-1}-A(t)^{-1}\right]x+\left[e^{(t-s)A(t)}-1\right]x \\ &= O((t-s)^\delta)+\left[e^{(t-s)A(t)}-1\right]x, \end{aligned}$$

so that the first part also follows.
(viii) Let $x \in E$. By (2.6) and (2.18), we can write

$$A(t)U(t,s)A(s)^{-1}x - U(t,s)x = Q_s\left((A(\cdot)U(\cdot,s)A(s)^{-1}x)\right)(t)$$
$$+ A(t)e^{(t-s)A(t)}\left[A(s)^{-1} - A(t)^{-1}\right]x + \left[e^{(t-s)A(t)} - e^{(t-s)A(s)}\right]x - \int_s^t Z(r,s)x\,dr,$$

and recalling (2.3), [3, Lemma 1.10 (i)] and Lemma 2.2 (i), the result follows.
(ix) If $h \in]0,(t-s)/2[$, we write

$$h^{-1}[U(t,s+h) - U(t,s)]A(s)^{-1}x = U(t,s+h)h^{-1}[1 - U(s+h,s)]A(s)^{-1}x$$
$$= -U(t,s+h)h^{-1}\int_s^{s+h} A(r)U(r,s)A(s)^{-1}x\,dr,$$

and by (i) and (vii), we obtain

$$h^{-1}[U(t,s+h) - U(t,s)]A(s)^{-1}x = \mathrm{o}(1) - U(t,s)x \quad \text{as} \quad h \downarrow 0.$$

If $h \in]-s,0[$, we write:

$$h^{-1}[U(t,s+h) - U(t,s)]A(s+h)^{-1}x = U(t,s)h^{-1}[U(s,s+h) - 1]A(s+h)^{-1}x$$
$$= -U(t,s)h^{-1}\int_{s+h}^{s} A(r)U(r,s+h)A(s+h)^{-1}x\,dr$$

and again (i) and (vii) yield

$$h^{-1}[U(t,s+h) - U(t,s)]A(s+h)^{-1}x = \mathrm{o}(1) - U(t,s)x \quad \text{as} \quad h \uparrow 0.$$

Theorem 2.3 is completely proved.

3. Existence of strong solutions. We are concerned here with existence of strong solutions of problem (0.1). We start with an obvious necessary condition.

Proposition 3.1. *Let u be a strong solution of (0.1) with $x \in E$, $f \in C([s,T],E)$; then $x \in \overline{D_{A(s)}}$.*

Proof: Evident by definition.

Next, we establish the usual variation of parameters formula for any strong solution.

Proposition 3.2. *Let u be a strong solution of (0.1) with $x \in \overline{D_{A(s)}}$, $f \in C([s,T],E)$; then u is given by*

$$u(t) = U(t,s)x + \int_s^t U(t,r)f(r)\,dr, \quad t \in [s,T]. \tag{3.1}$$

Proof: By definition, there exist three sequences $\{u_n\} \subset C^1([s,T],E) \cap C([s,T],D_A)$, $\{x_n\} \subset D_{A(s)}$ and $\{f_n\} \subset C([s,T],E)$ such that

$$u_n \to u, \;\; f_n \to f \;\; \text{in} \;\; C([s,T],E), \;\; x_n \to x \;\; \text{in} \;\; E \;\; \text{as} \;\; n \to \infty, \tag{3.2}$$

$$u_n' - A(\cdot)u_n(\cdot) = f_n \;\; \text{in} \;\; [s,T], \;\; u_n(s) = x_n. \tag{3.3}$$

By [4, Theorem 2.1], we can represent u_n as

$$u_n(t) = U(t,s)x_n + \int_s^t U(t,r)f_n(r)\,dr, \quad t \in [s,T], \tag{3.4}$$

and as $n \to \infty$, by (3.2) we get the result.

We prove now our existence result.

Theorem 3.3. *Let $x \in \overline{D_{A(s)}}$, $f \in C([s,T],E)$; then problem (0.1) has a unique strong solution u, which is given by (3.1).*

Proof: Uniqueness was proved in Proposition 3.2. To show existence, we choose any sequence $\{f_n\} \subset C^1([s,T],E)$ such that

$$f_n(s) = f(s) \;\; \forall n \in \mathbf{N}, \;\; f_n \to f \;\text{ in }\; C([s,T],E) \;\text{ as }\; n \to \infty.$$

Next, we select $\{x_n\} \subset D_{A(s)}$ such that

$$A(s)x_n + f(s) \in \overline{D_{A(s)}} \;\; \forall n \in \mathbf{N}, \;\; x_n \to x \;\text{ in }\; E \;\text{ as }\; n \to \infty;$$

this can be done by taking, e.g.,

$$x_n := n^2[R(n, A(s))]^2[x - A(s)^{-1}f(s)] + A(s)^{-1}f(s), \;\; n \in \mathbf{N}.$$

Now, by [3, Theorem 6.1], the problem

$$v'(t) - A(t)v(t) = f_n(t), \;\;\; t \in [s,T], \;\; v_n(s) = x_n$$

has a unique strict solution u_n, which can be represented by (3.4), due to [4, Theorem 2.1]. As $n \to \infty$, formula (3.4) reduces to (3.1), so that by definition, the function (3.1) is indeed a strong solution of (0.1).

4. Regularity of strong solutions. We collect here some regularity properties of the strong solution u of problem (0.1). As u is given by formula (3.1), we will study separately the two functions

$$(U_s x)(t) := U(t,s)x, \quad t \in [s,T], \tag{4.1}$$

$$(U_s * f)(t) := \int_s^t U(t,r)f(r)\,dr, \quad t \in [s,T]. \tag{4.2}$$

We start with the function (4.1). Our first result concerns time regularity.

Theorem 4.1. *Let $x \in E$ and $\beta \in]0,1[$. We have:*

(i) $U_s x \in C_0(]s,T],E) \cap C^1(]s,T],E)$ *and* $(U_s x)' \in C_1(]s,T],E)$;
(ii) $x \in \overline{D_{A(s)}}$ *if and only if* $U_s x \in C([s,T],E)$, *and in this case* $(U_s x)' \in C_1([s,T],E)$;
(iii) $x \in D_{A(s)}(\beta,\infty)$ *if and only if* $U_s x \in C^\beta([s,T],E)$ *and* $(U_s x)' \in C_{1-\beta}(]s,T],E)$;
(iv) $x \in D_{A(s)}(\beta)$ *if and only if* $U_s x \in h^\beta([s,T],E)$ *and* $(U_s x)' \in C_{1-\beta}([s,T],E)$;
(v) $x \in D_{A(s)^2}(\frac{1}{2},\infty)$ *if and only if* $U_s x \in C^{*,1}([s,T],E)$;
(vi) $x \in D_{A(s)^2}(\frac{1}{2})$ *if and only if* $U_s x \in h^{*,1}([s,T],E)$;
(vii) $x \in D_{A(s)}(1,\infty)$ *if and only if* $U_s x \in Lip([s,T],E)$ *and* $(U_s x)' \in C_0(]s,T],E)$.

Moreover, in each case $U_s x$ and $(U_s x)'$ depend continuously on x in the corresponding norms.

Proof: (i) Immediate consequence of Theorem 2.3 (i)-(v).
(ii) The first part is contained in Theorem 2.3 (ii). On the other hand, if $x \in \overline{D_{A(s)}}$, then by [3, Theorem 6.5], the problem

$$\begin{cases} u'(t) - A(t)u(t) = 0, & t \in]s,T] \\ u(s) = x, \end{cases} \tag{4.3}$$

has a unique classical solution; by Proposition 5.1 below and by Theorem 3.3, such solution is $U_s x$. Hence, again by [3, Theorem 6.5], we get the following representation formula:

$$\begin{aligned}(U_s x)' &= A(t)(U_s x)(t) = (1-Q_s)^{-1}\left(A(\cdot)e^{(\cdot - s)A(\cdot)}x\right)(t) \\ &= Q_s\left(A(\cdot)(U_s x)\right)(t) + A(t)e^{(t-s)A(t)}x \\ &= Q_s\left(A(\cdot)(U_s x)\right)(t) + \left[A(t)e^{(t-s)A(t)} - A(s)e^{(t-s)A(s)}\right]x \\ &\qquad + A(s)e^{(t-s)A(s)}x,\ t \in]s,T],\end{aligned} \tag{4.4}$$

where Q_s is the integral operator defined by (2.1)-(2.2). Recall now that $Q_s(A(\cdot)(U_s x)) \in B_{1-\delta}(s,T,E)$ by (i) and [3, Propositions 2.1 (vi), 2.6 (iii)(g), 2.4 (vii)]; thus, by [3, Lemma 1.10 (i)], we obtain

$$(t-s)(U_s x)'(t) = \mathrm{O}\left((t-s)^\delta\right) + (t-s)A(s)e^{(t-s)A(s)}x \quad \text{as} \quad t \downarrow s,$$

which implies, by [2, Lemma 2.4 (vi)],

$$(t-s)(U_s x)'(t) = \mathrm{o}(1) \quad \text{as} \quad t \downarrow s.$$

(iii) If $U_s x \in C^\beta([s,T],E)$, then we write

$$(U_s x)(t) - (U_s x)(s) = \left[e^{(t-s)A(s)} - 1\right]x + \int_s^t Z(r,s)x\,dr,$$

and by Lemma 2.2 (i) and $(1.3)_1$ we get $x \in D_{A(s)}(\beta \wedge \delta, \infty)$. If $\beta \le \delta$, we obtain $x \in D_{A(s)}(\beta,\infty)$; otherwise, again by Lemma 2.2 (i) and $(1.3)_1$ we get $x \in D_{A(s)}(\beta \wedge 2\delta, \infty)$. A finite number of iterations of this argument leads to the conclusion. If, conversely, $x \in D_{A(s)}(\beta,\infty)$, then we have

$$(U_s x)(t) - (U_s x)(r) = \int_r^t A(s)e^{(q-s)A(s)}x\,dq + \int_r^t Z(q,s)x\,dq, \quad s \le r \le t \le T, \tag{4.6}$$

which by $(1.3)_2$ and Lemma 2.2 (i), easily implies

$$(U_s x)(t) - (U_s x)(r) = \mathrm{O}((t-r)^\beta) \quad \text{as} \quad t-r \downarrow 0.$$

Moreover, from (4.4), we deduce (by [3, Propositions 2.1 (iv), 2.6 (iii)(d), 2.4 (iv) and Lemma 1.10 (i)])

$$(t-s)^{1-\beta}(U_s x)'(t) = \mathrm{O}(1) \quad \text{as} \quad t \downarrow s \iff x \in D_{A(s)}(\beta,\infty).$$

(iv) Quite similar to (iii) (compare with [3, Remark 6.7]).
(v) If $s < r < t \le T$, we have

$$U(t,s)x + U(r,s)x - 2U\left(\frac{t+r}{2},s\right)x = \tag{4.7}$$

$$\int_r^{(t+r)/2}\int_0^{(t-r)/2} A(s)^2 e^{(p+q-s)A(s)}x\,dp\,dq + \int_r^{(t+r)/2}\left[Z\left(q+\frac{t-r}{2},s\right) - Z(q,s)\right]x\,dq;$$

hence, by $(1.3)_2$ and Lemma 2.2 (iii), we conclude that

$$U(t,s)x + U(r,s)x - 2U\left(\frac{t+r}{2},s\right)x = \mathrm{O}(t-r) \quad \text{as} \quad t-r \downarrow 0 \iff x \in D_{A(s)^2}(\tfrac{1}{2},\infty).$$

(vi) Quite similar to (v).
(vii) Similar to (iii).

Concerning space regularity of the function (4.1), we have the following result:

Theorem 4.2. *Let $x \in E$ and $\beta \in]0,1[$. We have:*

(i) $U_s x \in C_1(]s,T], D_A)$;
(ii) $x \in \overline{D_{A(s)}}$ *if and only if* $U_s x \in C([s,T], \overline{D_A})$, *and in this case* $U_s x \in C_1([s,T], D_A)$;
(iii) $x \in D_{A(s)}(\beta,\infty)$ *if and only if* $U_s x \in B(s,T,D_A(\beta,\infty))$ *and* $U_s x \in C_{1-\beta}(]s,T], D_A)$;
(iv) $x \in D_{A(s)}(\beta)$ *if and only if* $U_s x \in C([s,T], D_A(\beta))$ *and* $U_s x \in C_{1-\beta}([s,T], D_A)$;
(v) $x \in D_{A(s)^2}(\frac{1}{2},\infty)$ *if and only if* $U_s x \in B(s,T,D_{A^2}(\frac{1}{2},\infty))$;
(vi) $x \in D_{A(s)^2}(\frac{1}{2})$ *if and only if* $U_s x \in C([s,T], D_{A^2}(\frac{1}{2}))$;
(vii) $x \in D_{A(s)}(1,\infty)$ *if and only if* $U_s x \in C_0(]s,T], D_A)$.

Moreover, in each case, $U_s x$ depends continuously on x in the corresponding norms.

Proof: (i) Immediate consequence of Theorem 2.3 (v).
(ii) The "if" part is obvious. Conversely, if $x \in \overline{D_{A(s)}} = \overline{D_{A(s)^2}}$ and $\{x_n\} \subset D_{A(s)^2}$ is such that $x_n \to x$ in E as $n \to \infty$, then $\{U_s x_n\} \subset C([s,T], D_A)$ by [3, Theorem 6.1], and $U_s x_n \to U_s x$ in $B(s,T,E)$ by Theorem 2.3 (i), so that $U_s x \in C([s,T], D_A)$. The second part is a consequence of Theorems 4.1 (ii) and 2.3 (v).
(iii) Again, the "if" part is evident. Conversely, let $x \in D_{A(s)}(\beta,\infty)$; then, by Proposition 5.1 below and Theorem 3.3, $U_s x$ is the classical solution of problem (4.3), so that, as in (4.4),

$$A(t)(U_s x)(t) = Q_s(A(\cdot)U_s x)(t) + A(t)e^{(t-s)A(t)}x. \tag{4.8}$$

Hence, if $\xi > 0$, we have, recalling (2.1) and (2.2)

$$\begin{aligned}\xi^{1-\beta}A(t)e^{\xi A(t)}U_s x(t) &= \xi^{1-\beta}\int_s^t A(t)^2 e^{(\xi+t-r)A(t)}\left[A(t)^{-1} - A(r)^{-1}\right]A(r)U(r,s)x\,dr \\ &\quad + \xi^{1-\beta}\left[A(t)e^{(\xi+t)A(t)} - A(s)e^{(\xi+t)A(s)}\right]x + \xi^{1-\beta}A(s)e^{(\xi+t)A(s)}x;\end{aligned} \tag{4.9}$$

thus, by [3, Lemmas 1.11 (i), 1.10 (i) and Theorem 6.4], we get, after easy calculations,

$$\sup_{\xi>0}\left\|\xi^{1-\beta}A(t)e^{\xi A(t)}U_s x(t)\right\|_E \le c\left\{(t-s)^\delta + 1\right\}\|x\|_{D_{A(s)}(\beta,\infty)},$$

so that $U_s x \in B(s,T,D_A(\beta,\infty))$. Moreover, by Theorems 4.1 (iii) and 2.3 (v) we immediately obtain the second part.
(iv) The "if" part is clear. Conversely, fix $\epsilon > 0$; if $t - s \le \epsilon^{1/\delta}$ by (4.9), as above, we get for small ξ

$$\left\|\xi^{1-\beta}A(t)e^{\xi A(t)}U_s x(t)\right\|_E \le c\left[(t-s)^\delta + \left\|\xi^{1-\beta}A(s)e^{\xi A(s)}x\right\|_E\right] \le c\epsilon,$$

whereas, if $t - s > \epsilon^{1/\delta}$, we have directly for small ξ

$$\begin{aligned}\left\|\xi^{1-\beta}A(t)e^{\xi A(t)}U_s x(t)\right\|_E &\le c\xi^{1-\beta}\|A(t)U(t,s)x\|_E \\ &\le c(t-s)^{\beta-1}\xi^{1-\beta} \le c\epsilon^{(\beta-1)\delta}\xi^{1-\beta} \le c\epsilon;\end{aligned}$$

hence, $U_s x \in C([s,T], D_A(\beta))$ by $(1.13)_1$. The remaining part of (iv) is quite similar to (iii) (compare with [3, Remark 6.7]).
(v) The "if" part is obvious. If $x \in D_{A(s)^2}(1/2,\infty)$, we use (4.8) and write for $\xi > 0$

$$\begin{aligned}\xi A(t)^2 e^{\xi A(t)}U_s x(t) &= \xi\int_s^t A(t)^3 e^{(\xi+t-r)A(t)}\left[A(t)^{-1} - A(r)^{-1}\right]A(r)U(r,s)x\,dr \\ &\quad + \xi\left[A(t)^2 e^{(\xi+t)A(t)} - A(s)^2 e^{(\xi+t)A(s)}\right]x + \xi A(s)^2 e^{(\xi+t)A(s)}x;\end{aligned} \tag{4.10}$$

as, in particular, $x \in D_{A(s)}(1-\delta/2, \infty)$, we proceed as in (4.9) and readily obtain the result.
(vi) Quite similar to (iv), using (4.10) instead of (4.9).
(vii) Quite similar to (iii).

Let us consider now the function (4.2); again we study its time and space regularity.

Theorem 4.3. *Let $f \in L^\infty(s,T,E)$. We have*

(i) $U_s * f \in C^{*,1}([s,T],E) \cap B(s,T,D_{A^2}(\frac{1}{2},\infty))$;

(ii) *if* $f \in C\left([s,T],\overline{D_A}\right)$ *then* $U_s * f \in h^{*,1}([s,T],\bar{D}_A) \cap C([s,T],D_{A^2}(\frac{1}{2}))$.

Moreover, $U_s * f$ depends continuously on $\|f\|_{L^\infty(s,T,E)}$ in the corresponding norms.

Proof: By (2.6), we easily have for $s \le r < t \le T$

$$\begin{aligned} &U_s * f(t) + U_s * f(r) - 2U_s * f(\tfrac{t+r}{2}) = \\ &\int_r^t U(t,q)f(q)\,dq - 2\int_r^{(t+r)/2} U(\tfrac{t+r}{2},q)f(q)\,dq \\ &+ \int_s^r \int_s^{(t+r)/2} \int_0^{(t-r)/2} A(q)^2 e^{(p+u-q)A(q)} f(q)\,du\,dp\,dq \\ &+ \int_s^r \int_r^{(t+r)/2} \left[z\left(p+\frac{t-r}{2},q\right) - Z(p,q)\right] f(q)\,dp\,dq; \end{aligned} \tag{4.10}$$

hence, by Theorem 2.3 (i) and Lemma 2.2 (ii), we deduce

$$\left\| U_s * f(t) + U_s * f(r) - 2U_s * f(\frac{t+r}{2}) \right\|_E \le c(t-r)\|f\|_{L^\infty(s,T,E)};$$

i.e, $U_s * f \in C^{*,1}([s,T],E)$. Next, let $s < t \le T$, for each $\xi > 0$ and for a.a. $q \in]s,t[$, we can write by Theorem 2.3 (v) and (2.10)

$$\begin{aligned} \xi A(t)^2 e^{\xi A(t)} U(t,q) f(q) &= \xi A(t) e^{\xi A(t)} \frac{\partial}{\partial t} U(t,q) f(q) \\ &= \xi A(t) e^{\xi A(t)} \left[A(q) e^{(t-q)A(q)} - A(t) e^{(t-q)A(t)} \right] f(q) \\ &\quad + \xi A(t)^2 e^{(\xi+t-q)A(t)} f(q) + \xi A(t) e^{\xi A(t)} Z(t,q) f(q). \end{aligned} \tag{4.11}$$

We now integrate over $]s,t[$: by [3, Lemma 1.10] and Lemma 2.2 (i), we easily obtain

$$\sup_{\xi>0} \left\| \xi A(t)^2 e^{\xi A(t)} \int_s^t U(t,q) f(q)\,dq \right\|_E \le c\|f\|_{L^\infty(s,T,E)};$$

i.e, $U_s * f \in B(s,T,D_{A^2}(\frac{1}{2},\infty))$.

(ii) For $s \le r < t \le T$ we split (4.10) as

$$
\begin{aligned}
&U_s * f(t) + U_s * f(r) - 2U_s * f(\frac{t+r}{2}) = \\
&\int_r^t \left[e^{(t-q)A(q)} - 1\right] f(q)\,dq + \int_r^t [f(q) - f(r)]\,dq \\
&+ \int_r^t \int_q^t Z(p,q) f(q)\,dp\,dq - 2\int_r^{(t+r)/2} \left[e^{((t+r)/2-q)A(q)} - 1\right] f(q)\,dq \\
&- 2\int_r^{(t+r)/2} [f(q) - f(r)]\,dq - 2\int_r^{(t+r)/2} \int_q^{(t+r)/2} Z(p,q) f(q)\,dp\,dq \\
&+ \int_s^r \int_s^{(t+r)/2} \int_0^{(t-r)/2} A(q)^2 e^{(p+u-q)A(q)} f(q)\,du\,dp\,dq \\
&+ \int_s^r \int_r^{(t+r)/2} \left[Z(p+\frac{t-r}{2}, q) - Z(p,q)\right] f(q)\,dp\,dq =: \sum_{i=1}^{8} I_i\,.
\end{aligned}
$$

Fix $\epsilon \in]0,1[$ and choose $\eta_\epsilon \in]0, \epsilon^{2/\delta}]$, such that (compare with $(1.13)_2$, $(1.13)_1$) :

$$\sup_{s \le q \le T} \left\| \left[e^{pA(q)} - 1\right] f(q) \right\|_E \le \epsilon, \quad \forall p \in]0, \eta_\epsilon], \tag{4.12}$$

$$\sup_{s \le q \le T} \left\| p^2 A(q)^2 e^{pA(q)} f(q) \right\|_E \le \epsilon, \quad \forall p \in]0, \eta_\epsilon^{1/2} + \eta_\epsilon], \tag{4.13}$$

$$\|f(q) - f(p)\|_E \le \epsilon, \quad \text{if} \quad |q - p| \le \eta_\epsilon\,. \tag{4.14}$$

We will show that if $0 < t - r \le \eta_\epsilon$, then

$$\left\| U_s * f(t) + U_s * f(r) - 2U_s * f(\frac{t+r}{2}) \right\|_E \le c\epsilon(t-r)\,. \tag{4.15}$$

By (4.12) and (4.14), recalling Lemma 2.2 (i)-(ii), we have

$$\|I_1\|_E + \|I_2\|_E + \|I_4\|_E + \|I_5\|_E \le c\epsilon(t-r),$$

$$\|I_3\|_E + \|I_6\|_E \le c(t-r)^{1+\delta}, \quad \|I_8\|_E \le c(t-r)^{1+\delta/2},$$

so that

$$\left\| \sum_{i=1}^{8} I_i \right\|_E \le c\epsilon(t-r) + \|I_7\|_E \quad \text{if} \quad 0 < t - r \le \eta_\epsilon\,. \tag{4.16}$$

We have only to estimate $\|I_7\|_E$ for $0 < t - r \le \eta_\epsilon$. We distinguish two cases: (a) $r - s \le \eta_\epsilon^{1/2}$, (b) $r - s > \eta_\epsilon^{1/2}$. In case (a), by (4.13), we have easily $\|I_7\|_E \le c\epsilon(t-r)$. In case (b), we write

$$I_7 = \left[\int_s^{r-\eta_\epsilon^{1/2}} + \int_{r-\eta_\epsilon^{1/2}}^r\right] \int_r^{(t+r)/2} \int_0^{(t-r)/2} A(q)^2 e^{(p+u-q)A(q)} f(q)\,du\,dp\,dq =: I_{7,1} + I_{7,2},$$

and again, clearly, $\|I_{7,2}\|_E \le c\epsilon(t-r)$; on the other hand,

$$\|I_{7,1}\|_E \le \int_r^{(t+r)/2}\int_0^{(t-r)/2}\int_s^{r-\eta_\epsilon^{1/2}} \left\|e^{(p+u-r)A(q)}\right\|_{\mathcal{L}(E)} \left\|A(q)^2 e^{(r-q)A(q)} f(q)\right\|_E \, dq\, du\, dp$$
$$\le c\eta^{-1/2}(t-r)^2 \le c\epsilon(t-r)$$

so that $\|I_7\|_E \le c\epsilon(t-r)$ if $0 < t-r \le \eta_\epsilon$. By (4.16), we obtain (4.15); i.e., $U_s * f \in h^{*,1}([s,T],\bar{D}_A)$. (Since $U_s * f(t) \in \overline{D_{A(t)}}$ $\forall t \in [s,T]$ by (i)). Next, fix $\epsilon > 0$ and choose $\eta_\epsilon \in]0,\epsilon^{1/\delta}]$, such that (compare with $(1.13)_1$)

$$\sup_{0<\xi\le 2\eta_\epsilon} \left\|\xi^2 A(q)^2 e^{\xi A(q)} f(q)\right\|_E \le \epsilon, \quad \forall q \in [s,T]. \tag{4.17}$$

We have two cases: (a) $t-s \le \eta_\epsilon$, (b) $t-s > \eta_\epsilon$. In case (a), for each $q \in]s,T[$ and $\xi > 0$ we have, similarly to (4.11),

$$\begin{aligned}\xi A(t)^2 e^{\xi A(t)} U(t,q) f(q) &= \xi A(t) e^{\xi A(t)} \left[A(q)e^{(t-q)A(q)} - A(t)e^{(t-q)A(t)}\right] f(q)\\ &\quad + \left[\xi A(t)^2 e^{(\xi+t-q)A(t)} - \xi A(q)^2 e^{(\xi+t-q)A(q)}\right] f(q)\\ &\quad + \xi A(q)^2 e^{(\xi+t-q)A(q)} f(q) + \xi A(t) e^{\xi A(t)} Z(t,q) f(q),\end{aligned}$$

and consequently, by [3, Lemma 1.10] and (4.17), we have, provided $\xi \le \eta_\epsilon$,

$$\left\|\xi A(t)^2 e^{\xi A(t)} \int_s^t U(t,q) f(q)\, dq\right\|_E \le c(t-s)^\delta + c\epsilon \le c\epsilon\,.$$

In case (b), we split

$$\xi A(t)^2 e^{\xi A(t)} \int_s^t U(t,q) f(q)\, dq = \xi A(t)^2 e^{\xi A(t)} \left[\int_s^{t-\eta_\epsilon} + \int_{t-\eta_\epsilon}^t\right] U(t,q) f(q)\, dq =: J_1 + J_2,$$

and as above, we have $\|J_2\|_E \le c\epsilon$ provided $\xi \le \eta_\epsilon$; concerning J_1, for each $q \in]s, t-\eta_\epsilon[$, we have, by [3, Theorem 6.6]:

$$\left\|\xi A(t)^2 e^{\xi A(t)} U(t,q) f(q)\right\|_E \le$$
$$\xi^\delta \sup_{(t+q)/2\le p\le t} [A(p)U(p,q)f(q)]_{D_{A(p)}(\delta,\infty)} \le c\xi^\delta (t-q)^{-\delta-1}\|f(q)\|_E,$$

so that if $\xi \le \epsilon^{1/\delta}\eta_\epsilon$

$$\|J_1\|_E \le c\int_s^{t-\eta_\epsilon} \xi^\delta (t-q)^{-\delta-1}\|f(q)\|_E\, dq \le c\xi^\delta \eta_\epsilon^{-\delta} \le c\epsilon\,.$$

Hence, we have shown that if ξ is sufficiently small,

$$\sup_{s\le t\le T} \left\|\xi A(t)^2 e^{\xi A(t)} \int_s^t U(t,q) f(q)\, dq\right\|_E \le c\epsilon\,;$$

i.e., by $(1.13)_1$, $U_s * f \in C([s,T], D_{A^2}(\frac{1}{2}))$.

Recalling formula (3.1), the above theorems obviously imply the following regularity result for the strong solution of problem (0.1).

Corollary 4.4. *Let $x \in \overline{D_{A(s)}}$, $f \in C([s,T],E)$, and let u be the strong solution of (0.1). We have:*

(i) $x \in D_{A(s)}(\beta,\infty)$, $\beta \in]0,1[$, *if and only if* $u \in C^{\beta}([s,T],E) \cap B(s,T,D_A(\beta,\infty))$;
(ii) $x \in D_{A(s)}(\beta)$, $\beta \in]0,1[$, *if and only if* $u \in h^{\beta}([s,T],E) \cap C([s,T],D_A(\beta))$;
(iii) $x \in D_{A(s)^2}(\frac{1}{2},\infty)$ *if and only if* $u \in C^{*,1}([s,T],E) \cap B(s,T,D_{A^2}(\frac{1}{2},\infty))$;
(iv) *if, in addition* $f \in C([s,T],\overline{D_A})$, *then* $x \in D_{A(s)^2}(\frac{1}{2})$ *if and only if* $u \in h^{*,1}([s,T],\bar{D}_A)$ $\cap C([s,T],D_{A^2}(\frac{1}{2}))$.

Moreover, u depends continuously on the data x, f in the corresponding norms.

Remark 4.5. In the autonomous case, $A(t) \equiv A$, parts (i)-(ii) of Corollary 4.4 were proved in [14, Theorems 3.1, 3.2, 3.3]; an extension to the non-autonomous case with constant domains is in [2, Theorems 6.5, 6.6, 7.4]. A similar result is proved in [1, Theorem 6.3] under assumptions which are independent of ours, as pointed out in [3, Section 7]. However, parts (iii)-(iv) of Corollary 4.4 are new, even in the autonomous case; in that situation, for instance, part (ii) implies by interpolation, that $u \in C^{1-\theta}([s,T],D_A(\theta))\ \forall\, \theta \in]0,1[$, provided $x \in D_{A(s)^2}(\frac{1}{2},\infty)$; this result was known only under the stronger assumption $x \in D_A$ (see [14, Theorem 3.4(c)]).

Remark 4.6. Corollary 4.4 has a counterpart when the evolution problem (0.1) is considered in L^p– rather than in C-spaces (see [9, Theorem 28]).

Remark 4.7. It can be shown by using (4.7) and (4.9), that if $x \in D_{A(s)}(\theta,\infty)$, $\theta \in]0,1[$, then the strong solution u of (0.1) satisfies

$$\sup_{t\in]s,T]} \left\{(t-s)^{1-\theta}[u]_{C^{*,1}([(t+s)/2,t],E)}\right\} + \sup_{t\in]s,T]} \left\{(t-s)^{1-\theta}[u]_{B((t+s)/2,t,D_{A^2}(1/2,\infty))}\right\} < \infty.$$

This property is closely related with certain function spaces introduced in [3, Definition 1.4].

Remark 4.8 A further increase of the smoothness of the data turns the strong solution u, given by (3.1), into a strict or classical one (see [3, Theorems 6.1-6.4], [4, Theorem 2.1] and Theorem 5.2 below).

5. Classical solutions. In this short section, we make a few remarks concerning classical solutions of (0.1) which exist under very general assumptions on the data (see [3, Theorems 6.3, 6.4]).

Proposition 5.1. *Let u be a classical solution of (0.1). If $x \in \overline{D_{A(s)}}$ and $f \in C([s,T],E)$, then u is also a strong solution of (0.1).*

Proof: By Theorem 3.3, the strong solution v of (0.1) exists. Let $\{v_n\} \subset C^1([s,T],E) \cap C([s,T],D_A)$ be such that

$$v_n(s) \to x \text{ in } E \text{ and } v_n \to v,\ \ f_n := v_n' - A(\cdot)v_n(\cdot) \to f \text{ in } C([s,T],E) \text{ as } n \to \infty.$$

Then the function $u_n := u - v_n$ is a strict solution of

$$\begin{cases} u_n'(t) - A(t)u_n(t) = f(t) - f_n(t), & t \in [s+\epsilon,T] \\ u_n(s+\epsilon) = u(s+\epsilon) - v_n(s+\epsilon), \end{cases}$$

where $\epsilon \in]0, T-s[$. By [4, Theorem 1.1], we get the estimate

$$\|u_n(t)\|_E \le c\left\{\|u(s+\epsilon) - v_n(s+\epsilon)\|_E + \int_{s+\epsilon}^t \|f(r) - f_n(r)\|_E\, dr\right\},\ \ t \in [s+\epsilon, T].$$

As $n \to \infty$, we deduce

$$\|u(t) - v(t)\|_E \le c\|u(s+\epsilon) - v(s+\epsilon)\|_E \ \ \forall \epsilon \in]0, T-s[,\ \ \forall t \in [s+\epsilon, T],$$

and as $\epsilon \downarrow 0$, we obtain $u \equiv v$.

We extend now the representation formula (3.1) to (certain) classical solutions of (0.1).

Theorem 5.2. *Let u be a classical solution of (0.1) with $x \in \overline{D_{A(s)}}$ and $f \in C(]s,T], E) \cap L^1(s,T,E)$. Then u is given by (3.1).*

Proof: By assumption, $u \in C([s,T], E)$ and in addition, u is a strict solution of

$$\begin{cases} u'(t) - A(t)u(t) = f(t), & t \in [s+\epsilon, T] \\ u(s+\epsilon) = u(s+\epsilon), \end{cases}$$

where $\epsilon \in]0, T-s[$; therefore, by [4, Theorem 2.1],

$$u(t) = U(t, s+\epsilon)u(s+\epsilon) + \int_{s+\epsilon}^t U(t,r)f(r)\, dr,\ \ \forall \epsilon \in]0, T-s[,\ \ \forall t \in [s+\epsilon, T].$$

Thus, by Theorem 2.3 (i)-(ii), as $\epsilon \downarrow 0$, we get (3.1).

Remark 5.3. The same result holds for the classical solution of (0.1) if $x \in \overline{D_{A(s)}}$ and f belongs to $I_\mu(s,T,E)$ for some $\mu \in [1, 1+\delta[$; i.e., $f \in B_\mu(s,T,E)$ and there exists the limit

$$\int_s^t f(r)\, dr := \lim_{a \downarrow s} \int_a^t f(r)\, dr,\ \ t \in]s, T],$$

(see [3, formula (1.1)]). Indeed, in this case, we write for $a \in]s, t[$

$$\int_a^t U(t,r)f(r)\, dr = \int_a^t [U(t,r) - U(t,s)]f(r)\, dr + U(t,s)\int_a^t f(r)\, dr,$$

so that by Theorem 2.3 (iv), we easily check the existence of the limit

$$\int_s^t U(t,r)f(r)\, dr := \lim_{a \downarrow s} \int_a^t U(t,r)f(r)\, dr,\ \ t \in]s, T].$$

Hence, formula (3.1) still makes sense and it is easy to see, by the same argument as before, that the classical solution u is given by (3.1).

6. Examples. Let Ω be a bounded open set of $\mathbf{R}^n$, $n \ge 1$, with boundary $\partial\Omega$ of class C^{2m}, $m \ge 1$. We introduce the differential operators

$$E(t,x,D) := \sum_{|\gamma| \le 2m} a_\gamma(t,x)D^\gamma,\ \ (t,x) \in [0,T] \times \bar{\Omega}, \tag{6.1}$$

$$B(t,x,D) = \{B_j(t,x,D)\}_{j=1,\dots,m} := \Big\{ \sum_{|\beta|\le m_j} b_{j\beta}(t,x)D^\beta \Big\}_{j=1,\dots,m}, \quad (t,x) \in [0,T] \times \partial\Omega, \tag{6.2}$$

where $m_1,\dots,m_m \in \mathbf{N}$ with $0 \le m_1 \le m_2 \le \cdots \le m_m \le 2m-1$, under the following assumptions:

$$\begin{cases} a_\gamma \in C^\mu\left([0,T], C(\bar\Omega)\right), \ |\gamma| \le 2m; \ b_{j\beta} \in \bigcap_{r=0}^{2m-m_j} C^{\mu_{jr}}\left([0,T], C^r(\partial\Omega)\right), \\ \qquad |\beta| \le m_j, \ j=1,\dots,m, \\ \text{where } \mu, \mu_{jr} \in]0,1] \ (r=0,1,\dots,2m-m_j; \ j=1,\dots,m); \\ \qquad \text{if } m_j = 0 \text{ we assume } b_{j\beta} \equiv 1. \end{cases} \tag{6.3}$$

$$\begin{cases} \text{(ellipticity). There exist } \theta_0 \in]\pi/2, \pi[\text{ and } \nu > 0 \text{ such that} \\ \qquad \nu\left(|\xi|^{2m} + r^{2m}\right) \le \left|\sum_{|\gamma|\le 2m} a_\gamma(t,x)\xi^\gamma - (-1)^m e^{i\theta} r^{2m}\right| \\ \qquad\qquad \forall (t,x) \in [0,T] \times \bar\Omega, \ \forall \theta \in [-\theta_0,\theta_0], \ \forall \xi \in \mathbf{R}^n, \ \forall r \in \mathbf{R}. \end{cases} \tag{6.4}$$

$$\begin{cases} \text{(root conditoin). If } (t,x) \in [0,T] \times \partial\Omega, \ \theta \in [-\theta_0,\theta_0], \ \xi \in \mathbf{R}^n, \ r \in \mathbf{R} \\ \text{with } (\xi,r) \ne (0,0), \ (\xi|\nu(x)) = 0, \text{ then the polynomial} \\ \qquad \varsigma \to \sum_{|\gamma|=2m} a_\gamma(t,x)(\xi + \varsigma\nu(x))^\gamma - (-1)^m e^{i\theta} r^{2m} \\ \text{has exactly } m \text{ roots } \varsigma_j^+(t,x,\theta,\xi,r) \text{ with positive imaginary part (here } \nu(x) \text{ is the unit} \\ \text{outward normal vector at } x \text{ and } (\cdot\,|\,\cdot) \text{ is the scalar product in } \mathbf{R}^n). \end{cases} \tag{6.5}$$

$$\begin{cases} \text{(complementing condition). If } (t,x) \in [0,T] \times \partial\Omega, \ \ \theta \in [-\theta_0,\theta_0], \\ \xi \in \mathbf{R}^n, \ \ r \in \mathbf{R} \text{ with } (\xi,r) \ne (0,0), \ \ (\xi\,|\,\nu(x)) = 0, \text{ then the } m \text{ polynomials} \\ \qquad \varsigma \to \sum_{|\beta|=m_j} b_{j\beta}(t,x)(\xi + \varsigma\nu(x))^\beta, \ \ j=1,\dots,m, \\ \text{are linearly independent modulo } \varsigma \to \prod_{j=1}^m (\varsigma - \varsigma_j^+(t,x,\theta,\xi,r)). \end{cases} \tag{6.6}$$

Consider the non-homogeneous elliptic boundary value problem

$$\begin{cases} \lambda u - E(t,\cdot,D)u = f & \text{in } \ \Omega, \\ B_j(t,\cdot,D)u = g_j & \text{on } \ \partial\Omega, \ j=1,\dots,m, \end{cases} \tag{6.7}$$

with fixed $t \in [0,T]$ and prescribed data $f, g_1, \dots, g_m$. The following result is well known (see [16, Theorems 5.5.2-4.9.1] and [10, Theorem 4.1]).

Proposition 6.1. *Let $p \in]1,\infty[$. Under assumptions (6.1),...,(6.6), there exists $\lambda_0 > 0$, such that if $|\lambda| > \lambda_0$ and $|\arg(\lambda - \lambda_0)| \le \theta_0$, then for each $f \in L^p(\Omega)$ and $g_j \in W^{2m-m_j-1/p,p}(\partial\Omega)$, $j = 1,\dots,m$, problem (6.7) has a unique solution $u \in W^{2m,p}(\Omega)$, moreover, there exists $M_p > 0$, such that*

$$\sum_{r=0}^{2m} |\lambda - \lambda_0|^{1-r/2m} \|D^r u\|_{L^p(\Omega)} \le$$

$$M_p \left\{ \|f\|_{L^p(\Omega)} + \sum_{j=1}^{m} \sum_{r=0}^{2m-m_j} |\lambda - \lambda_0|^{1-(m_j+r)/2m} \|D^r \tilde g_j\|_{L^p(\Omega)} \right\},$$

where $\tilde g_j$ is any function in $W^{2m-m_j,p}(\Omega)$ satisfying

$$\tilde g_j\big|_{\partial\Omega} = g_j \ \ (j=1,\dots,m).$$

A refinement of the above result is the following one ([15], see also [5, proof of Theorem 1.2]).

Proposition 6.2. *Under assumptions (6.1),...,(6.6), there exists $\lambda_1 > 0$, such that if $|\lambda| > \lambda_1$ and $|\arg(\lambda - \lambda_1)| \le \theta_0$, then for each $q \in]n, \infty[$ and for each $f \in L^q(\Omega)$ and $g_j \in W^{2m-m_j-1/q,q}(\partial\Omega)$, $j = 1, \dots, m$, problem (6.7) has a unique solution $u \in W^{2m,q}(\Omega)$; moreover, there exist $N_q > 0$, $K_q > 1$, such that*

$$\begin{aligned}
&\sum_{r=0}^{2m-1} |\lambda - \lambda_1|^{1-r/2m} \|D^r u\|_{C(\bar\Omega)} + |\lambda - \lambda_1|^{n/2mq} \sup_{x_0 \in \bar\Omega} \left\|D^{2m} u\right\|_{L^q\left(\Omega \cap B(x_0, |\lambda-\lambda_1|^{-1/2m})\right)} \\
&\le N_q |\lambda|^{n/2mq} \Big\{ \sup_{x_0 \in \bar\Omega} \|f\|_{L^q\left(\Omega \cap B(x_0, K_q|\lambda - \lambda_1|^{-1/2m}\right)} \\
&+ \sum_{j=1}^{m} \sum_{r=0}^{2m-m_j} |\lambda - \lambda_1|^{1-(m_j+r)/2m} \sup_{x_0 \in \bar\Omega} \|D^r \tilde g_i\|_{L^q\left(\Omega \cap B(x_0, K_q|\lambda-\lambda_1|^{-1/2m})\right)} \Big\},
\end{aligned}$$

where $\tilde g_j$ is any function in $W^{2m-m_j,q}(\Omega)$ satisfying

$$\tilde g_j\big|_{\partial\Omega} = g_j \quad (j = 1, \dots, m).$$

Set now $E := C(\bar\Omega)$, and for $t \in [0, T]$,

$$\begin{cases} D_{A(t)} := \Big\{ u \in \bigcap_{q \in]n,\infty[} W^{2m,q}(\Omega) : E(t, \cdot, D)u \in C(\bar\Omega), \\ \qquad\qquad B_j(t, \cdot, D)u = 0 \text{ on } \partial\Omega, \quad j = 1, \dots, m \Big\} \\ A(t)u := E(t, \cdot, D)u - (\lambda_1 + 1)u. \end{cases} \tag{6.8}$$

Then, we have

Theorem 6.3. *Under assumptions (6.1),...,(6.6), the operators $\{A(t)\}$ defined by (6.8) satisfy Hypotheses I, II of Section 1 in the space $E = C(\bar\Omega)$, provided $\mu \in]0, 1]$, $\mu_{jr} \in]1 - (m_j + r)/2m, 1]$ for $r = 0, 1, \dots, 2m - m_j$ and $j = j_0, \dots, m$, where*

$$j_0 := \min\{j : m_j > 0\}. \tag{6.9}$$

Proof: By Proposition 6.2, we immediately obtain that if $\lambda \in S_{\theta_0}$ and $f \in C(\bar\Omega)$, then $u = R(\lambda, A(t))f$ exists and

$$\|u\|_{C(\bar\Omega)} \le \frac{c}{1 + |\lambda|} \|f\|_{C(\bar\Omega)},$$

so that Hypothesis I is fulfilled. Concerning Hypothesis II, if we set for $f \in C(\bar\Omega)$

$$v := A(s)^{-1} f, \quad u := R(\lambda, A(t))[\lambda - A(s)]v,$$

then we have to estimate the E-norm of

$$u - v = A(t) R(\lambda, A(t)) \left[A(t)^{-1} - A(s)^{-1}\right] f.$$

Now, $u, v \in \bigcap_{q \in]n,\infty[} W^{2m,q}(\Omega)$ and solve respectively,

$$\begin{cases} (\lambda + \lambda_1 + 1)u - E(t, \cdot, D)u = \lambda v - f & \text{in } \Omega \\ B_j(t, \cdot, D)u = 0 & \text{on } \partial\Omega, \ j = 1, \dots, m, \end{cases} \tag{6.10}$$

$$\begin{cases} (\lambda_1+1)v - E(s,\cdot,D)v = -f & \text{in } \Omega, \\ B_j(s,\cdot,D)v = 0 & \text{on } \partial\Omega,\ j=1,\dots,m, \end{cases} \tag{6.11}$$

so that $u - v \in \bigcap_{q\in]n,\infty[} W^{2m,q}(\Omega)$ and solves the non-homogenous problem

$$\begin{cases} (\lambda+\lambda_1+1)(u-v) - E(t,\cdot,D)(u-v) = [E(t,\cdot,D) - E(s,\cdot,D)]v \quad \text{in } \Omega, \\ B_j(t,\cdot,D)(u-v) = \begin{cases} [B_j(s,\cdot,D) - B_j(t,\cdot,D)]v & \text{if } m_j > 0 \\ \qquad \text{on } \partial\Omega,\ j=1,\dots,m \\ 0 \quad \text{if } m_j = 0. \end{cases} \end{cases}$$

Hence, Propositon 6.2 yields (by extending the coefficients $b_{j\beta}(t,x)$ on $[0,T]\times\bar\Omega$)

$$\begin{aligned} &|\lambda+1|\,\|u-v\|_{C(\bar\Omega)} \le \\ &C_q|\lambda+1|^{n/2mq}\Big\{ \sup_{x_0\in\bar\Omega} \Big\|[(E(t,\cdot,D) - E(s,\cdot,D)]v\Big\|_{L^q(\Omega\cap B(x_0,K_q|\lambda-\lambda_1|^{-1/2m}))} \\ &+\sum_{j=j_0}^{m}\sum_{r=0}^{2m-m_j} |\lambda+1|^{1-(m_j+r)/2m} \\ &\cdot \sup_{x_0\in\bar\Omega} \|D^r[B_j(t,\cdot,D) - B_j(s,\cdot,D)]v\|_{L^q(\Omega\cap B(x_0,K_q|\lambda-\lambda_1|^{-1/2m}))} \Big\}, \end{aligned}$$

where j_0 is defined by (6.9). By an easy calculation, we find

$$\begin{aligned} &|\lambda+1|\,\|u-v\|_{C(\bar\Omega)} \le \\ &c_q|\lambda+1|^{n/2mq}\Big\{|t-s|^{\mu}|\lambda+1|^{-n/2mq}\|v\|_{C^{2m-1}(\bar\Omega)} + |t-s|^{\mu}\|D^{2m}v\|_{L^q(\Omega)} \\ &+\sum_{j=j_0}^{m}\sum_{r=0}^{2m-m_j} |\lambda+1|^{1-(m_j+r)/2m-n/2mq}|t-s|^{\mu_{jr}}\|v\|_{C^{2m-1}(\bar\Omega)} \\ &+\sum_{j=j_0}^{m} |t-s|^{\mu_{j0}}\|D^{2m}v\|_{L^q(\Omega)}\Big\}. \end{aligned}$$

On the other hand, by (6.11) and Proposition 6.1,

$$\|v\|_{C^{2m-1}(\bar\Omega)} + \|D^{2m}v\|_{L^q(\Omega)} \le c_q\|f\|_{C(\bar\Omega)},$$

and therefore, we deduce

$$\begin{aligned} &\|u-v\|_{C(\bar\Omega)} \le \\ &c_q\|f\|_{C(\bar\Omega)}\left\{|t-s|^{\mu}|\lambda+1|^{n/2mq-1} + \sum_{j=j_0}^{m}\sum_{r=0}^{2m-m_j}|t-s|^{\mu_{jr}}|\lambda+1|^{-(m_j+r)/2m}\right\}. \end{aligned}$$

Here, $q \in]n,\infty[$ is arbitrary. Hence, if we choose $\mu > 0$, $\mu_{jr} + (m_j+r)/2m > 1$ we get Hypothesis II, with the pairs (α_i, β_i) given by

$$(\mu,\epsilon),\ \ (\mu_{jr}, 1-(m_j+r)/2m)\ \ (r = 0,1,\dots,2m-m_j;\ j = j_0,\dots,m) \tag{6.13}$$

for any $\epsilon > 0$.

Remark 6.4(a). The above result is not optimal, since there is an (arbitrarily) small gap between the regularity of the coefficients of $E(t,x,D)$ and the maximal regularity we can obtain according to the results of [3] for the strict solution u of the parabolic problem (0.1), with $\{A(t)\}$ given by (6.8). Indeed, fix for simplicity $\mu \in]0, 1/2m]$ and $\mu_{jr} := 1-(mj+r)/2m+\mu$ $(r = 0,\ldots,2m-m_j,\ j = j_0,\ldots,m)$; then the best result we can get for u provided all assumptions of [3, Theorem 6.1] are fulfilled, is $\partial u/\partial t,\ E(\cdot,\cdot,D)u \in C^\delta([0,T], C(\bar\Omega))$ where δ is defined by (1.6); now, in this case, by (6.13), $\delta = \mu - \epsilon$ for an arbitrary $\epsilon > 0$. Thus, the coefficients are C^μ in t, but the time derivative of u is just $C^{\mu-\epsilon}$.

(b) This gap disappears if we choose $E := L^p(\Omega)$, $p \in]1,\infty[$, and

$$\begin{cases} D_{A(t)} := \left\{u \in W^{2m,p}(\Omega) : B_j(t,\cdot,D)u = 0 \text{ on } \partial\Omega,\ \ j = 1,\ldots,m\right\}, \\ A(t) := E(t,\cdot,D)u - (\lambda_0+1)u; \end{cases} \tag{6.14}$$

in this case, by repeating the proof of Theorem 6.3, we obtain Hypotheses I and II, with the pairs (α_i, β_i) given by

$$(\mu, 0),\ \ (\mu_{jr}, 1-(mj+r)/2m)\ \ (r = 0,\ldots,2m-m_j;\ j = j_0,\ldots,m).$$

Thus, choosing μ and μ_{jr} as in (a), we get now $\delta = \mu$, so that the regularity result is optimal.

Remark 6.5. Theorem 6.3 applies also to elliptic systems satisfying the assumptions of [10, Section 5], provided the coefficients of all zero-order boundary operators are independent of t. This restriction is indispensible; consider, for example, the system

$$E(D) := \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & 1+\frac{d^2}{dx^2} \end{pmatrix}$$

with the boundary conditions

$$B(t,D) := \begin{pmatrix} 1 & 1+t \\ 0 & \frac{d}{dx} \end{pmatrix}$$

in the interval $]0,\pi/2[$. If we set $E = [C([0,\pi/2])]^2$ and

$$\begin{cases} D_{A(t)} := \Big\{u \in \left[C^2([0,\pi/2])\right]^2 : u_1(0)+(1+t)u_2(0) = u_1(\pi/2)+(1+t)u_2(\pi/2) \\ \qquad\qquad = u_2'(0) = u_2'(\pi/2) = 0\Big\} \\ A(t)u := (u_1'', u_2''+u_2), \end{cases}$$

then it is easy to see that $\{A(t)\}$ fulfills Hypothesis I and in addition we have

$$t \to R(\lambda, A(t)) \in C^1([0,T], \mathcal{L}(E)) \qquad \forall\,\lambda \in S_{\theta_0},\ \forall\, T > 0,$$

where θ_0 is any number in $]\pi/2,\pi[$, and

$$\overline{D_{A(t)}} = \left\{u \in [C([0,\pi/2])]^2 : u_1(0)+(1+t)u_2(0) = u_1(\pi/2)+(1+t)u_2(\pi/2) = 0\right\}$$

(actually, it can be shown that $\{A(t)\}$ satisfies the assumptions of [1]).

Now, by [3, Lemma 7.10], the range of the operator $\frac{d}{dt}A(t)^{-1}$ must be contained in $\overline{D_{A(t)}}$ for each t; but, on the other hand, choosing $f(x) = (1, 4\sin x\cos^2 x)$, we check that

$$[A(t)^{-1}f](x) =$$
$$\left(-\frac{x^2}{2} + (\frac{\pi}{4} + \frac{1}{2} - \frac{2}{\pi})x - \frac{\pi}{4} - t[(\frac{2}{\pi} - \frac{1}{2})x + \frac{\pi}{4}], \sin x + \cos x[\frac{\pi}{4} - \frac{1}{2}(x + \sin x\cos x)]\right)$$

and hence,

$$\left[\frac{d}{dt}A(t)^{-1}f\right](x) = \left(-(\frac{2}{\pi} - \frac{1}{2})x + \frac{\pi}{4}, 0\right).$$

Since this function does not belong to $\overline{D_{A(t)}}$, Hypothesis II cannot hold.

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