Existence and Maximal Time Regularity for Linear Parabolic Integrodifferential Equations

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The linear equation $v'(t)-A(t)v(t)+\int_0^t B(t,s)v(s)\,ds=f(t)$, $t\in[0,T]$, with the initial datum $u(0)=x$, in a Banach space E is considered: here $\{A(t)\}$ is a family of generators of analytic semigroups in E with domains $D_{A(t)}$ which may vary with t and be not dense in E, and $\{B(t,s)\}$ is a family of closed linear operators with $D_{B(t,s)}\supseteq D_{A(s)}$. Sharp existence and Hölder regularity results, as well as a representation formula, are obtained by a perturbation argument for strict solutions $v\in C^1([0,T],E)$, provided f is Hölder continuous, $x\in D_{A(0)}$, and suitable compatibility conditions involving x and $f(0)$ hold; similar results are also proved for classical solutions $v\in C^1(]0,T],E)$ when $x\in\overline{D_{A(0)}}$ and f is Hölder continuous in any closed subinterval of $]0,T]$, with a singularity at $t=0$.

0. Introduction

This paper is concerned with continuously differentiable solutions of the linear integrodifferential Cauchy problem

$$v'(t)-A(t)v(t)-\int_0^t B(t,s)v(s)\,ds=f(t),\qquad t\in[0,T],$$
$$v(0)=x \tag{0.1}$$

in a Banach space E; here $x\in E$, $f:[0,T]\to E$ is a continuous function, and $\{A(t)\}_{t\in[0,T]}$, $\{B(t,s)\}_{0\leqslant s<t\leqslant T}$ are two families of closed linear operators in E. We consider here the parabolic case of (0.1), i.e., we suppose that for each $t\in[0,T]$, $A(t)$ is the infinitesimal generator of an analytic semigroup $\{e^{\xi A(t)}\}_{\xi\geqslant 0}$; as the domains $D_A(t)$ are possibly not dense in E, these semigroups need not be strongly continuous at $\xi=0$. About $\{B(t,s)\}$ we require that the domains $D_{B(t,s)}$ contain $D_{A(s)}$ for $0\leqslant s<t\leqslant T$, and in addition we assume a Hölder condition on $t\mapsto B(t,s)A(s)^{-1}$ (not uniformly in s).

There are a lot of papers considering the problem (0.1) under different assumptions: we quote here just a few of them, referring to Acquistapace and Terreni [4] for a more detailed list of references. In Lunardi and

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Sinestrari [10] and in [4] the problem (0.1) is considered respectively in the constant-domain case (i.e. $D_{A(t)} = D_{A(0)}$ $\forall t \in [0,T]$) and in the variable-domain case; the method in both papers consists of a fixed-point argument, which is based on the sharp existence and regularity results known for the linear nonintegral Cauchy problem in the case of constant domains (Sinestrari [12], Acquistapace and Terreni [2]) as well as variable domains (Acquistapace and Terreni [1]). This method leads to very precise smoothness results and does not require the construction of the fundamental solution, or resolvent operator, of (0.1); this is the main difference with respect to other papers such as Friedman and Shinbrot [6], Tanabe [15], and Prüss [11]. In both [10] and [4] only strict solutions are treated, i.e. solutions $v(t)$ such that $v'(t)$ and $A(t)v(t)$ are continuous in the whole interval $[0,T]$, whereas in [6], [15], [11]—under stronger assumptions—classical solutions are also studied, i.e. continuous functions $v(t)$ such that $v'(t), A(t)v(t)$ are continuous in $]0,T]$ and the equation (0.1) holds in $]0,T]$.

In the present paper we deal with both strict and classical solutions (whose exact meaning is given in Definitions 1.4 and 1.5 below), proving existence, uniqueness, and maximal regularity: this means that u', $A(\cdot)u(\cdot)$ are as smooth as f is, provided some necessary and sufficient compatibility conditions on the data x, f hold. Concerning strict solutions, we find again, and improve slightly, the results of [10] and [4]; concerning classical solutions we generalize Prüss's definition in [11], proving existence and uniqueness in a larger class of functions, and allowing data f which are Hölder continuous in any closed subinterval of $]0,T]$, with a singularity at $t=0$. Our method consists, as in the mentioned papers, of considering the integral term of (0.1) as a perturbation of the linear nonintegral Cauchy problem; but, instead of looking for a fixed-point technique, we try to obtain a representation formula for the solution by solving a suitable integral-type equation. This is also the method of [11], where an operator-valued integral equation is solved in order to find the resolvent operator, whereas our equation is vector-valued and yields directly the strict or classical solution of (0.1), without passing through the construction of the resolvent operator; by a careful analysis of this formula we also get our maximal regularity results. A heuristic derivation of our representation formula can be found at the beginning of Section 3.

Let us describe now the subject of the next sections. In Section 1 we list our notation and hypotheses and prove some preliminary results; Section 2 contains a survey of the linear nonintegral Cauchy problem in the variable domain case; Section 3 is devoted to a series of technical lemmata as a prelude to our main theorems; in Section 4 we study the properties of strict and classical solutions of (0.1), again assuming variable domains; Section 5

concerns the case of constant domains; finally in Section 6 we give an example.

1. Notation, Assumptions, and Preliminaries

Let E be a Banach space and let $T>0$. We will use the following function spaces:

a. $C([0,T],E)$ and, for $1 \leqslant p \leqslant \infty$, $L^p(0,T,E)$;
b. for $0<\delta<1$, the Hölder spaces $C^\delta([0,T],E)$;
c. $C^1([0,T],E) = \{u \in C([0,T],E) : u' = du/dt \in C([0,T],E)\}$;
d. for $0<\delta<1$, $C^{1,\delta}([0,T],E) = \{u \in C^1([0,T],E) : u' \in C^\delta([0,T],E)\}$, which are Banach spaces with their usual norms;
e. for $\mu \geqslant 0$, $C_\mu(]0,T],E) = \{u :]0,T] \to E$ continuous: $t \mapsto t^\mu u(t) \in L^\infty(0,T,E)\}$, which is a Banach space with norm

$$\|u\|_{C_\mu(]0,T],E)} = \sup_{t\in]0,T]} t^\mu \|u(t)\|_E;$$

f. $C(]0,T],E) := \bigcap_{\epsilon\in]0,T]} C([\epsilon,T],E)$, and, for $0<\delta<1$, $C^\delta(]0,T],E)$, $C^1(]0,T],E)$, and $C^{1,\delta}(0,T,E)$, which are defined similarly.

If F is another Banach space, we denote by $\mathscr{L}(E,F)$ [or $\mathscr{L}(E)$ if $E=F$] the Banach space of bounded linear operators $E \to F$, with its usual norm.

If $A : D_A \subseteq E \to E$ is a linear operator, we denote by $\rho(A)$ its resolvent set and by $R(\lambda, A)$ its resolvent operator $(\lambda I - A)^{-1}$.

Let us list now our assumptions about the operators $\{A(t)\}$ and $\{B(t,s)\}$ mentioned in the introduction. Set

$$\Delta = \{(t,s) \in [0,T]^2 : s<t\}. \tag{1.1}$$

Hypothesis I. For each $t \in [0,T]$, $A(t) : D_{A(t)} \subseteq E \to E$ is a closed linear operator such that:

i. $\rho(A(t)) \supseteq S_{\theta_0} := \{\lambda \in \mathbb{C} : |\arg\lambda| \leqslant \theta_0\} \cup \{0\}$ for some $\theta_0 \in]\pi/2, \pi[$;
ii. $\|R(\lambda, A(t))\|_{\mathscr{L}(E)} \leqslant M/(1+|\lambda|)$ $\forall t \in [0,T]$, $\forall \lambda \in S_{\theta_0}$, for some $M>0$.

Hypothesis II. For each $\lambda \in S_{\theta_0}$, we have $t \mapsto R(\lambda, A(t)) \in C^1([0,T], \mathscr{L}(E))$ and

$$\left\| \frac{\partial}{\partial t} R(\lambda, A(t)) \right\|_{\mathscr{L}(E)} \leqslant \frac{N}{1+|\lambda|^\alpha} \qquad \forall t \in [0,T], \quad \forall \lambda \in S_{\theta_0},$$

for some $N>0$, $\alpha \in]0,1[$.

Hypothesis III.

$$\left\| \frac{d}{dt} A(t)^{-1} - \frac{d}{dr} A(r)^{-1} \right\|_{\mathscr{L}(E)} \leqslant L|t-r|^{\alpha} \qquad \forall t, r \in [0,T]$$

for some $L > 0$.

In particular, under the above assumptions, for each $t \in [0,T]$, $A(t)$ generates a bounded analytic semigroup $\{e^{\xi A(t)}\}_{\xi \geqslant 0}$ which is possibly not strongly continuous at $\xi = 0$; moreover $t \mapsto e^{\xi A(t)}$ is in $C^1([0,T], \mathscr{L}(E))$ for fixed $\xi > 0$, and by its usual representation via a Dunford integral, it is easily seen that

$$\left\| \frac{\partial}{\partial t} e^{\xi A(t)} \right\|_{\mathscr{L}(E)} \leqslant \text{const}\, \xi^{\alpha - 1} \qquad \forall \xi > 0, \quad \forall t \in [0,T]. \tag{1.2}$$

Hypothesis IV. For each $(t,s) \in \Delta$, $B(t,s) : D_{B(t,s)} \subseteq E \to E$ is a closed linear operator and $D_{B(t,s)} \supseteq D_{A(s)}$ $\forall (t,s) \in \Delta$.

Hypothesis V.

$$\| B(t,s) A(s)^{-1} \|_{\mathscr{L}(E)} \leqslant \frac{K}{(t-s)^{1-\beta}} \qquad \forall (t,s) \in \Delta$$

for some $K > 0$, $\beta \in]0,1[$.

Hypothesis VI. For each $\epsilon \in]0,\beta]$ there exists $H_\epsilon > 0$ such that

$$\| B(t,s) A(s)^{-1} - B(r,s) A(s)^{-1} \|_{\mathscr{L}(E)} \leqslant H_\epsilon \frac{(t-r)^{\epsilon}}{(r-s)^{1+\epsilon-\beta}} \qquad \forall (t,s),(r,s) \in \Delta$$

with $r \leqslant t$.

These assumptions will guarantee the existence and regularity of strict solutions of (0.1). In the study of classical solutions we will sometimes need in addition:

Hypothesis VII. For each $\epsilon \in]0,\beta]$ there exists $Q_\epsilon > 0$ such that

$$\| B(t,s) A(s)^{-1} - B(t,\sigma) A(\sigma)^{-1} \|_{\mathscr{L}(E)} \leqslant Q_\epsilon \frac{(s-\sigma)^{\epsilon}}{(t-s)^{1+\epsilon-\beta}} \qquad \forall (t,s),(t,\sigma) \in \Delta$$

with $\sigma \leqslant s$.

REMARK 1.1. Hypotheses I, II, and III assure the solvability of the nonintegral version of (0.1), i.e. the case $B(t,s) \equiv 0$ (see Kato and Tanabe [8], Acquistapace and Terreni [1]); however, Hypothesis III can be modified and weakened (Yagi [17,18]). Hypotheses IV, V, and VI are very close to, and slightly more restrictive than, the corresponding ones in [4], but they generalize those of [11] relative to $\{B(t,s)\}$, even when also Hypothesis VII is assumed. On the other hand, concerning $\{A(t)\}$, Hypothesis III is replaced in [11] by that of [17], which is independent of it.

Example 1.2. It is easily seen that if we choose $B(t,s) = (t-s)^{\beta-1} A(s)$ or $B(t,s) = [(\partial/\partial t) e^{\xi A(t)}]_{\xi = t-s} A(s)$, then Hypotheses IV–VII are fulfilled (compare with Lemma 2.3 below).

Definition 1.3. We set

$$C([0,T], D_A) = \{ u \in C([0,T],E) : u(t) \in D_{A(t)}\ \forall t \in [0,T], \text{ and } A(\cdot)u(\cdot) \in C([0,T],E) \}$$

(this space was denoted by $C([0,T], D_{A(\cdot)})$ in [4]). By Hypothesis I, it is a Banach space with norm

$$\|u\|_{C([0,T],D_A)} = \| A(\cdot) u(\cdot) \|_{C([0,T],E)}.$$

Similarly we define $C(]0,T], D_A)$ and, for $\mu \geqslant 0$, $C_\mu(]0,T], D_A)$. For the sake of simplicity, from now on we will simply write Au, $R(\lambda, A)u$ to mean $A(\cdot)u(\cdot)$, $R(\lambda, A(\cdot))u(\cdot)$.

We now define our solutions.

Definition 1.4. Let $f \in C([0,T],E)$, $x \in D_{A(0)}$. We say that $v : [0,T] \to E$ is a *strict solution* of (0.1) if:

i. $v \in C^1([0,T],E) \cap C([0,T],D_A)$;
ii. $v(0) = x$ and (0.1) holds in $[0,T]$.

Definition 1.5. Let $f \in C(]0,T],E)$, $x \in E$. We say that $v : [0,T] \to E$ is a *classical solution* of (0.1) if:

i. $v \in C([0,T],E) \cap C^1(]0,T],E) \cap C(]0,T],D_A)$;
ii. there exists the integral

$$\int_0^t B(t,s) v(s)\, ds := \lim_{\eta \to 0^+} \int_\eta^t B(t,s) v(s)\, ds = \lim_{\eta \to 0^+} \int_\eta^t \left[B(t,s) A(s)^{-1} \right] A(s) v(s)\, ds,$$

and $t \mapsto \int_0^t B(t,s) v(s)\, ds \in C(]0,T],E)$;
iii. $v(0) = x$, and (0.1) holds in $]0,T]$.

REMARK 1.6. Strict and classical solutions are called strong and strict, respectively, in [11]; we also note that our definition of classical solutions is more general than in [11], where such solutions are required to be in $C_1(]0,T], D_A)$.

We will frequently use certain interpolation spaces between $D_{A(t)}$ and E, which are defined as follows:

Definition 1.7. For each $t \in [0,T]$ and $\theta \in]0,1[$ we set

$$D_{A(t)}(\theta,\infty) = \left\{ x \in E : \sup_{\xi>0} \xi^{-\theta} \|(e^{\xi A(t)} - 1)x\|_E < \infty \right\};$$

it is a Banach space with norm

$$\|x\|_{D_{A(t)}(\theta,\infty)} = \|x\|_E + \sup_{\xi>0} \xi^{-\theta} \|(e^{\xi A(t)} - 1)x\|_E.$$

Obviously if $0 < \sigma \leqslant \theta < 1$

$$D_{A(t)} \subseteq D_{A(t)}(\theta,\infty) \subseteq D_{A(t)}(\sigma,\infty) \subseteq \overline{D_{A(t)}} \qquad \forall t \in [0,T].$$

When $\theta = 0$ or $\theta = 1$, Definition 1.7 would give $D_{A(t)}(0,\infty) = E$, $D_{A(t)}(1,\infty) \supseteq D_{A(t)}$ (without equality in general). However, we adopt the following convention:

Convention 1.8. For each $t \in [0,T]$ we set

$$D_{A(t)}(0,\infty) = \overline{D_{A(t)}}, \qquad D_{A(t)}(1,\infty) = D_{A(t)}.$$

We finish this section with a useful lemma:

Lemma 1.9. *Under Hypothesis I, for each* $\theta, \sigma \in [0,1]$ *we have*

$$\|e^{\xi A(t)}\|_{\mathscr{L}(D_{A(t)}(\theta,\infty), D_{A(t)}(\sigma,\infty))} \leqslant \text{const}\, \xi^{(\theta-\sigma)\wedge 0} \qquad \forall \xi > 0, \quad \forall t \in [0,T].$$

PROOF. See [12, Proposition 1.13 and 1.14]. □

2. The Nonintegral Problem

We collect in this section some propositions about the Cauchy problem

$$u'(t) - A(t)u(t) = f(t), \qquad t \in [0,T],$$
$$u(0) = x, \qquad (2.1)$$

where $x \in E$ and $f : [0,T] \to E$ are prescribed data and $\{A(t)\}_{t\in[0,T]}$ is a family of operators satisfying Hypotheses I, II, and III. Most of the results are proved in [1] and in [4, Appendix]. First of all, for any $\phi \in L^1(0,T,E)$

set

$$T\phi(t) = \int_0^t e^{(t-s)A(t)} \phi(s)\, ds, \qquad t \in [0,T], \qquad (2.2)$$

$$P\phi(t) = \int_0^t P(t,s)\phi(s)\, ds, \qquad t \in [0,T], \qquad (2.3)$$

where (compare with Example 1.2)

$$P(t,s) := \left[\frac{\partial}{\partial t} e^{\xi A(t)}\right]_{\xi = t-s}, \qquad (t,s) \in \Delta. \qquad (2.4)$$

We recall that by (1.2)

$$\|P(t,s)\|_{\mathscr{L}(E)} \leqslant \text{const}\,(t-s)^{\alpha-1} \qquad \forall (t,s) \in \Delta. \qquad (2.5)$$

Our first statement concerns strict solutions of (2.1).

Proposition 2.1. *Under Hypotheses I, II, III, fix* $\delta \in]0,\alpha]$ *and let* $x \in D_{A(0)}$, $f \in C^\delta([0,T],E)$. *The following assertions are true:*

i. *if a strict solution of* (2.1) *exists, then the vector*

$$y_0 := A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \qquad (2.6)$$

belongs to $\overline{D_{A(0)}}$;

ii. *if, conversely* $y_0 \in \overline{D_{A(0)}}$, *then there exists a unique strict solution* u *of* (2.1), *which is given by*

$$u(t) = e^{tA(t)}x + Tg(t), \qquad t \in [0,T], \qquad (2.7)$$

where $g = (1+P)^{-1}[f - P(\cdot,0)x]$, *i.e.,* g *is the unique solution of the integral equation*

$$g(t) + Pg(t) = f(t) - P(t,0)x, \qquad t \in [0,T]. \qquad (2.8)$$

Assume in addition $y_0 \in \overline{D_{A(0)}}$. *Then we have:*

iii. *there exists* $C_1 > 0$ *such that*

$$\|u'\|_{C([0,t],E)} + \|Au\|_{C([0,t],E)} \leqslant C_1\left\{\|A(0)x\|_E + \|f\|_{C^\delta([0,t],E)}\right\} \qquad \forall t \in]0,T]; \qquad (2.9)$$

iv. $u', Au \in C^\delta(]0,T],E)$;

v. $u', Au \in C^\delta([0,T],E)$ *if and only if* $y_0 \in D_{A(0)}(\delta,\infty)$;

vi. *if* $y_0 \in D_{A(0)}(\delta,\infty)$, *there exists* $C_2 > 0$ *such that*

$$\|u'\|_{C^\delta([0,t],E)} + \|Au\|_{C^\delta([0,t],E)} \leqslant C_2\left\{\|A(0)x\|_E + \|f\|_{C^\delta([0,t],E)} + \|y_0\|_{D_{A(0)}(\delta,\infty)}\right\} \qquad \forall t \in]0,T]. \qquad (2.10)$$

PROOF. Parts i, ii, iv, and v are proved in [1, Sections 2 and 5]. The proof of iii and vi in the case $t=T$ is in [4, Appendix]; the general case $t \in]0,T]$ follows in the same manner. □

Let us consider now classical solutions of (2.1).

Proposition 2.2. *Under Hypotheses I, II, III, fix $\delta \in]0,\alpha]$ and $x \in E$, $f \in L^1(0,T,E) \cap C^\delta(]0,T],E)$. The following assertions are true:*

i. *if a classical solution of (2.1) exists, then $x \in \overline{D_{A(0)}}$;*

ii. *if, conversely, $x \in \overline{D_{A(0)}}$, then there exists a unique classical solution u of (2.1), which again is given by (2.7) with $g=(1+P)^{-1}[f-P(\cdot,0)x]$.*

Assume in addition $x \in \overline{D_{A(0)}}$. Then we have:

iii. *if $x \in D_{A(0)}(\theta,\infty)$, $\theta \in [0,1]$, there exists $C_3(\theta)>0$ such that*

$$\|A(t)u(t)\|_E \leqslant C_3(\theta)\{t^{\theta-1}\|x\|_E + t^{-1}\|f\|_{L^1(0,t,E)} + \|f\|_{C([t/2,t],E)} + t^\delta [f]_{C^\delta([t/2,t],E)}\} \quad \forall t \in]0,T]; \tag{2.11}$$

iv. $u', Au \in C^\delta(]0,T],E)$.

PROOF. Parts ii and iv are proved in [1, Section 4] and [4, Appendix]. Let us prove i. By repeating the argument used in the proof of [1, Proposition 1.9], one finds that any classical solution u of (2.1) must satisfy

$$u(t) = e^{tA(0)}x + o(1) \quad \text{as} \quad t \to 0^+;$$

as $u \in C([0,T],E)$ and $u(0)=x$ by definition, it follows that

$$e^{tA(0)}x - x = o(1) \quad \text{as } t \to 0^+, \qquad \text{i.e.} \quad x \in \overline{D_{A(0)}}.$$

Finally we have to verify iii: in order to do this, we need some lemmata, in which Hypotheses I, II, III are always assumed.

Lemma 2.3. *For each $\epsilon \in]0,\alpha]$ there exists $C_4(\epsilon)>0$ such that*

$$\|P(t,s)-P(r,s)\|_{\mathscr{L}(E)} \leqslant C_4(\epsilon)\frac{(t-r)^\epsilon}{(r-s)^{1+\epsilon-\beta}} \qquad \forall (t,s),(r,s) \in \Delta$$

with $r \leqslant t$, where $P(t,s)$ is defined by (2.4).

PROOF. It is [1, Lemma 3.2]. □

Lemma 2.4. *Let P be the operator defined in (2.3). Then there exist C_5, $C_6>0$ such that:*

i. $\|P\phi\|_{L^1(0,t,E)} \leqslant C_5 t^\alpha \|\phi\|_{L^1(0,t,E)}$ $\|t \in]0,T]$,

ii. $\|P\phi(t)\|_E \leqslant C_6\{t^{\alpha-1}\|\phi\|_{L^1(0,t,E)} + t^\alpha\|\phi\|_{C([t/2,t],E)}\}$ $\forall t \in]0,T]$.

PROOF. i: By (2.5) we have

$$\int_0^t \|P\phi(r)\|_E dr \leqslant c\int_0^t\int_0^r (r-s)^{\alpha-1}\|\phi(s)\|_E ds\,dr = c\int_0^t\int_s^t (r-s)^{\alpha-1}dr\|\phi(s)\|_E ds \leqslant ct^\alpha\|\phi\|_{L^1(0,t,E)}.$$

ii: Similarly

$$\|P\phi(t)\|_\epsilon \leqslant c\left\{\int_0^{t/2} + \int_{t/2}^t\right\}(t-s)^{\alpha-1}\|\phi(s)\|_E ds \leqslant c\left(\frac{t}{2}\right)^{\alpha-1}\int_0^{t/2}\|\phi(s)\|_E ds + c\int_{t/2}^t (t-s)^{\alpha-1}ds\,\|\phi\|_{C([t/2,t],E)},$$

which implies the result. □

Lemma 2.5.

i. *For each $\epsilon \in]0,\alpha[$ there exists $C_7(\epsilon)>0$ such that*

$$\|P\phi\|_{C^\epsilon([t/2,t],E)} \leqslant C_7(\epsilon)\{t^{\alpha-\epsilon-1}\|\phi\|_{L^1(0,t,E)} + \|\phi\|_{C([t/4,t],E)}\} \quad \forall t \in]0,T].$$

ii. *There exists $C_8>0$ such that for each $\delta \in]0,\alpha]$ we have*

$$\|P\phi\|_{C^\alpha([t/2,t],E)} \leqslant C_8\left\{t^{-1}\|\phi\|_{L^1(0,t,E)} + \|\phi\|_{C([t/4,t],E)} + \frac{t^\delta}{\delta}[\phi]_{C^\alpha([t/4,t],E)}\right\} \quad \forall t \in]0,T].$$

PROOF. i: It follows by repeating, with a more precise specification of the constants, the estimate used in the proof of [1, Proposition 3.5(iv)].

ii: It follows by repeating the argument of the proof of [4, Lemma A.1(i)]. □

Lemma 2.6. *Let $g=(1+P)^{-1}\phi$ be the unique solution of the equation $g+Pg=\phi$.*

i. *There exists $C_9>0$ such that*

$$\|(1+P)^{-1}\phi\|_{L^1(0,t,E)} \leqslant C_9\|\phi\|_{L^1(0,t,E)} \qquad \forall t \in]0,T].$$

ii. *There exists* $C_{10} > 0$ *such that*

$$\left\|\left[(1+P)^{-1}\phi\right](t)\right\|_E \leqslant C_{10}\left\{t^{\alpha-1}\|\phi\|_{L^1(0,t,E)} + t^{\alpha}\|\phi\|_{C([t/2,t],E)}\right\} \qquad \forall t \in \left]0,T\right].$$

iii. *For each* $\epsilon \in \left]0,\alpha\right]$ *there exists* $C_{11}(\epsilon) > 0$ *such that*

$$\left\|(1+P)^{-1}\phi\right\|_{C^{\epsilon}([t/2,t],E)} \leqslant C_{11}(\epsilon)\left\{t^{\alpha-\epsilon-1}\|\phi\|_{L^1(0,t,E)} + \|\phi\|_{C^{\epsilon}([t/16,t],E)}\right\} \qquad \forall t \in \left]0,T\right].$$

PROOF. In the proof of [1, Proposition 3.6] it is shown that

$$\left[(1+P)^{-1}\phi\right](t) = \phi(t) + \int_0^t R(t,s)\phi(s)\,ds, \qquad t \in [0,T],$$

where $R(t,s)$ is a suitable kernel satisfying (2.5); hence i and ii follow as in the proof of Lemma 2.4. To prove iii, note that $g = (1+P)^{-1}\phi = -Pg + \phi$; hence the result follows easily by using Lemma 2.5i–ii and parts i–ii above. □

Lemma 2.7. *Let* $x \in E$. *Then:*

i. *there exists* $C_{12} > 0$ *such that*

$$\|P(t,0)x\|_E \leqslant C_{12} t^{\alpha-1}\|x\|_E \qquad \forall t \in \left]0,T\right];$$

ii. *there exists* $C_{13} > 0$ *such that*

$$\|P(\cdot,0)x\|_{L^1(0,t,E)} \leqslant C_{13} t^{\alpha}\|x\|_E \qquad \forall t \in \left]0,T\right];$$

iii. *for each* $\epsilon \in \left]0,\alpha\right]$ *there exists* $C_{14}(\epsilon) > 0$ *such that*

$$\|P(\cdot,0)x\|_{C^{\epsilon}([t/2,t],E)} \leqslant C_{14}(\epsilon) t^{\alpha-\epsilon-1}\|x\|_E \qquad \forall t \in \left]0,T\right].$$

If in addition $x \in D_{A(0)}(\theta,\infty)$, $\theta \in [\alpha,1]$, *then:*

iv. *there exists* $C_{15} > 0$ *such that*

$$\|P(t,0)x\|_E \leqslant C_{15} t^{\theta-1}\|x\|_{D_{A(0)}(\theta,\infty)} \qquad \forall \in \left]0,T\right];$$

v. *there exists* $C_{16} > 0$ *such that*

$$\|P(\cdot,0)x\|_{L^1(0,t,E)} \leqslant C_{16} t^{\theta}\|x\|_{D_{A(0)}(\theta,\infty)} \qquad \forall t \in \left]0,T\right];$$

vi. *for each* $\epsilon \in \left]0,\alpha\right]$ *there exists* $C_{17}(\epsilon) > 0$ *such that*

$$\|P(\cdot,0)x\|_{C^{\epsilon}([t/2,t],E)} \leqslant C_{17}(\epsilon) t^{\theta-\epsilon-1}\|x\|_{D_{A(0)}(\theta,\infty)} \qquad \forall t \in \left]0,T\right].$$

PROOF. Parts i and ii are quite easy; part iii follows directly by Lemma 2.3. Suppose now $x \in D_{A(0)}(\theta,\infty)$, $\theta \in [\alpha,1]$: then parts iv and v follow by the proof of [1, Proposition 3.3(ii)–(iii)].

Finally let us prove vi: We can write

$$P(t,0)x = \frac{1}{2\pi i}\int_\gamma e^{t\lambda}\frac{\partial}{\partial t}R(\lambda,A(t))x\,d\lambda, \qquad t \in \left]0,T\right],$$

where $\gamma = \gamma^- \cup \{0\} \cup \gamma^+$, $\gamma^{\pm} = \{\lambda \in \mathbb{C} : \arg\lambda = \pm\theta'\}$ with fixed $\theta' \in \left]\pi/2,\theta_0\right[$. Hence if $0 < t/2 \leqslant s \leqslant r \leqslant t \leqslant T$ we have

$$\begin{aligned}
&P(t,0)x - P(r,0)x \\
&= \frac{1}{2\pi i}\int_\gamma e^{r\lambda}\left[\frac{\partial}{\partial r}R(\lambda,A(r)) - \frac{\partial}{\partial s}R(\lambda,A(s))\right]x\,d\lambda \\
&\quad + \frac{1}{2\pi i}\int_\gamma\int_s^r \lambda e^{\lambda\sigma}\,d\sigma\frac{\partial}{\partial s}R(\lambda,A(s))x\,d\lambda \\
&= \frac{1}{2\pi i}\int_\gamma e^{r\lambda}\left\{\left[\frac{\partial}{\partial r}R(\lambda,A(r)) - \frac{\partial}{\partial s}R(\lambda,A(s))\right]\right. \\
&\qquad \times\left[(1-e^{rA(0)})x + \left[A(0)^{-1} - A(r)^{-1}\right]\right. \\
&\qquad\qquad \left.\times A(0)e^{rA(0)} + \frac{1}{\lambda}A(0)e^{rA(0)}x\right] \\
&\qquad + \frac{1}{\lambda}\left[\frac{d}{dr}A(r)^{-1} - \frac{d}{ds}A(s)^{-1}\right]A(0)e^{rA(0)}x \\
&\qquad - \left[R(\lambda,A(r)) - R(\lambda,A(s))\right]\frac{d}{dr}A(r)^{-1}A(0)e^{rA(0)}x \\
&\qquad - R(\lambda,A(s))\left[\frac{d}{dr}A(r)^{-1} - \frac{d}{ds}A(s)^{-1}\right]A(0)e^{rA(0)}x \\
&\qquad \left. + \frac{\partial}{\partial s}R(\lambda,A(s))\left[A(s)^{-1} - A(r)^{-1}\right]A(0)e^{rA(0)}x\right\}d\lambda \\
&\quad + \frac{1}{2\pi i}\int_s^r\int_\gamma \lambda e^{\lambda\sigma}\left\{\frac{\partial}{\partial s}R(\lambda,A(s))\right. \\
&\qquad \times\left[(1-e^{sA(0)})x + \left[A(0)^{-1} - A(s)^{-1}\right]\right. \\
&\qquad\qquad \left.\times A(0)e^{sA(0)}x + \frac{1}{\lambda}A(0)e^{sA(0)}x\right] \\
&\qquad + \frac{1}{\lambda}\frac{d}{ds}A(s)^{-1}A(0)e^{sA(0)}x \\
&\qquad \left. - R(\lambda,A(s))\frac{d}{ds}A(s)^{-1}A(0)e^{sA(0)}x\right\}d\lambda\,d\sigma.
\end{aligned}$$

Consequently we easily deduce by Lemma 1.9

$$\begin{aligned}
&\|P(r,0)x - P(s,0)x\|_E \\
&\quad \leqslant c\left\{ \frac{(r-s)^{\alpha}}{t^{1-\theta}} + \int_s^r \left[\frac{s^{\theta}}{\sigma^{2-\alpha}} + \frac{1}{\sigma^{1-\alpha} s^{1-\theta}} \right] d\sigma \right\} \|x\|_{D_{A(0)}(\theta,\infty)} \\
&\qquad + \left\| (e^{(r-s)A(s)} - 1) e^{sA(s)} \frac{d}{ds} A(s)^{-1} A(0) e^{sA(0)} x \right\|_E \\
&\quad \leqslant c\left\{ \frac{(r-s)^{\alpha}}{t^{1-\theta}} + \|e^{sA(s)}\|_{\mathscr{L}(E, D_{A(s)}(\epsilon,\infty))} \left\| \frac{d}{ds} A(s)^{-1} \right\|_{\mathscr{L}(E)} \right. \\
&\qquad\qquad\qquad \left. \times \frac{(r-s)^{\epsilon}}{t^{1-\theta}} \right\} \|x\|_{D_{A(0)}(\theta,\infty)} \\
&\quad \leqslant c(\epsilon) \frac{(r-s)^{\epsilon}}{t^{1+\epsilon-\theta}} \|x\|_{D_{A(0)}(\theta,\infty)},
\end{aligned}$$

and the proof is complete. □

Let us go back now to the proof of proposition 2.2(iii). By (2.7) it follows that the function Au is given by

$$\begin{aligned}
A(t)u(t) = {} & A(t)e^{tA(t)}x \\
& + \int_0^t A(t) e^{(t-s)A(t)} [g(s) - g(t)] \, ds + (e^{tA(t)} - 1) g(t), \\
& \qquad t \in \,]0, T],
\end{aligned}$$

where $g = (1+P)^{-1}(f - P(\cdot,0)x)$. Hence, by collecting the results of the above lemmata, one obtains in a standard way the desired estimate (2.11) with the interval $[t/16, t]$ in place of $[t/2, t]$ on the right-hand side: it is clear however that suitable modifications to the proofs of the lemmata would lead to the right estimate. The proof of Proposition 2.2 is now complete. □

REMARK 2.8. By Proposition 2.2(iii) it follows that for a classical solution of (2.1) we can have $u \in C_{\mu}(]0,T], D_A]$ for any $\mu \geqslant 0$. Indeed, pick $y \in E$, $p > 0$ and set

$$f(t) = y \sin t^{-p}, \qquad T \in \,]0, T];$$

then clearly $f \in L^{\infty}(0,T,E) \cap C^{\infty}(]0,T], E)$, and it is readily seen that

$$t^{\delta}[f]_{C^{\delta}([t/2,t],E)} \leqslant c(\delta, p) t^{-\delta p} \qquad \forall t \in \,]0, T];$$

hence if p is large enough, we get by (2.11) that Au can blow up at 0 with any power $\mu \geqslant 0$.

The preceding remark motivates the introduction of the (possibly infinite) number

$$[f]_{\mu,\delta,t} = \sup_{r \in]0,t]} \left\{ r^{\mu+\delta} [f]_{C^{\delta}([r/2,r],E)} + r^{\mu} \|f\|_{C([r/2,r],E)} \right\},$$
$$\mu \in [0,\infty[, \quad \delta \in \,]0,1[, \quad t \in \,]0,T]. \tag{2.12}$$

Then we have:

Corollary 2.9. *Under hypotheses I, II, III fix* $\delta \in \,]0,1[$, *let* $x \in \overline{D_{A(0)}}$, $f \in L^1(0,T,E) \cap C^{\delta}(]0,T],E)$, *and let u be the classical solution of* (2.1). *If* $\mu \geqslant 0$, *we have* $u \in C_{\mu}(]0,T], D_A)$ *provided*

$$x \in D_{A(0)}(1-\mu,\infty), \quad f \in L^{1/\mu}(0,T,E), \quad [f]_{\mu\delta,T} < \infty \quad \text{if} \quad \mu \in [0,1[,$$
$$[f]_{\mu,\delta,T} < \infty \qquad \text{if} \quad \mu \in [1,\infty[$$

(*when* $\mu = 0$, $L^{1/\mu}$ *stands for* L^{∞}). *In addition there exists* $C_{18}(\mu) > 0$ *such that*

$$\begin{aligned}
&\|Au\|_{C_{\mu}(]0,t],E)} \\
&\quad \leqslant C_{18}(\mu) \left\{ \|x\|_{D_{A(0)}(1-\mu,\infty)} \right. \\
&\qquad\qquad \left. + \|f\|_{L^{1 \vee (1/\mu)}(0,t,E)} + [f]_{\mu,\delta,t} \right\} \\
&\qquad\qquad\qquad \forall t \in \,]0,T].
\end{aligned} \tag{2.13}$$

PROOF. It is an easy consequence of Proposition 2.2(iii) and (2.12). □

3. Technicalities

We study now the integrodifferential problem (0.1): in order to find strict or classical solutions, we treat (0.1) as a perturbation of the nonintegral problem (2.1), and try to apply the results of Propositions 2.1 and 2.2. From now on we will denote by $v(t)$ any solution of (0.1) with assigned data x, f, and by $u(t)$ the corresponding solution of (2.1) with the same data.

In this section we prove some technical results which are the basis of the existence and regularity theorems for strict and classical solutions. Let us start with some heuristic considerations. Let v by any strict solution of (0.1); then v also solves

$$v'(t) - A(t)v(t) = f(t) + Sv(t), \qquad t \in [0,T],$$
$$v(0) = x,$$

where we have set

$$Sv(t)=\int_0^t B(t,s)v(s)\,ds, \qquad t\in[0,T]. \tag{3.1}$$

Now we can split $v=u+z$, where u is the strict solution of (2.1) and z is the strict solution of

$$z'(t)-A(t)z(t)=Sv(t), \qquad t\in[0,T],$$
$$z(0)=0;$$

hence, by (2.7) (with $f\equiv Sv$, $x=0$), we have $z(t)=Tg(t)$, with $g=(1+P)^{-1}Sv$, so that v must satisfy

$$v(t)=u(t)+\left[T(1+P)^{-1}Sv\right](t), \qquad T\in[0,T],$$

or

$$A(t)v(t)=A(t)u(t)+A(t)\left\{\left[T(1+P)^{-1}SA^{-1}\right]Av\right\}(t), \qquad t\in[0,T].$$

This is an equation of the form

$$Av=Au+RAv, \tag{3.2}$$

and it is of integral type, since the operator R is given by

$$\begin{aligned} R\phi(t)&=A(t)\left[T(1+P)^{-1}SA^{-1}\phi\right](t)\\ &=A(t)\int_0^t e^{(t-s)A(t)}\left[(1+P)^{-1}SA^{-1}\phi\right](s)\,ds, \qquad t\in[0,T]. \end{aligned} \tag{3.3}$$

If we can solve (3.2), then we get for v the representation formula

$$V=A^{-1}(1-R)^{-1}Au,$$

which, together with the formula (2.7) for u, represents v in terms of the data x,f. To give sense to the above argument, we need some technical lemmata which analyze the properties of the operators S and R defined by (3.1) and (3.3).

Lemma 3.1. *Under Hypotheses I, IV, V, VI let S be defined by* (3.1). *The following assertions are true*:

i. *if* $\theta\in[0,1[$ *and* $\phi\in C_\theta(]0,T],D_A)$, *then* $S\phi\in L^1(0,T,E)$ *and there exists* $C_{19}>0$ *such that*
$$\|S\phi\|_{L^1(0,t,E)}\leqslant C_{19}t^{1-\theta+\beta}\|\phi\|_{C_\theta([0,t],D_A)} \qquad \forall t\in\,]0,T];$$

ii. *if* $\theta\in[\beta,1[$ *and* $\phi\in C_\theta(]0,T],D_A)$, *then* $S\phi\in C_{\theta-\beta}(]0,T],E)$ *and there exists* $C_{20}>0$ *such that*
$$\|S\phi\|_{C_{\theta-\beta}(]0,t],E)}\leqslant C_{20}\|\phi\|_{C_\theta(]0,t],D_A)} \qquad \forall t\in\,]0,T];$$

iii. *if* $\theta\in\,]0,\beta[$ *and* $\phi\in C_\theta(]0,T],D_A)$, *then* $S\phi\in C^{\beta-\theta}([0,T],E)$, $S\phi(0)=0$, *and there exists* $C_{21}>0$ *such that*
$$[S\phi]_{C^{\beta-\theta}([0,t],E)}\leqslant C_{21}\|\phi\|_{C_\theta(]0,t],D_A)} \qquad \forall t\in\,]0,T];$$

iv. *if* $\phi\in C_0(]0,T],D_A)$, *then for each* $\epsilon\in\,]0,\beta[$ *we have* $S\phi\in C^\epsilon([0,T],E)$, $S\phi(0)=0$, *and there exists* $C_{22}(\epsilon)>0$ *such that*
$$\|S\phi\|_{C^\epsilon([0,t],E)}\leqslant C_{22}(\epsilon)\|\phi\|_{C_0(]0,t],D_A)} \qquad \forall t\in\,]0,T];$$

v. *if* $\theta\in[0,1[$ *and* $\phi\in C_\theta(]0,T],D_A)$, *then for each* $\epsilon\in\,]0,\beta[$ *we have* $S\phi\in C^\epsilon(]0,T],E)$, *and there exists* $C_{23}(\epsilon)>0$ *such that*
$$t^{\theta-\beta}\|S\phi\|_{C([t/2,t],E)}+t^\theta[S\phi]_{C^\epsilon([t/2,t],E)}\leqslant C_{23}(\epsilon)\|\phi\|_{C_\theta(]0,T],D_A)} \qquad \forall t\in\,]0,T].$$

Assume in addition Hypothesis VII, let $\theta\in[1,\ 1+\beta[$, *and let* $\phi\in C_\theta(]0,T],D_A)$ *be such that*
$$\int_0^T A(s)\phi(s)\,ds:=\lim_{\eta\to0^+}\int_\eta^T A(s)\phi(s)\,ds \text{ exists in } E;$$

then we have:

vi. $S\phi(t):=\lim_{\eta\to0^+}\int_\eta^t B(t,s)\phi(s)\,ds$ *exists in* E $\forall t\in\,]0,T]$, *and there exists* $C_{24}>0$ *such that*
$$\|S\phi(t)\|_E\leqslant C_{24}\left\{t^{\beta-\theta}\|\phi\|_{C_\theta(]0,t],D_A)}+t^{\beta-1}\left\|\int_0^t A(s)\phi(s)\,ds\right\|_E\right\} \qquad \forall t\in\,]0,T];$$

vii. *for each* $\epsilon\in\,]0,\beta[$ *we have* $S\phi\in C^\epsilon(]0,T],E)$, *and there exists* $C_{25}(\epsilon)>0$ *such that*
$$[S\phi]_{C^\epsilon([t/2,t],E)}\leqslant C_{25}(\epsilon)\times\left\{t^{-\theta}\|\phi\|_{C_\theta(]0,t],D_A)}+t^{\beta-1}\left\|\int_0^t A(s)\phi(s)\,ds\right\|_E\right\} \qquad \forall t\in\,]0,T].$$

PROOF. i: Let $\theta\in[0,1]$. If $0<r\leqslant t\leqslant T$, we have

$$\begin{aligned}\|S\phi(r)\|_E&\leqslant\int_0^r\frac{c}{(r-s)^{1-\beta}}\frac{1}{s^\theta}\|\phi\|_{C_\theta(]0,t],D_A)}\\ &\leqslant cr^{\beta-\theta}\|\phi\|_{C_\theta(]0,t],D_A)},\end{aligned} \tag{3.4}$$

which implies (i).

ii: It follows by (3.4) provided $\theta \in [\beta,1[$.

iii–iv: Let $\theta \in [0,\beta[$. If $0<\sigma<r\leqslant t\leqslant T$, we have for each $\epsilon \in]0,\beta[$

$$\begin{aligned}\|S\phi(r)-S\phi(\sigma)\|_E &\leqslant \int_0^r \|B(r,s)A(s)^{-1}\|_{\mathscr{L}(E)}\|A(s)\phi(s)\|_E\,ds\\ &+\int_0^\sigma \|B(r,s)A(s)^{-1}-B(\sigma,s)A(s)^{-1}\|_{(E)}\|A(s)\phi(s)\|_E\,ds\\ &\leqslant c(\epsilon)\left\{\int_0^r \frac{ds}{(r-s)^{1-\beta}s^\theta}+(r-\sigma)^\epsilon\int_0^\sigma \frac{ds}{(\sigma-s)^{1+\epsilon-\beta}s^\theta}\right\}\|\phi\|_{C_\theta(]0,t],D_A)},\\ &\qquad \forall \epsilon\in]0,\beta[. \qquad (3.5)\end{aligned}$$

If $\theta=0$ we get (iv); otherwise, if $\theta\in]0,\beta[$, by choosing $\epsilon=\beta-\theta$ we get (iii).

v: Let $0<t/2\leqslant \sigma<r\leqslant t\leqslant T$. By (3.5) we obtain

$$\|S\phi(r)-S\phi(\sigma)\|_E\leqslant c(\epsilon)\{t^{-\theta}(r-\sigma)^\beta+(r-\sigma)^\epsilon t^{\beta-\epsilon-\theta}\}\|\phi\|_{C_\theta(]0,t],D_A)} \qquad \forall\epsilon\in]0,\beta[,$$

which, taking into account (3.4), implies (v).

vi: By using Hypothesis VII, we can write for $0<\eta<t\leqslant T$

$$\begin{aligned}\int_\eta^t B(t,s)\phi(s)\,ds &= \int_\eta^t [B(t,s)A(s)^{-1}-B(t,0)A(0)^{-1}]A(s)\phi(s)\,ds\\ &+B(t,0)A(0)^{-1}\int_\eta^t A(s)\phi(s)\,ds, \qquad (3.6)\end{aligned}$$

so that $S\phi(t)=\lim_{\eta\to 0^+}\int_\eta^t B(t,s)\phi(s)\,ds$ exists. In addition, by (3.6) we easily get

$$\|S\phi(t)\|_E\leqslant c(\epsilon)\left\{\int_0^t \frac{ds}{(t-s)^{1+\epsilon-\beta}s^{\theta-\epsilon}}\|\phi\|_{C_\theta(]0,t],D_A)} + t^{\beta-1}\left\|\int_0^t A(s)\phi(s)\,ds\right\|_E\right\},$$

which yields (vi).

vii: If $0<t/2\leqslant\sigma<r\leqslant t\leqslant T$, instead of (3.5) we can write

$$\begin{aligned}\|S\phi(r)-S\phi(\sigma)\|_E &\leqslant \int_0^r\|B(r,s)A(s)^{-1}\|_{\mathscr{L}(E)}\|A(s)\phi(s)\|_E\,ds\\ &+\int_0^\sigma\|B(r,s)A(s)^{-1}-B(\sigma,s)A(s)^{-1}\\ &\qquad -B(r,0)A(0)^{-1}+B(\sigma,0)A(0)^{-1}\|_{\mathscr{L}(E)}\\ &\qquad\times\|A(s)\phi(s)\|_E\,ds\\ &+\|B(r,0)A(0)^{-1}-B(\sigma,0)A(0)^{-1}\|_{\mathscr{L}(E)}\\ &\qquad\times\left\|\int_0^\sigma A(s)\phi(s)\,ds\right\|_E. \qquad (3.7)\end{aligned}$$

Now fix $\epsilon\in]0,\beta[$, pick $\gamma\in]\epsilon,\beta[$, and set $\rho=\epsilon/\gamma$. As

$$\begin{aligned}&\|B(r,s)A(s)^{-1}-B(\sigma,s)A(s)^{-1}-B(r,0)A(0)^{-1}+B(\sigma,0)A(0)^{-1}\|_{\mathscr{L}(E)}\\ &\qquad\leqslant c(\epsilon)\min\left\{\frac{(r-\sigma)^\gamma}{(\sigma-s)^{1+\gamma-\beta}},\frac{s^\gamma}{(\sigma-s)^{1+\gamma-\beta}}\right\},\end{aligned}$$

the second term on the right-hand side of (3.7) does not exceed

$$c(\epsilon)(r-\sigma)^{\gamma\rho}\int_0^\sigma\frac{s^{\gamma(1-\rho)}}{(\sigma-s)^{1+\gamma-\beta}}\,ds\cdot\|\phi\|_{C_\theta(]0,t],D_A)};$$

hence (3.7) implies

$$\begin{aligned}\|Su(r)-Su(\sigma)\|_E &\leqslant c(\epsilon)\{t^{-\theta}(r-\sigma)^\beta+t^{\beta-\theta-\epsilon}(r-\sigma)^\epsilon\}\|\phi\|_{C_\theta(]0,t],D_A)}\\ &+c(\epsilon)t^{\beta-\epsilon-1}(r-\sigma)^\epsilon\sup_{\sigma\in[t/2,t]}\left\|\int_0^\sigma A(s)\phi(s)\,ds\right\|_E,\end{aligned}$$

and the result follows, since

$$\left\|\int_0^\sigma A(s)\phi(s)\,ds\right\|_E\leqslant\left\|\int_0^t A(s)\phi(s)\,ds\right\|_E+ct^{1-\theta}\|\phi\|_{C_\theta(]0,t],D_A)} \qquad \forall\sigma\in[t/2,t]. \quad \square$$

Lemma 3.2. *Let* $\xi,\ \eta\in]0,1[$; *then*

$$\lim_{\omega\to+\infty} t^\eta\int_0^t\frac{e^{-\omega(t-s)}\,ds}{(t-s)^{1-\xi}s^\eta}=0 \qquad \textit{uniformly in}\quad t\in]0,T].$$

PROOF. Fix $\epsilon \in]0,T]$. If $t \in]0,\epsilon]$ we have

$$t^{\eta}\int_0^t \frac{e^{-\omega(t-s)}\,ds}{(t-s)^{1-\xi}s^{\eta}} = t^{\eta}\int_0^t \frac{e^{-\omega\sigma}\,d\sigma}{\sigma^{1-\xi}(t-\sigma)^{\eta}} \leqslant ct^{\xi} \leqslant c\epsilon^{\xi},$$

whereas if $t \in [\epsilon, T]$, by taking ω larger than a suitable ω_ϵ, we get

$$t^{\eta}\int_0^t \frac{e^{-\omega(t-s)}\,ds}{(t-s)^{1-\xi}s^{\eta}} \leqslant t^{\eta}e^{-\omega\epsilon/2}\int_0^{t-\epsilon/2}\frac{ds}{(t-s)^{1-\xi}s^{\eta}}$$

$$+\frac{t^{\eta}}{(t-\epsilon/2)^{\eta}}\int_{t-\epsilon/2}^t \frac{ds}{(t-s)^{1-\xi}}$$

$$\leqslant cT^{\xi}e^{-\omega\epsilon/2} + \frac{2^{\eta}}{\xi}\epsilon^{\xi} \leqslant c\epsilon^{\xi}.$$

□

Lemma 3.3. *Under Hypotheses I, II, III let $\theta \in [0,1[$, $\delta \in]0,\alpha]$. Denote by $g = (1+P)^{-1}\phi$ the unique solution of the equation $g + Pg = \phi$, with P defined by (2.3). If ϕ belongs to:*

i. $C_\theta(]0,T],E)$,
ii. $L^1(0,T,E) \cap C^\delta(]0,T],E)$,
iii. $C^\delta([0,T],E)$,

then g belongs to the same space.

PROOF. Parts ii and iii are proved in [1, Proposition 3.6] and [4, Lemma A.2]. Let us prove i. For each $\phi \in C_\theta(]0,T],E)$ define

$$\|\phi\|_{\theta,\omega,t} = \sup_{r\in]0,t]} r^{\theta}e^{-\omega r}\|\phi(r)\|_E, \qquad \omega > 0, \quad t \in]0,T]. \tag{3.8}$$

Clearly $\|\cdot\|_{\theta,\omega,t}$ and $\|\cdot\|_{C_\theta(]0,t],E)}$ are equivalent norms in $C_\theta(]0,t],E)$ (uniformly in $t \in]0,T]$). Moreover if $0 < r \leqslant t \leqslant T$,

$$r^{\theta}e^{-\omega r}\|P\phi(r)\|_E \leqslant cr^{\theta}\int_0^r \frac{e^{-\omega(r-s)}\,ds}{(r-s)^{1-\alpha}s^{\theta}}\|\phi\|_{\theta,\omega,t}, \tag{3.9}$$

so that

$$\|P\phi\|_{\theta,\omega,t} \leqslant c \sup_{r\in]0,t]}\left\{r^{\theta}\int_0^r \frac{e^{-\omega(r-s)}\,ds}{(r-s)^{1-\alpha}s^{\theta}}\right\}\|\phi\|_{\theta,\omega,t} \qquad \forall t \in]0,T].$$

By Lemma 3.2 we get for large ω

$$\|P\phi\|_{\theta,\omega,t} \leqslant 1/2\|\phi\|_{\theta,\omega,t} \qquad \forall t \in]0,T],$$

so that $(1+P)^{-1}$ exists in $\mathscr{L}(C_\theta(]0,t],E))$ $\forall t \in]0,T]$. □

Lemma 3.4. *Under Hypotheses I, II, III let T be defined by (2.2). The following assertions are true:*

i. *if $\theta \in]0,1[$, $\delta \in]0,\alpha]$, and $\phi \in C_\theta(]0,T],E) \cap C^\delta(]0,T],E)$, then $T\phi \in C^{1,\delta}(]0,T],E) \cap C^\delta(]0,T],D_A) \cap C^{1-\theta}([0,T],E)$ and $T\phi(0) = 0$;*
ii. *if $\phi \in C_0(]0,T],E)$, then $T\phi \in C^\gamma([0,T],E)$ $\forall\gamma \in]0,1[$;*
iii. *if $\delta \in]0,\alpha]$ and $\phi \in C^\delta([0,T],E)$ with $\phi(0) = 0$, then $T\phi \in C^{1,\delta}([0,T],E) \cap C^\delta([0,T],D_A)$ and $(T\phi)'(0) = T\phi(0) = 0$.*

PROOF. ii–iii: By [1, Proposition 3.7] and [4, Lemma A.3].
i: Again by [1, Proposition 3.7] and [4, Lemma A.3] we get $T\phi \in C^{1,\delta}(]0,T],E) \cap C^\delta(]0,T],D_A)$. The fact that $T\phi \in C^{1-\delta}([0,T],E)$ and $T\phi(0) = 0$ can be shown as in the proof of [1, Proposition 3.7(ii)]. □

Lemma 3.5. *Under Hypotheses I–VI let R be defined by (3.3). The following assertions are true:*

i. *if $\theta \in [\beta,1[$ and $\phi \in C_\theta(]0,T],E)$, then for each $\epsilon \in]0,\beta[$ we have $R\phi \in C_{\theta-\epsilon}(]0,T],E)$, and there exists $C_{26}(\epsilon) > 0$ such that*
$$\|R\phi(t)\|_E \leqslant C_{26}(\epsilon)t^{\epsilon-\theta}\|\phi\|_{C_\theta(]0,t],E)} \qquad \forall t \in]0,T];$$
ii. *if $\theta \in]0,\beta[$ and $\phi \in C_\theta(]0,T],E)$, then $R\phi \in C^{(\beta-\theta)\wedge\alpha}([0,T],E)$, $R\phi(0) = 0$, and there exists $C_{27} > 0$ such that*
$$\|R\phi\|_{C^{(\beta-\theta)\wedge\alpha}([0,t],E)} \leqslant C_{27}\|\phi\|_{C_\theta(]0,t],E)} \qquad \forall t \in]0,T];$$
iii. *if $\phi \in C_0(]0,t],E)$, then for each $\epsilon \in]0,\beta[$ $R\phi \in C^{\epsilon\wedge\alpha}([0,T],E)$ we have $R\phi(0) = 0$, and there exists $C_{28}(\epsilon) > 0$ such that*
$$\|R\phi\|_{C^{\epsilon\wedge\alpha}([0,t],E)} \leqslant C_{28}(\epsilon)\|\phi\|_{C_0(]0,t],E)} \qquad \forall t \in]0,T];$$
iv. *if $\theta \in [0,1[$ and $\phi \in C_\theta(]0,T],E)$, then for each $\epsilon \in]0,\beta[$ we have $R\phi \in C^{\epsilon\wedge\alpha}(]0,T],E)$.*

Assume in addition Hypothesis VII, let $\theta \in [1, 1+\beta[$, and let $\phi \in C_\theta(]0,T],E)$ be such that

$$\int_0^t \phi(s)\,ds := \lim_{\eta\to 0^+}\int_\eta^T \phi(s)\,ds \text{ exists in } E;$$

then we have:

v. *for each* $\epsilon \in]0, \beta[$ *we have* $R\phi \in C^{\epsilon\wedge\alpha}(]0,T], E)$, *and there exists* $C_{29}(\epsilon) > 0$ *such that*

$$\|R\phi(t)\|_E \leq C_{29}(\epsilon)\left\{t^{\epsilon-\theta}\|\phi\|_{C_\theta(]0,t],E)} + t^{\beta-1}\left\|\int_0^t \phi(s)\,ds\right\|_E\right\} \quad \forall t \in]0,T].$$

PROOF. Let $\theta \in [0,1[$ and $\phi \in C_\theta(]0,T],E)$. By Lemma 3.1, for each $\epsilon \in]0,\beta[$ we have

$$SA^{-1}\phi \in \begin{cases} \{f \in C^\epsilon([0,T],E) : f(0)=0\} & \text{if } \theta = 0, \\ C^\epsilon(]0,T],E) \cap \{f \in C^{\beta-\theta}([0,T],E) : f(0)=0\} & \text{if } \theta \in]0,\beta[, \\ C^\epsilon(]0,T],E) \cap C_{\theta-\beta}(]0,T],E) & \text{if } \theta \in [\beta,1[. \end{cases} \tag{3.10}$$

Hence by Lemma 3.3, $(1+P)^{-1}SA^{-1}\phi$ has the same properties with ϵ replaced by $\epsilon\wedge\alpha$. Thus by Lemma 3.4 we get for each $\epsilon \in]0,\beta[$

$$T(1+P)^{-1}SA^{-1}\phi \in \begin{cases} \{f \in C^{\epsilon\wedge\alpha}([0,T],D_A) : f(0)=0\} & \text{if } \theta=0, \\ C^{\epsilon\wedge\alpha}(]0,T],D_A) \cap \{f \in C^{(\beta-\theta)\wedge\alpha}([0,T],D_A) : f(0)=0\} & \text{if } \theta \in]0,\beta[, \\ C^{\epsilon\wedge\alpha}(]0,T],D_A) \cap \{f \in C^{1-\theta+\beta}([0,T],E) : f(0)=0\} & \text{if } \theta \in [\beta,1[. \end{cases} \tag{3.11}$$

This in particular implies iv. Next, by Proposition 2.2ii the function $u = T(1+P)^{-1}SA^{-1}\phi$ is the classical solution of

$$\begin{aligned} u'(t) - A(t)u(t) &= [SA^{-1}\phi](t), \quad T \in]0,T], \\ u(0) &= 0, \end{aligned} \tag{3.12}$$

so that by (2.11)

$$\begin{aligned} \|R\phi(t)\|_E &= \|A(t)u(t)\|_E \\ &\leq C(\epsilon)\{t^{-1}\|SA^{-1}\phi\|_{L^1(0,t,E)} \\ &\quad + \|SA^{-1}\phi\|_{C([t/2,t],E)} + t^\epsilon[SA^{-1}\phi]_{C^\epsilon([t/2,t],E)}\} \quad \forall t \in]0,T]. \end{aligned} \tag{3.13}$$

Hence by Lemma 3.1i–v we deduce that

$$\|R\phi(t)\|_E \leq C(\epsilon)t^{\epsilon-\theta}\|\phi\|_{C_\theta(]0,t],E)} \quad \forall t \in]0,T];$$

this proves i. In addition by Proposition 2.1ii, if $\theta \in [0,\beta[$ the solution u of (3.12) is strict, since $SA^{-1}\phi$ is Hölder continuous and $y_0 = 0$ in this case. Therefore Proposition 2.1v yields

$$u \in \begin{cases} C^{1,\epsilon\wedge\alpha}([0,T],E) \cap C^{\epsilon\wedge\alpha}([0,T],D_A) \quad \forall \epsilon \in]0,\beta[& \text{if } \theta=0, \\ C^{1,(\beta-\theta)\wedge\alpha}([0,T],E) \cap C^{(\beta-\theta)\wedge\alpha}([0,T],D_A) & \text{if } \theta \in]0,\beta[, \end{cases}$$

and by Proposition 2.1vi $Au = R\phi$ satisfies

$$\|R\phi\|_{C^{\epsilon\wedge\alpha}([0,t],E)} \leq C(\epsilon)\|SA^{-1}\phi\|_{C^\epsilon([0,t],E)} \quad \forall \epsilon \in]0,\beta[\quad \text{if } \theta=0,$$

$$\|R\phi\|_{C^{(\beta-\theta)\wedge\alpha}([0,t],E)} \leq C\|SA^{-1}\phi\|_{C^{\beta-\theta}([0,t],E)} \quad \text{if} \theta \in]0,\beta[.$$

This, together with (3.11), implies ii and iii.

Let us prove v. By Lemma 3.1vi–vii we have $\forall t \in]0,T]$

$$\|[SA^{-1}\phi](t)\|_E \leq c\left\{t^{\beta-\theta}\|\phi\|_{C_\theta(]0,t],E)} + t^{\beta-1}\left\|\int_0^t \phi(s)\,ds\right\|_E\right\},$$

$$[SA^{-1}\phi]_{C^\epsilon([t/2,t],E)} \leq c(\epsilon)\left\{t^{-\theta}\|\phi\|_{C_\theta(]0,t],E)} + t^{\beta-\epsilon-1}\left\|\int_0^t \phi(s)\,ds\right\|_E\right\} \quad \forall \epsilon \in]0,\beta[.$$

In particular $SA^{-1}\phi \in L^1(0,T,E) \cap C^\epsilon(]0,T],E)$ $\forall \epsilon \in]0,\beta[$, so that again $u = T(1+P)^{-1}SA^{-1}\phi$ is the classical solution of (3.12), and $Au = R\phi$ satisfies (3.13): consequently we get for each $\epsilon \in]0,\beta[$

$$\begin{aligned} \|R\phi(t)\|_E \leq c(\epsilon)\Bigg\{ & t^{\epsilon-\theta}\|\phi\|_{C_\theta(]0,t],E)} + t^{-1}\int_0^t s^{\beta-1}\left\|\int_0^s \phi(\sigma)\,d\sigma\right\|_E ds \\ & + t^{\beta-1}\left\|\int_0^t \phi(\sigma)\,d\sigma\right\|_E\Bigg\} \quad \forall t \in]0,T], \end{aligned}$$

which easily implies the result, since

$$\left\|\int_0^s \phi(\sigma)\,d\sigma\right\|_E \leq \left\|\int_0^t \phi(\sigma)\,d\sigma\right\|_E + cs^{1-\theta} \quad \forall s \in]0,t], \quad \forall t \in]0,T]. \quad \square$$

Lemma 3.6. *Under Hypotheses I–VI. Let $\theta \in [0,1[$. The following assertions are true.*

i. $(1-R)^{-1}$ *exists and belongs to* $\mathscr{L}(C_\theta(]0,T],E))$;

ii. *if* $\phi \in C_\theta(]0,T],E)$ *then there exists* $C_{30} > 0$ *such that*

$$\left\|(1-R)^{-1}\phi\right\|_{C_\theta(]0,t],E)} \leqslant C_{30}\|\phi\|_{C_\theta(]0,t],E)} \qquad \forall t \in]0,T].$$

PROOF. If $\phi \in C_\theta(]0,T],E)$, then by Lemma 3.5

$$R\phi \in \begin{cases} C^\epsilon([0,T],E) \quad \forall \epsilon \in]0,\beta[& \text{if } \theta = 0, \\ C^{\beta-\theta}([0,T],E) & \text{if } \theta \in]0,\beta[, \\ C_{\theta-\beta}(]0,T],E) & \text{if } \theta \in [\beta,1[, \end{cases}$$

so that in any case $R\phi \in C_\theta(]0,T],E)$; we want to show that if ω is sufficiently large,

$$\|R\phi\|_{\theta,\omega,t} \leqslant \tfrac{1}{2}\|\phi\|_{\theta,\omega,t} \qquad \forall t \in]0,T], \quad \forall \phi \in C_\theta(]0,T],E), \tag{3.14}$$

where $\|\cdot\|_{\theta,\omega,t}$ is defined in (3.8); this will imply both i and ii. Now if $0 \leqslant r \leqslant t \leqslant T$, we have by (3.13)

$$\begin{aligned} &r^\theta e^{-\omega r}\|R\phi(r)\|_E \\ &\quad \leqslant c(\epsilon) r^\theta e^{-\omega r}\left\{r^{-1}\|SA^{-1}\phi\|_{L^1(0,r,E)} + \|SA^{-1}\phi\|_{C([r/2,r],E)} \right. \\ &\qquad\qquad \left. + r^\epsilon [SA^{-1}\phi]_{C^\epsilon([r/2,r],E)}\right\} \\ &\quad = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}). \end{aligned}$$

By Lemma 3.2

$$\begin{aligned} (\mathrm{I}) &\leqslant c(\epsilon) r^{\theta-1}\int_0^r e^{-\omega(r-\sigma)}\int_0^\sigma \frac{e^{-\omega(\sigma-s)}\,ds}{(\sigma-s)^{1-\beta}s^\theta}\,d\sigma\,\|\phi\|_{\theta,\omega,t} \\ &\leqslant c(\epsilon) r^{\theta-1}\int_0^r \sigma^{-\theta}\left\{\sigma^\theta\int_0^\sigma \frac{e^{-\omega(\sigma-s)}\,ds}{(\sigma-s)^{1-\beta}s^\sigma}\right\} d\sigma\,\|\phi\|_{\theta,\omega,t} \\ &\leqslant o(1)\|\phi\|_{\theta,\omega,t} \qquad \text{as} \quad \omega \to \infty. \end{aligned}$$

In order to estimate (II) we have, again by Lemma 3.2,

$$\begin{aligned} (\mathrm{II}) &= c(\epsilon) r^\theta e^{-\omega r}\|SA^{-1}\phi\|_{C([r/2,r],E)} \\ &\leqslant c(\epsilon) 2^\theta \sup_{\theta \in [r/2,r]}\left\{\sigma^\theta\int_0^\sigma \frac{e^{-\omega(\sigma-s)}\,ds}{(\sigma-s)^{1-\beta}s^\theta}\right\}\|\phi\|_{\theta,\omega,t} \\ &\leqslant o(1)\|\phi\|_{\theta,\omega,t} \qquad \text{as} \quad \omega \to \infty. \end{aligned}$$

On the other hand concerning (III) note that if $r \in]0,t]$ and $r/2 \leqslant s \leqslant \sigma \leqslant r$ we have

$$\begin{aligned} &c(\epsilon) r^{\theta+\epsilon}\frac{\|[SA^{-1}\phi](\sigma) - [SA^{-1}\phi](s)\|_E}{(\sigma-s)^\epsilon} \\ &\quad \leqslant c(\epsilon) r^{\theta+\epsilon}\left\{\left[\int_s^\sigma \frac{e^{-\omega(\sigma-\rho)}\,d\rho}{(\sigma-\rho)^{1-\beta}\rho^\theta}\right]^{\epsilon/\beta + (1-\epsilon/\beta)} \right. \\ &\qquad\qquad \left. + e^{-\omega(\sigma-s)}(\sigma-s)^\epsilon\int_0^s \frac{e^{-\omega(s-\rho)}\,d\rho}{(s-\rho)^{1+\epsilon-\beta}\rho^\theta}\right\}\|\phi\|_{\theta,\omega,t} \\ &\quad \leqslant c(\epsilon) r^\epsilon\left\{r^\theta\left[\frac{(\sigma-s)^\beta}{(r/2)^\theta}\right]^{\epsilon/\beta}\left[\int_0^{\sigma-s}\frac{e^{-\omega\xi}\,d\xi}{\xi^{1-\beta}(r/2)^\theta}\right]^{1-\epsilon/\beta} \right. \\ &\qquad\qquad \left. + (\sigma-s)^\epsilon 2^\theta\left[s^\theta\int_0^s \frac{e^{-\omega(s-\rho)}\,d\rho}{(s-\rho)^{1+\epsilon-\beta}\rho^\theta}\right]\right\}\|\phi\|_{\theta,\omega,t} \\ &\quad \leqslant c(\epsilon)(\sigma-s)^\epsilon\{\omega^{\epsilon-\beta} + o(1)\}\|\phi\|_{\theta,\omega,t} \qquad \text{as} \quad \omega \to \infty. \end{aligned}$$

This implies

$$(\mathrm{III}) \leqslant o(1)\|\phi\|_{\theta,\omega,t} \qquad \text{as} \quad \omega \to \infty. \tag{3.18}$$

By (3.15), (3.16), (3.17), and (3.18) we get (3.14) and hence the result. □

REMARK 3.7. By Lemma 3.6 it follows also that $(1-R)^{-1} \in \mathscr{L}(C([0,t],E))$ and that there exists $C_{31} > 0$ such that

$$\left\|(1-R)^{-1}\phi\right\|_{C([0,t],E)} \leqslant C_{31}\|\phi\|_{C([0,t],E)} \qquad \forall \phi \in C([0,T],E), \ \forall t \in]0,T].$$

4. Strict and Classical Solutions

We are now ready to prove the main theorems of the present paper. We start with the existence result for strict solutions.

Theorem 4.1. *Under Hypotheses I–VI fix* $\delta \in]0,\alpha]$ *and let* $x \in D_{A(0)}$, $f \in C^\delta([0,T],E)$. *The following assertions are true:*

i. *if a strict solution of* (0.1) *exists, then the vector* y_0, *defined by* (2.6), *belongs to* $\overline{D_{A(0)}}$;

ii. *if, conversely,* $y_0 \in \overline{D_{A(0)}}$, *then there exists a unique strict solution* v *of* (0.1), *which is given by*

$$v(t) = A(t)^{-1}z(t), \qquad t \in [0,T], \tag{4.1}$$

where $z=(1-R)^{-1}Au$, *i.e.* z *is the unique solution of the equation*

$$z(t)-(Rz)(t)=A(t)u(t), \qquad t\in[0,T],$$

and u *is the strict solution of* (2.1).

Assume in addition $y_0\in\overline{D_{A(0)}}$. *Then we have:*

iii. *there exists* $C_{32}>0$ *such that*

$$\|v'\|_{C([0,t],E)}+\|Av\|_{C([0,t],E)}+\|Sv\|_{C([0,t],E)}$$
$$\leqslant C_{32}\{\|A(0)x\|_E+\|f\|_{C^\delta([0,t],E)}\} \qquad \forall t\in]0,T];$$

iv. $v', Av\in C^{\delta\wedge\epsilon}(]0,T],E)$ $\forall\epsilon\in]0,\beta[$;

v. *if* $\epsilon\in]0,\beta[$, *then* $v', Av\in C^{\delta\wedge\epsilon}([0,T],E)$ *if and only if* $y_0\in D_{A(0)}(\delta\wedge\epsilon,\infty)$;

vi. *if* $\epsilon\in]0,\beta[$ *and* $y_0\in D_{A(0)}(\delta\wedge\epsilon,\infty)$, *then there exists* $C_{33}(\epsilon)>0$ *such that*

$$\|v'\|_{C^{\delta\wedge\epsilon}([0,t],E)}+\|Av\|_{C^{\delta\wedge\epsilon}([0,t],E)}+\|Sv\|_{C^{\epsilon}([0,t],E)}$$
$$\leqslant C_{33}(\epsilon)\left\{\|A(0)x\|_E+\|f\|_{C^\delta([0,t],E)}+\|y_0\|_{D_{A(0)}(\delta\wedge\epsilon,\infty)}\right\}$$
$$\forall t\in]0,T].$$

PROOF. i: If v is a strict solution of (0.1), then Sv is Hölder continuous and $Sv(0)=0$ by Lemma 3.1iv; hence v is a strict solution of (2.1) with f replaced by $f+Sv$, and i follows by Proposition 2.1i.

ii: The heuristic argument at the beginning of Section 3 is now quite justified, so that if v solves (0.1) then necessarily v is given by (4.1) with $z=(1-R)^{-1}Au$, where u is the strict solution of (2.1) (which exists by Proposition 2.1ii). This in particular implies uniqueness. Conversely, let us show that the function (4.1) is in fact the strict solution of (0.1): indeed, as $Au\in C([0,T],E)$, by Remark 3.7 the same is true for z, so that $v=A^{-1}z\in C([0,T],D_A)$. Next, we have

$$v=A^{-1}z=A^{-1}[Rz+Au]=T(1+P)^{-1}SA^{-1}z+u; \tag{4.2}$$

as $SA^{-1}z\in C^\epsilon([0,T],E)$ $\forall\epsilon\in]0,\beta[$ and $SA^{-1}z(0)=0$ (Lemma 3.1iv), we easily deduce by Lemma 3.3iii and Lemma 3.4iii that $T(1+P)^{-1}SA^{-1}z\in C^{1,\delta\wedge\epsilon}([0,T],E)\cap C^{\delta\wedge\epsilon}([0,T],D_A)$ $\forall\epsilon\in]0,\beta[$ and $[T(1+P)^{-1}SA^{-1}z](0)=0$. Thus (4.2) yields $v\in C^1([0,T],E)$, $v(0)=u(0)=x$, and

$$v'=\left[T(1+P)^{-1}SA^{-1}z\right]'+u'=AT(1+P)^{-1}SA^{-1}z+SA^{-1}z-Au+f$$
$$=Av+SA^{-1}z+f=Av+Sv+f.$$

This proves ii.

iii: By Remark 3.7

$$\|Av\|_{C([0,t],E)}=\left\|(1-R)^{-1}Au\right\|_{C([0,t],E)}\leqslant C\|Au\|_{C([0,t],E)}$$
$$\forall t\in]0,T], \tag{4.3}$$

and iii follows by (2.9) and Lemma 3.1iv.

iv–v: We know from the proof of ii that $T(1+P)^{-1}SA^{-1}z\in C^{1,\delta\wedge\epsilon}([0,T],E)\cap C^{\delta\wedge\epsilon}([0,T],D_A)$ $\forall\epsilon\in]0,\beta[$; hence the results follow by (4.2) and Proposition 2.1iv–v.

vi: We have

$$\|Av\|_{C^{\delta\wedge\epsilon}([0,t],E)}=\|z\|_{C^{\delta\wedge\epsilon}([0,t],E)}\leqslant\|Rz\|_{C^{\delta\wedge\epsilon}([0,t],E)}+\|Au\|_{C^{\delta\wedge\epsilon}([0,t],E)}$$
$$\forall t\in]0,T];$$

on the other hand by Lemma 3.5iii and (4.3)

$$\|Rz\|_{C^{\delta\wedge\epsilon}([0,t],E)}\leqslant C(\epsilon)\|_{C([0,t],E)}\leqslant C(\epsilon)\|Au\|_{C([0,t],E)} \quad \forall t\in]0,T],$$

so that

$$\|Av\|_{C^{\delta\wedge\epsilon}([0,t],E)}\leqslant C(\epsilon)\|Au\|_{C^{\delta\wedge\epsilon}([0,t],E)} \qquad \forall t\in]0,T]. \tag{4.4}$$

Consequently, recalling Lemma 3.1iv,

$$\|v'\|_{C^{\delta\wedge\epsilon}([0,t],E)}\leqslant\|Av\|_{C^{\delta\wedge\epsilon}([0,t],E)}+\|Sv\|_{C^{\delta\wedge\epsilon}([0,t],E)}+\|f\|_{C^\delta([0,t],E)}$$
$$\leqslant C(\epsilon)\{\|Au\|_{C^{\delta\wedge\epsilon}([0,t],E)}+\|f\|_{C^\delta([0,t],E)}\} \qquad \forall t\in]0,T]. \tag{4.5}$$

The result follows by (4.5) and (2.10). □

REMARK 4.2. The results of Theorem 4.1, except for the representation formula (4.1), are essentially known [4, Theorems 2.1, 2.2]: there is some difference in parts iv–v, because of the slightly different assumptions on the smoothness of $B(t,s)$, as we already pointed out in Remark 1.1.

Let us study now classical solutions of (0.1). First we consider data x, f such that the classical solution u of the corresponding nonintegral problem (2.1) is in $C_\mu(]0,T],D_A)$ with $\mu\in[0,1[$.

Theorem 4.3. *Under Hypotheses I–VI, fix* $\mu\in[0,1[$, $\delta\in]0,\alpha]$, *let* $x\in D_{A(0)}(1-\mu,\infty)$, *and let* $f\in L^{1/\mu}(0,T,E)\cap C^\delta(]0,T],E)$ *be such that* $[f]_{\mu,\delta,T}<\infty$, *where* $[\cdot]_{\mu,\delta,T}$ *is defined in* (2.12) (*if* $\mu=0$, $L^{1/\mu}$ *stands for* L^∞). *The following assertions are true:*

i. *there exists a classical solution* v *of* (0.1), *which is given by* (4.1) *with* $z=(1-R)^{-1}Au$, *where* u *is the unique classical solution of* (2.1);

ii. *such a solution belongs to* $C_\mu(]0,T], D_A)$ *and is unique in this class*;
iii. *there exists* $C_{34}(\mu) > 0$ *such that*

$$\|v'\|_{C_\mu(]0,t],E)} + \|Av\|_{C_\mu(]0,t],E)} + \|Sv\|_{C_{(\mu-\beta)\vee 0}(]0,t],E)} \leqslant C_{34}(\mu)\left\{\|x\|_{D_{A(0)}(1-\mu,\infty)} + \|f\|_{L^{1/\mu}(0,t,E)} + [f]_{\mu,\delta,t}\right\} \quad \forall t \in]0,T];$$

iv. $v', Av \in C^{\delta\wedge\epsilon}(]0,T], E)\ \forall \epsilon \in]0,\beta[$.

PROOF. i–ii: if $v \in C_\mu(]0,T], D_A)$ is a classical solution of (0.1), then by Lemma 3.1i–v we have $Sv \in L^1(0,T,E) \cap C^\epsilon(]0,T],E)\ \forall \epsilon \in]0,\beta[$; hence v is a classical solution of (2.1) with f replaced by $f + Sv$, and again the argument used at the beginning of Section 3 applies. Thus v is given by (4.1) with $z = (1-R)^{-1}Au$; note that $Au \in C_\mu(]0,T],E)$ by Corollary 2.9, so that $z \in C_\mu(]0,T],E)$ by Lemma 3.6i. This in particular implies uniqueness in $C_\mu(]0,T], D_A)$. On the other hand, the function v given by (4.1) is in $C_\mu(]0,T], D_A)$ and is in fact a classical solution of (0.1): indeed, (4.2) still holds; in addition, by Lemma 3.1, $SA^{-1}z$ satisfies (3.10) with $\theta = \mu$, so that by Lemma 3.3 and Lemma 3.4 it is easily seen that $T(1+P)^{-1}SA^{-1}z$ satisfies (3.11) with $\theta = \mu$. Consequently i and ii follow by (4.2) as in the proof of Theorem 4.1ii.

iii: By Lemma 3.6ii

$$\|Av\|_{C_\mu(]0,t],E)} = \left\|(1-R)^{-1}Au\right\|_{C_\mu(]0,t],E)} \leqslant C\|Au\|_{C_\mu(]0,t],E)} \quad \forall t \in]0,T],$$

and iii follows by (2.11) and Lemma 3.1ii, iii, or iv.

iv: It also follows by (4.2) and the properties of $T(1+P)^{-1}SA^{-1}z$ and u, given by (3.11) (with $\phi = z$) and by Proposition 2.2iv. □

We consider now the most general case which can be treated by our method, i.e. data x, f such that the solution u of (2.1) is in $C_\mu(]0,T],E)$ with $\mu \in [1, 1+\beta[$; we need in this case the additional Hypothesis VII.

Theorem 4.4. *Under Hypotheses I–VI and VII, fix* $\mu \in [1, 1+\beta[$, $\delta \in]0,\alpha]$, *let* $x \in \overline{D_{A(0)}}$, *and let* $f \in L^1(0,T,E) \cap C^\delta(]0,T],E)$ *be such that* $[f]_{\mu,\delta,T} < \infty$, *where* $[\cdot]_{\mu,\delta,T}$ *is defined in* (2.12). *The following assertions are true*:

i. *there exists a classical solution* v *of* (0.1), *which is given by*

$$v(t) = u(t) + A(t)^{-1}w(t), \quad t \in]0,T] \tag{4.6}$$

where $w = (1-R)^{-1}RAu$, *i.e.*, w *is the unique solution of the equation*

$$w(t) - [Rw](t) = A(t)\left[T(1+P)^{-1}Su\right](t), \quad t \in]0,T],$$

and u *is the classical solution of* (2.1);

ii. *such a solution belongs to* $C_\mu(]0,T], D_A)$ *and is unique in this class*;
iii. *there exists* $C_{35}(\mu) > 0$ *such that*

$$\|v'\|_{C_\mu(]0,t],E)} + \|Av\|_{C_\mu(]0,t],E)} + \|Sv\|_{C_\mu(]0,t],E)} \leqslant C_{35}(\mu)\left\{\|x\|_E + \|f\|_{L^1(0,t,E)} + [f]_{\mu,\delta,t}\right\} \quad \forall t \in]0,T];$$

iv. $v', Av \in C^{\delta\wedge\epsilon}(]0,T], E)\ \forall \epsilon \in]0,\beta[$.

PROOF. i–ii: If $v \in C_\mu(]0,T], D_A)$ is a classical solution of (0.1), then by Lemma 3.1vi–vii $Sv \in L^1(0,T,E) \cap C^\epsilon(]0,T],E)\ \forall \epsilon \in]0,\beta[$; hence v is a classical solution of (2.1) with f replaced by $f + Sv$. Thus by Proposition 2.2ii

$$v(t) = e^{tA(t)}x + \left[T(1+P)^{-1}(f + Sv - P(\cdot,0)x)\right](t) = u(t) + \left[T(1+P)^{-1}Sv\right](t), \quad t \in [0,T],$$

which implies

$$Av = Au + RAv.$$

Now we cannot deduce (4.1), since by Corollary 2.9 $Au \in C_\mu(]0,T],E)$ with $\mu[1, 1+\beta[$, and $(1-R)^{-1}$ does not operate in this space. So we use a device of Prüss [11, (39)]: set

$$w = Av - Au; \tag{4.7}$$

then w solves

$$w = Rw + RAu. \tag{4.8}$$

We claim that the function RAu belongs to $C_\theta(]0,T],E)$ with a suitable $\theta \in]0,1[$. Indeed, since u solves (2.1), by Lemma 3.5v and (2.13) we have for each $\epsilon \in]0,\beta[$

$$\begin{aligned}\|[RAu](t)\|_E &\leqslant C(\epsilon)\left\{t^{\epsilon-\mu}\|Au\|_{C_\mu(]0,t],E)} + t^{\beta-1}\left\|\int_0^t [u'(s) - f(s)]\,ds\right\|_E\right\} \\ &\leqslant C(\epsilon)\left\{t^{\epsilon-\mu}\left[\|x\|_E + \|f\|_{L^1(0,t,E)} + [f]_{\mu,\delta,t}\right] + t^{\beta-1}\left[\|u(t)\|_E + \|x\|_E + \|f\|_{L^1(0,t,E)}\right]\right\} \quad \forall t \in]0,T].\end{aligned} \tag{4.9}$$

By [1, Theorem 2.1] we have

$$\|u(t)\|_E \leqslant C\{\|x\|_E + \|f\|_{L^1(0,t,E)}\} \quad \forall t \in]0,T], \tag{4.10}$$

and consequently

$$\|[RAu](t)\|_E \leqslant C(\epsilon)t^{\epsilon-\mu}\{\|x\|_E + \|f\|_{L^1(0,t,E)} + [f]_{\mu,\delta,t}\}$$
$$\forall\epsilon \in]0,\beta[, \quad \forall t \in]0,T].$$

Hence our claim is proved by choosing $\theta = \mu - \epsilon$ with any $\epsilon \in]\mu-1, \beta[$. Therefore by Lemma 3.5 we can solve (4.8) in $C_{\mu-\epsilon}(]0,T],E)$, i.e., there exists $w = (1-R)^{-1}RAu \in C_{\mu-\epsilon}(]0,T],E)$ $\forall\epsilon \in]\mu-1,\beta[$; in addition, by Proposition 2.2iv and Lemma 3.4iv, we have $w \in C^{\delta\wedge\epsilon}(]0,T],E)$ $\forall\epsilon \in]0,\beta[$.

Hence if $v \in C_\mu(]0,T],D_A)$ is a classical solution of (0.1), then v is given by (4.6); this in particular yields uniqueness in $C_\mu(]0,T],D_A)$. Conversely it is easy to verify that the function (4.6) is in fact a classical solution of (0.1): indeed we have

$$v = u + A^{-1}w = u + T(1+P)^{-1}SA^{-1}w + T(1+P)^{-1}Su,$$

so that by Lemmata 3.1, 3.3, and 3.5, $v(0) = x$, $v \in C^1(]0,T],E)$, and

$$v' = Au + f + AT(1+P)^{-1}SA^{-1}w + SA^{-1}w + AT(1+P)^{-1}Su + Su$$
$$= Av + f + SA^{-1}w + Su = Av + Sv + f.$$

Note that $Sv = Su + SA^{-1}w$ makes sense because of Lemma 3.1vi, i (since $w \in C_{\mu-\epsilon}(]0,T],E)$ $\forall\epsilon \in]\mu-1,\beta[$). As $u \in C_\mu(]0,T],D_A)$ and $w \in C_{\mu-\epsilon}(]0,T],E)$, we have $v \in C_\mu(]0,T],D_A)$. Parts i and ii are proved.

iii: We have by (2.13), Lemma 3.6ii, and (4.9)

$$\|Av\|_{C_\mu(]0,t],E)} \leqslant \|Au\|_{C_\mu(]0,t],E)} + \|w\|_{C_{\mu-\epsilon}(]0,t],E)}$$
$$\leqslant C\Big\{\|x\|_E + \|f\|_{L^1(0,t,E)} + [f]_{\mu,\delta,t} + \|RAu\|_{C_{\mu-\epsilon}(]0,t],E)}\Big\}$$
$$\leqslant C(\epsilon)\{\|x\|_E + \|f\|_{L^1(0,t,E)} + [f]_{\mu,\delta,t}\}$$
$$\forall\epsilon \in]\mu-1,\beta[, \quad \forall t \in]0,T].$$

Next

$$\|v'\|_{C_\mu(]0,t],E)} \leqslant \|Av\|_{C_\mu(]0,t],E)} + \|Sv\|_{C_\mu(]0,t],E)} + [f]_{\mu,\delta,t} \qquad \forall t \in]0,T];$$

on the other hand, by Lemma 3.1vi, i and (4.10),

$$\|Sv\|_{C_\mu(]0,t],E)} \leqslant \|Su\|_{C_{\mu-\beta}(]0,t],E)} + \|SA^{-1}w\|_{C_{(\mu-\epsilon-\beta)\vee 0}(]0,t],E)}$$
$$\leqslant C\Big\{\|Au\|_{C_\mu(]0,t],E)} + \Big\|\int_0^t [u'(s) - f(s)]\,ds\Big\|_E + \|w\|_{C_{\mu-\epsilon}(]0,t],E)}\Big\}$$
$$\leqslant C(\epsilon)\{\|x\|_E + \|f\|_{L^1(0,t,E)} + [f]_{\mu,\delta,t}\}$$
$$\forall\epsilon \in]\mu-1,\beta[, \quad \forall t \in]0,T],$$

and iii follows.

iv: It follows by the properties of u (Proposition 2.2iv), those of w established above, and those of $Sv = Su + SA^{-1}w$ (Lemma 3.1vii, v). □

REMARK 4.5. The representation formulas (4.6) and (4.1) formally coincide, since by (4.6) we have

$$v = u + A^{-1}(1-R)^{-1}RAu = A^{-1}(1-R)^{-1}[(1-R)Au + RAu]$$
$$= A^{-1}(1-R)^{-1}Au,$$

which is (4.1); on the other hand we have seen that (4.6) makes sense whenever $u \in C_\mu(]0,T],D_A)$ with $\mu < 1+\beta$, whereas (4.1) requires $u \in C_\mu(]0,T],D_A)$ with $\mu < 1$.

5. The Constant-Domain Case

In this section we study the problem (0.1) under the additional assumption that the domains $D_{A(t)}$ do not depend on t: this allows a considerable weakening of the smoothness requirements about $\{A(t)\}$. Namely, we replace Hypotheses II and III by:

Hypothesis II′. $D_{A(t)} = D_{A(0)}$ $\forall t \in [0,T]$.

Hypothesis III′. $\|1 - A(t)A(r)^{-1}\|_{\mathscr{L}(E)} \leqslant L|t-r|^\alpha$ $\forall t, r \in [0,T]$ for some $L > 0$, $\alpha \in]0,1[$.

These assumptions guarantee the solvability of (2.1): see Tanabe [14], Sobolevskii [13], and Acquistapace and Terreni [2].

The method for solving (0.1) is always the same: we treat (0.1) as a perturbation of (2.1) and use the sharp existence and regularity results proved in [2] for the constant-domain case of (2.1). In this way we will obtain results for (0.1) which are completely analogous, even from the formal point of view, to those of Theorems 4.1, 4.2, and 4.4.

The results for the nonintegral problem can be summarized as follows. Define for any $\phi \in L^1(0,T,E)$

$$T\phi(t) = \int_0^t e^{(t-s)A(s)}\phi(s)\,ds, \qquad t \in [0,T], \tag{5.1}$$

$$P\phi(t) = \int_0^t P(t,s)\phi(s)\,ds, \qquad t \in [0,T], \tag{5.2}$$

where

$$P(t,s) = \big[1 - A(t)A(s)^{-1}\big]A(s)e^{(t-s)A(s)}, \qquad (t,s) \in \Delta \tag{5.3}$$

[Δ is defined by (1.1)]. From now on these definitions of $T, P, P(t,s)$ replace (2.2), (2.3), and (2.4). Note that again $P(t,s)$ satisfies (2.5).

Proposition 5.1. *Under Hypotheses I, II′, and III′ let the operators $P(t,s), P, T$ be defined by* (5.3), (5.2), *and* (5.1); *then all assertions of Proposition* 2.1 *and* 2.2 *are true, provided in the former case we redefine the vector y_0 by*

$$y_0 := A(0)x + f(0). \tag{5.4}$$

From now on, the definition of y_0 by (5.4) *replaces* (2.6).

PROOF. Concerning the assertions of Proposition 2.1, parts i, ii, iv, and v are proved in [2, Theorems 4.2, 4.3, 4.7]; part iii is proved in [2, Theorem 4.4] only when $t = T$, but the general case can be shown quite similarly. Part v can be obtained by a more detailed review of the proofs of [2, Theorem 4.7 and Lemma 3.5(vi)]. Concerning the assertions of Proposition 2.2, parts i, ii and iv are proved in [2, Theorems 5.3 and 5.4] (the assumptions of f are slightly stronger there, but this is not really used in those proofs). A special case of part iii, namely when $x \in D_{A(0)}(\theta, \infty)$, $\theta \in]0,1[$, and $f \in C^{\delta}([0,T], E)$, and only when $t = T$, is proved in [2, Theorem 7.5]; that proof can be easily extended to any $t \in]0, T]$ and to general data x, f. □

Let us go back now to the problem (0.1). The heuristic argument at the beginning of Section 3 can be repeated once more, so that we are led to analyze the operators S, defined by (3.1), and

$$R = AT(1+P)^{-1}SA^{-1}, \tag{5.5}$$

where T and P are defined by (5.1) and (5.2). Clearly for S Lemma 3.1 holds unchanged, whereas about R we have first of all:

Lemma 5.2. *Under Hypotheses I, II′, III′ let the operators P, T be defined by* (5.2), (5.1). *Then all assertions of Lemmata* 3.3 *and* 3.4 *are true.*

PROOF. Parts i and ii of Lemma 3.3 are proved in [2, Lemma 3.4(ii)–(v)], whereas part ii is proved in [2, Lemma 3.4(iv)] under slightly stronger assumptions (but the proof still works under the present ones). Parts i, ii, and iii of Lemma 3.4 are proved in [2, Lemma 3.5]. □

Since Lemma 5.2 yields exactly the same conclusions as Lemmata 3.3 and 3.4, it is clear that the operator R defined by (5.5) satisfies the same properties as stated in Lemma 3.5; similarly $(1-R)^{-1}$ satisfies the properties stated in Lemma 3.6 and Remark 3.7.

Thus we can conclude with the following theorem which summarizes the properties of strict and classical solutions of (0.1) in the constant-domain case.

Theorem 5.3. *Under Hypotheses I, II′, III′, IV, V, VI let the operators P, T be defined by* (5.2), (5.3). *Then all assertions of Theorems* 4.1 *and* 4.3 *are true, provided in the former case we redefine the vector y_0 by* (5.4). *If in addition Hypothesis VII holds, then all assertions of Theorem* 4.4 *are true.*

□

PROOF. Exactly as in Section 4.

REMARK 5.4. The constant-domain version of the result of Theorem 4.1, except for the representation formula (4.1), was essentially known, although in a slightly less precise formulation, which however allows somewhat weaker assumptions about $\{B(t,s)\}$: see Lunardi and Sinestrari [10] and [4, Remark 2.3].

6. An Example

The following example has been considered (concerning only strict solutions) in [4, Section 5]. Fix $n \geqslant 2$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega$ of class C^3. Let $A(t,x,D), \Gamma(t,x,D), B(t,s,x,D)$ be differential operators respectively defined by

$$A(t,x,D) = \sum_{ij=1}^{n} a_{ij}(t,x) D_{x_i} D_{x_j} + \sum_{i=1}^{n} b_i(t,x) D_{x_i} + c(t,x) I, \qquad (t,x) \in [0,T] \times \bar{\Omega}, \tag{6.1}$$

$$\Gamma(t,x,D) = \sum_{i=1}^{n} B_i(t,x) D_{x_i} + \gamma(t,x) I, \qquad (t,x) \in [0,T] \times \partial\Omega, \tag{6.2}$$

$$B(t,s,x,D) = \sum_{i=1}^{n} p_i(t,s,x) D_{x_i} + q(t,s,x) I, \qquad (t,s,x) \in \Delta \times \bar{\Omega}, \tag{6.3}$$

where, as in (1.1), $\Delta = \{(t,s) \in [0,T]^2 : 0 \leqslant s < t \leqslant T\}$; we make the following assumptions:

$$a_{ij}, b_i, c, \frac{\partial a_{ij}}{\partial t}, \frac{\partial b_i}{\partial t}, \frac{\partial c}{\partial t} \in C([0,T] \times \bar{\Omega}, \mathbb{C}), \tag{6.4a}$$

$$a_{ij}(\cdot, x), b_i(\cdot, x), c(\cdot, x) \in C^{1,\alpha}([0,T], \mathbb{C}) \tag{6.4b}$$

with norms independent of x, where $\alpha \in]0,1[$;

$$\operatorname{Re} \sum_{ij=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geqslant N|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n, \quad \forall (t,x) \in [0,T]\times\overline{\Omega}, \tag{6.5}$$

where $N > 0$;

$$B_i, \gamma \in C^1([0,T]\times \partial\Omega, \mathbb{R}), \tag{6.6a}$$

$$B_i(t,\cdot), \gamma(t,\cdot), \frac{\partial B_i}{\partial t}(t,\cdot), \frac{\partial \gamma}{\partial t}(t,\cdot) \in C^2(\partial\Omega, \mathbb{R}) \tag{6.6b}$$

with norms independent of t,

$$B_i(\cdot,x), \gamma(\cdot,x) \in C^{1,\alpha}([0,T],\mathbb{R}) \tag{6.6c}$$

with norms independent of x;

$$\gamma(t,x) \geqslant 0, \quad \sum_{i=1}^{n} B_i(t,x)\nu_i(x) > 0 \qquad \forall (t,x) \in [0,T]\times \partial\Omega, \tag{6.7}$$

where $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the exterior unit normal vector at $x \in \partial\Omega$;

$$p_i, q : \Delta \times \overline{\Omega} \to \mathbb{C} \text{ are measurable functions,} \tag{6.8a}$$

$$p_i(t,s,\cdot), q(t,s,\cdot) \in C^1(\overline{\Omega}, \mathbb{C}), \tag{6.8b}$$

$$\sum_{i=1}^{n} |p_i(t,s,x)| + |q(t,s,x)| \leqslant K(t-s)^{\beta-1} \qquad \forall (t,s,x) \in \Delta \times \overline{\Omega}, \tag{6.9a}$$

$$\sum_{i=1}^{n} |p_i(t,s,x) - p_i(r,s,x)| + |q(t,s,x) - q(r,s,x)| \leqslant H_\epsilon \cdot \frac{(t-r)^\epsilon}{(r-s)^{1+\epsilon-\beta}}$$

$$\forall \epsilon \in]0,\beta], \quad \forall (t,s),(r,s) \in \Delta \text{ with } r \leqslant t, \tag{6.9b}$$

where $K, H_\epsilon > 0$, $\beta \in]0,1[$.

We consider the integrodifferential problem

$$\begin{aligned} u_t(t,x) - A(t,x,D)u(t,x) - \int_0^t B(t,s,x,D)u(s,x)\,ds = f(t,x), \\ (t,x) \in [0,T]\times\overline{\Omega}, \\ \Gamma(t,x,D)u(t,x) = 0, \qquad (t,x) \in [0,T]\times\partial\Omega \\ u(0,x) = \psi(x), \qquad x \in \overline{\Omega} \end{aligned} \tag{6.10}$$

with prescribed data f, ψ.

Set $E = C(\overline{\Omega})$, $\|u\|_E = \sup_{x\in\overline{\Omega}}|u(x)|$; define for each $t \in [0,T]$

$$\begin{aligned} D_{A(t)} = \Big\{ u \in \bigcap_{q \in]1,\infty[} H^{2,q}(\Omega) : \\ A(t,\cdot,D)u \in C(\overline{\Omega}),\ \Gamma(t,\cdot,D)u = 0 \text{ on } \partial\Omega \Big\}, \\ A(t)u = A(t,\cdot,D)u, \end{aligned} \tag{6.11}$$

where $H^{2,q}(\Omega)$ is the usual Sobolev space. Next, define for each $(t,s) \in \Delta$

$$D_{B(t,s)} = \{u \in C(\overline{\Omega}) : B(t,s,\cdot,D)u \in C(\overline{\Omega})\},$$

$$B(t,s)u = B(t,s,\cdot,D)u.$$

In [4, Proposition 3.4] it is shown that there exists $\omega > 0$ such that the operators $\{A(t) - \omega I\}_{t\in[0,T]}$ fulfil Hypotheses I, II, and III of Section 1; moreover it is easy to see that $\{B(t,s)\}_{(t,s)\in\Delta}$ and $\{A(t) - \omega I\}_{t\in[0,T]}$ also satisfy Hypotheses IV, V, and VI. When in addition Hypothesis VII is required, we will assume the following:

$$\sum_{i=1}^{n} |p_i(t,s,x) - p_i(t,\sigma,x)| + |q(t,s,x) - q(t,\sigma,x)| \leqslant Q_\epsilon \frac{(s-\sigma)^\epsilon}{(t-s)^{1+\epsilon-\beta}}$$

$$\forall \epsilon \in]0,\beta[, \quad \forall (t,s),(t,\sigma) \in \Delta \text{ with } \sigma \leqslant s. \tag{6.13}$$

By the results of Acquistapace and Terreni [3], we have $\forall t \in [0,T]$

$$\overline{D_{A(t)}} = \overline{D_{A(t)-\omega I}} = C(\overline{\Omega})$$

and for each $\delta \in]0,1[$

$$D_{A(t)}(\delta,\infty) = \begin{cases} C^{2\delta}(\overline{\Omega}) & \text{if } \delta \in]0,\tfrac{1}{2}[, \\ \{u \in C^{*,1}(\overline{\Omega}) : \\ \quad \sup\{\sigma^{-1}|u(x) - u(x - \sigma\beta(x))| : \\ \quad x \in \partial\Omega,\ \sigma > 0,\ x - \sigma\beta(x) \in \overline{\Omega}\} < \infty\} & \text{if } \delta = \tfrac{1}{2}, \\ \{u \in C^{1,2\delta-1}(\overline{\Omega}) : \\ \quad \sum_{i=1}^{n} \beta_i(t,x) D_{x_i}u(t,x) \\ \quad + \gamma(t,x)u(t,x) = 0 \quad \forall x \in \partial\Omega\} & \text{if } \delta \in]\tfrac{1}{2},1[, \end{cases} \tag{6.14}$$

where $\beta(x)$ is the vector $(\beta_1(x), \ldots, \beta_n(x))$ and the Zygmund class $C^{*,1}(\overline{\Omega})$ is defined by

$$C^{*,1}(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega}) : \sup\left\{ \frac{|u(x)+u(y)-2u((x+y)/2)|}{|x-y|} : x, y, \frac{x+y}{2} \in \overline{\Omega},\ x \neq y \right\} < \infty \right\};$$

it is a Banach space with its natural norm.

Let us apply the results of Section 4 to the problem (6.10): we will just consider strict solutions and the most general case of classical solutions, i.e. the case treated in Theorem 4.4.

Theorem 6.1. *Assume that* (6.1)–(6.9) *hold, and let* $\{A(t)\}$, $\{B(t,s)\}$ *be defined by* (6.11) *and* (6.12). *Let* $f \in C([0,T] \times \overline{\Omega})$ *and* $\psi \in \bigcap_{q \in]1,\infty[} H^{2,q}(\Omega)$ *be such that*

$$f(\cdot, x) \in C^{\delta}([0,T], \mathbb{C}) \qquad \text{with norm independent of } x,$$

$$A(0, \cdot, D)\psi \in C(\overline{\Omega}) \ \text{ and } \ \Gamma(0, \cdot, D)\psi = 0 \qquad \text{on } \partial\Omega,$$

where $\delta \in]0, \alpha]$. *Then*:

i. *the problem* (6.10) *has a unique strict solution* $v \in C^1([0,T] \times \overline{\Omega})$, *such that*

$$D_{x_i} v(t, \cdot) \in C^{\sigma}(\overline{\Omega}, \mathbb{C})$$

with norms independent of t, $\forall \sigma \in]0,1[$, $i = 1, \ldots, n$,

$$A(\cdot, \cdot, D)v \in C([0,T] \times \overline{\Omega});$$

ii.

$$\|v_t\|_{C([0,T]\times\overline{\Omega})} + \|A(\cdot,\cdot,D)\|_{C([0,T]\times\overline{\Omega})} + \left\| \int_0^t B(\cdot, s, \cdot, D) v(s, \cdot)\, ds \right\|_{C([0,T]\times\overline{\Omega})}$$

$$\leqslant C\left\{ \|f\|_{C([0,T]\times\overline{\Omega})} + \sup_{x\in\overline{\Omega}} \|f(\cdot,x)\|_{C^{\delta}([0,T])} + \|A(0,\cdot,D)\psi\|_{C(\overline{\Omega})} \right\};$$

iii. *the strict solution* v *satisfies in addition*

$$v_t(\cdot, x),\ D_{x_i} v(\cdot, x),\ A(\cdot, x, D)v(\cdot, x) \in C^{\delta\wedge\epsilon}(]0,T])$$

uniformly in x, $\forall \epsilon \in]0, \beta[$, $i = 1, \ldots, n$;

iv. *if* $\epsilon \in]0, \beta[$, *the strict solution* v *satisfies*

$$v_t(\cdot, x),\ D_{x_i} v(\cdot, x),\ A(\cdot, x, D)v(\cdot, x) \in C^{\delta\wedge\epsilon}([0,T])$$

uniformly in x, $i = 1, \ldots, n$, *if and only if the function* $g := f(0, \cdot) + A(0, \cdot, D)\psi$ *satisfies*

$$g \in C^{2(\delta\wedge\epsilon)}(\overline{\Omega}) \qquad \text{if} \quad \delta\wedge\epsilon \in]0, \tfrac{1}{2}[,$$

$$g \in C^{*,1}(\overline{\Omega}) \quad \text{and}$$

$$\sup\{\sigma^{-1}|g(x) - g(x - \sigma\beta(x))| : x \in \partial\Omega,\ \sigma > 0,\ x - \sigma\beta(x) \in \overline{\Omega}\} < \infty \qquad \text{if} \quad \delta\wedge\epsilon = \tfrac{1}{2},$$

$$g \in C^{1,2(\delta\wedge\epsilon)-1}(\overline{\Omega}) \quad \text{and}$$

$$\sum_{i=1}^n \beta_i(0,x) D_{x_i} g(x) + \gamma(0,x) g(x) + \sum_{i=1}^n \frac{\partial\beta_i}{\partial t}(0,x) D_{x_i}\psi(x) + \frac{\partial\gamma}{\partial t}(0,x)\psi(x) = 0 \quad \text{on } \partial\Omega$$

$$\text{if} \quad \delta\wedge\epsilon \in]\tfrac{1}{2}, 1[; \tag{6.15}$$

v. *if* $\epsilon \in]0, \beta[$ *and* $g = f(0,\cdot) + A(0,\cdot,D)\psi$ *satisfies* (6.15), *then*

$$\sup_{x\in\overline{\Omega}} \|v_t(\cdot,x)\|_{C^{\delta\wedge\epsilon}([0,T])} + \sup_{x\in\overline{\Omega}} \|A(\cdot,x,D)v(\cdot,x)\|_{C^{\delta\wedge\epsilon}([0,T])} + \sup_{x\in\overline{\Omega}} \left\| \int_0^t B(\cdot,s,x,D)v(s,x)\, ds \right\|_{C^{\delta\wedge\epsilon}([0,T])}$$

$$\leqslant \begin{cases} C(\epsilon)\{\|g\|_{C^{2(\delta\wedge\epsilon)}(\overline{\Omega})} + \|\psi\|_{H^{2,n/[1-2(\delta\wedge\epsilon)]}(\Omega)}\} \\ \qquad \text{if} \quad \delta\wedge\epsilon \in]0, \tfrac{1}{2}[, \\ C(\epsilon,\rho)\{\|g\|_{C^{*,1}(\overline{\Omega})} + \|\psi\|_{H^{2,n/(1-\rho)}(\Omega)}\} \quad \forall\rho \in]0,1[\\ \qquad \text{if} \quad \delta\wedge\epsilon = \tfrac{1}{2}, \\ C(\epsilon)\{\|g\|_{C^{1,2(\delta\wedge\epsilon)-1}(\overline{\Omega})} + \|\psi\|_{H^{2,n/[2-2(\delta\wedge\epsilon)]}(\Omega)}\} \\ \qquad \text{if} \quad \delta\wedge\epsilon \in]\tfrac{1}{2}, 1[. \end{cases}$$

PROOF. Parts i, ii, iv are proved in [4, Theorem 3.5]. Estimate ii is an easy consequence of the results of Section 4. Let us prove v: it is sufficient to estimate $\|A(0,\cdot,D) + f(0,\cdot) - w\|_{D_{A(0)}(\delta\wedge\epsilon,\infty)}$, where $w := [(d/dt)A(t)^{-1}]_{t=0} A(0)\psi$ solves (see [4, proof of Proposition 3.4])

$$A(0,\cdot,D)w = F := \sum_{ij=1}^n \frac{\partial a_{ij}}{\partial t}(0,\cdot) D_{x_i} D_{x_j}\psi + \sum_{i=1}^n \frac{\partial b}{\partial t}(0,\cdot) D_{x_i}\psi + \frac{\partial c}{\partial t}(0,\cdot)\psi \qquad \text{in } \overline{\Omega}$$

$$\times$$

$$\Gamma(0,\cdot,D)w = G := -\sum_{i=1}^n \beta_i(0,\cdot) D_{x_i}\psi - \gamma(0,\cdot)\psi \qquad \text{on } \partial\Omega.$$

Now

$$F \in \bigcap_{q \in]1,\infty[} H^{2,q}(\Omega), \qquad G \in \bigcap_{q \in]1,\infty[} H^{1-1/q,q}(\partial\Omega),$$

and by the well-known estimates of Agnon, Douglis, and Nirenberg [5] (see also Triebel [16, Section 5.5.2]) we have for each $q \in]1,\infty[$

$$\begin{aligned}\|w\|_{H^{2,q}(\Omega)} &\leqslant c(q)\{\|F\|_{L^q(\Omega)} + \|g\|_{H^{1-1/q,q}(\partial\Omega)} + \|w\|_{L^q(\Omega)}\} \\ &\leqslant c(q)\|\psi\|_{H^{2,q}(\Omega)}.\end{aligned}$$

Hence by (6.15) and Sobolev's theorem we easily get

$$\|w\|_{D_{A(0)}(\delta\wedge\epsilon,\infty)} \leqslant \begin{cases} C(\epsilon)\|\psi\|_{H^{2,n/[1-2(\delta\wedge\epsilon)]}(\Omega)} & \text{if} \quad \delta\wedge\epsilon \in]0,\frac{1}{2}[, \\ C(\epsilon,\rho)\|\psi\|_{H^{2,n/(1-\rho)}(\Omega)} \quad \forall\rho \in]0,1[& \text{if} \quad \delta\wedge\epsilon = \frac{1}{2}, \\ C(\epsilon)\|\psi\|_{H^{2,n/[2-2(\delta\wedge\epsilon)]}(\Omega)} & \text{if} \quad \delta\wedge\epsilon \in]\frac{1}{2},1[, \end{cases}$$

and the result follows. □

Theorem 6.2. *Assume that* (6.1)–(6.9) *and* (6.13) *hold, and let* $\{A(t)\}, \{B(t,s)\}$ *be defined by* (6.11) *and* (6.12); *let* $\psi \in C([0,T]\times\overline{\Omega})$ *and let* $f:]0,T]\times\overline{\Omega} \to \mathbb{C}$ *be such that*

$$f(t,\cdot) \in C(\overline{\Omega}) \quad \forall t \in]0,T], \qquad f(\cdot,x) \in C^\delta(]0,T]) \quad \textit{uniformly in } x,$$

$$[f]_{\mu,\delta,T} = \sup_{t\in]0,T]}\left\{t^\mu\left[\sup_{r\in[t/2,t]}\|f(r,\cdot)\|_{C(\overline{\Omega})} + t^\delta \sup_{t/2\leqslant\sigma<r\leqslant t}(r-\sigma)^{-\delta}\|f(r,\cdot)-f(\sigma,\cdot)\|_{C(\overline{\Omega})}\right]\right\} < \infty,$$

where $\delta \in]0,\alpha]$, $\mu \in [0,1+\beta[$. *Then*

i. *problem* (6.10) *has a classical solution* v *which satisfies*

$$\sup_{t\in]0,T]} t^\mu\|A(t,\cdot,D)v(t,\cdot)\|_{C(\overline{\Omega})} < \infty$$

and is unique in this class;

ii.

$$\begin{aligned} &t^\mu\|v_t(t,\cdot)\|_{C(\overline{\Omega})} + t^\mu\|A(t,\cdot,D)v(t,\cdot)\|_{C(\overline{\Omega})} \\ &\quad + t^\mu\left\|\int_0^t B(t,s,\cdot,D)v(s,\cdot)\,ds\right\|_{C(\overline{\Omega})} \\ &\leqslant c(\mu)\left\{\|\psi\|_{C(\overline{\Omega})} + \int_0^t\|f(r,\cdot)\|_{C(\overline{\Omega})}\,dr + [f]_{\mu,\delta,t}\right\} \qquad \forall t \in]0,T]; \end{aligned}$$

iii. $v_t(\cdot,x), D_{x_i}v(\cdot,x), A(\cdot,x,D)v(\cdot,x) \in C^{\delta\wedge\epsilon}(]0,T])$ *uniformly in* x $\forall\epsilon \in]0,\beta[$, $i=1,\ldots,n$.

PROOF. It is a straightforward consequence of the results of Section 4. □

REMARK 6.3. We can consider the problem (6.10) with Dirichlet boundary conditions, i.e. $\Gamma(t,x,D) \equiv I$. In this case we have (Lunardi [9])

$$\overline{D_{A(t)}} = \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\} \qquad \forall t \in [0,T]$$

and $\forall t \in [0,T]$

$$D_{A(t)}(\delta,\infty) = \begin{cases} \{u \in C^{2\delta}(\overline{\Omega}) : u|_{\partial\Omega} = 0\} & \text{if} \quad \delta \in]0,\frac{1}{2}[, \\ \{u \in C^{*,1}(\overline{\Omega}) : u|_{\partial\Omega} = 0\} & \text{if} \quad \delta = \frac{1}{2}, \\ \{u \in C^{1,2\delta-1}(\overline{\Omega}) : u|_{\partial\Omega} = 0\} & \text{if} \quad \delta \in]\frac{1}{2},1[. \end{cases}$$

REMARK 6.4. The problem (6.10) can be settled also in $E = L^p(\Omega)$, $1 < p < \infty$; then we can choose as $B(t,s,x,D)$ a second-order differential operator with coefficient of class C^2. In this case it is known (Grisvard [7], Triebel [16, Section 4.3.3]) that

$$\overline{D_{A(t)}} = L^p(\Omega) \qquad \forall t \in [0,T],$$

and, $\forall t \in [0,T]$,

$$D_{A(t)}(\delta,\infty) = \begin{cases} B_\infty^{2\delta,p}(\Omega) & \text{if} \quad \delta \in \left]0,\frac{1}{2}\left(1+\frac{1}{p}\right)\right[, \\ \{u \in B_\infty^{2\delta,p}(\Omega) : \Gamma(t,x,D)u(x) = 0 \; \forall x \in \partial\Omega\} & \text{if} \quad \delta \in \left]\frac{1}{2}\left(1+\frac{1}{p}\right),1\right[; \end{cases}$$

the Besov-Nikolsky spaces $B_\infty^{s,p}(\Omega)$ $(s > 0,\ p \in]1,\infty[)$ are defined by

$$B_\infty^{s,p}(\Omega) = \begin{cases} \left\{u|_\Omega : u \in L^p(\mathbb{R}^n),\ \sup_{t>0}\int_{\mathbb{R}^n}\left|t^{-s}[u(x+te^i)-u(x)]\right|^p dx < \infty,\ i=1,\ldots,n\right\} & \text{if} \quad s \in]0,1[, \\ \left\{u|_\Omega : u \in L^p(\mathbb{R}^n),\ \sup_{t>0}\int_{\mathbb{R}^n}\left|t^{-1}[u(x+te^i) + u(x-te^i)-2u(x)]\right|^p dx < \infty,\ i=1,\ldots,n\right\} & \text{if} \quad s = 1, \\ \{u|_\Omega : u \in H^{k,p}(\mathbb{R}^n) : D^\alpha u \in B_\infty^{\sigma,p}(\mathbb{R}^n),\ |\alpha| \leqslant k\} & \text{if} \quad s = k+\sigma > 1 \quad \text{with } k \in \mathbb{N},\ \sigma \in]0,1]. \end{cases}$$

We recall that when $s > 1/p$ any function in $B^{s,p}_{\infty}(\Omega)$ has a trace on $\partial\Omega$.

Similarly we can treat Dirichlet boundary conditions in $E = L^p(\Omega)$: in this case the domains $D_{A(t)}$ are constant, so that the theory of Section 5 applies. We have again [7], [16, Section 4.3.3] that

$$\overline{D_{A(t)}} = L^p(\Omega) \qquad \forall t \in [0,T],$$

$$D_{A(t)}(\delta,\infty) = \begin{cases} B^{2\delta,p}_{\infty}(\Omega) & \text{if} \quad \delta \in \left]0, \frac{1}{2p}\right[, \\ \{u \in B^{2\delta,p}_{\infty}(\Omega) : u|_{\partial\Omega} = 0\} & \text{if} \quad \delta \in \left]\frac{1}{2p}, 1\right[. \end{cases}$$

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