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Infinite horizon optimal control problems with non-compact control space. Existence results and dynamic programming

Tesi di Dottorato

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Introduction

In optimal control theory, infinite horizon problems may be difficult to treat especially if associated with large classes of admissible controls or with state constraints. Such problems arise naturally in economic applications, and - in some cases - crucial questions such as the existence of solutions to the problem (namely of optimal controls) are left aside due to the technical issues that their handling involves, despite a large literature testifying to the interest of the scientific community about the subject.

Two such cases are the so called Ramsey-Skiba type models and the Shallow Lake type models. The Ramsey-Skiba model dates back to Ramsey ([30], 1928) and is what is called in economy a utility maximization problem. The original formulation by Ramsey is now a classic in economic theory. In this formulation, the state equation has a dynamics that is globally concave as a function of space, and the objective functional to be maximized involves a concave increasing function of the control. In 1978, Skiba ([34]) proposed a convex-concave dynamics version of the Ramsey model, which has since also become quite popular in growth theory. The non-concavity assumption about the dynamics determines a significant difference not only because it increases the descriptive capacity of the model, but also because it raises technical challenges. On the lines of the work by Skiba, the analysis has been developed in papers such as [3], [29], and [18]. Nevertheless, a proper existence theorem had not been provided.

The archetypal Shallow Lake model was introduced in 2003 by Mäler et al. ([28]) in the context of environmental economics. It has developed its own literature with papers including [26], [36], [24], and [25]. The dynamics of this problem also has convex-concave behaviour, while the objective functional - representing the social benefit of a community resulting as a trade off between different interests - involves a function of the state and the control which is unbounded both from above and from below. A deep study of the dynamics of optimal paths has been carried out through the analysis of necessary conditions for optimality, both from the side of the domain (Pontryagin's Maximum Principle, see [36] and [24]) and from the side of the range (Hamilton-Jacobi-Bellman equation and Dynamic Programming, see [26]) of the objective functional; but, again, no results were known about the existence of an optimal control.

It is noted that both models are interested by the Skiba phenomenon, as discussed in [34] and [36]. Even though the treatment of the subject goes beyond the scope of this thesis, the fact is in itself remarkable, since the possibility of having indifference points is a widely discussed topic; a partial literature includes basic papers such as [33] and [14] (plus the fundamental paper of Sikba [34]), as well as the more recent papers [23], [9], [37], [11], and [10] (in addition to the already cited [36]).

The technical difficulties implied in the treatment of the existence problem relate both to the lack of compactness and to the infinite horizon setting. These features determine the absence of good *a priori* estimates which are usually the starting point of a proof based on the application of one or more compactness theorems. In the following we make a comparison between the proofs of some traditional existence results - namely the Filippov theorem for Mayer problems and the Filippov-Cesari theorem for Bolza problems - and the methods we have developed in order to face the issues that the above referred models present when addressing the problem of existence.

The already mentioned characteristics of these models add up to, in the case of the Ramsey-Skiba setting, a state constraint which produces further complications in many respects of any general treatment of the subject, beyond the question of the existence of *optima*. The technical implications of our specific set of assumptions (including the assumption about the non-concavity of the dynamics), as well as the other results obtained in the analysis of the Ramsey-Skiba model are discussed in more detail in the introduction of Chapter 1.

The following outline of the Filippov-Cesari theory provides some details that aim to highlight the intimate nature of this important construction and the profound differences with our approach and in the respective operational contexts.

The Filippov-Cesary theory concerns finite horizon problems. Compactness of the control space is assumed in the first and less powerful version of the theory; in the enhanced version of the theory, due to Cesari, such assumption is dropped in favour of a coercivity assumption on the objective functional which plays essentially the same role in the sense that it ensures that, roughly speaking, minimizing controls can be treated as if they took values in a compact space. A state equation of the form

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{1}$$

is considered, with the assumption that f satisfies

$$|f(t, x, u)| \leq 1 + |x| + |u|, \tag{2}$$

plus some Liptschitz-continuity hypothesis that guarantees the well posedness of (1) for any measurable function u.

It is assumed that the domain of the admissible controls is a finite interval of "times" that is not fixed, so that the objective functional will depend on a set and on a function defined on that set. This choice about the domains is made to include minimal-time problems in the setting.

Hence it is natural to expect that the objective functional depends separately on the control function and on the initial and final times and states. Actually, the first existence result, due to Filippov, has been provided for a Mayer functional of the type:

$$J(u) = \Phi\left(a_u, b_u, x_u^0, x_u^1\right), \tag{3}$$

defined for a function $u \in L^1([a_u, b_u]; U)$ such that $U \subseteq \mathbb{R}^M$ is compact and the vector of the corresponding terminal times and states $\left(a_u, b_u, x_u^0, x_u^1\right)$ remains in a fixed compact set $S \subseteq \mathbb{R}^{2N+2}$. The function $\Phi : S \to \mathbb{R}$ is assumed to be continuous. Denoting with $x(\cdot; a, y, u)$ the unique solution to (1) such that $x(a) = y$, we can write $x_u^1 = x\left(b_u; a_u, x_u^0, u\right)$.

Cesari's method extends the existence result to problems with a functional involving an integral term that depends separately on the control and the associated state. This method is powerful enough to substitute the compactness hypothesis on the control space U with a coercivity hypothesis on the integral part of the objective functional.

Namely, the objective functional to be minimized has the form:

$$J(u) = \int_{a_u}^{b_u} L(s, x(s), u(s))\,\mathrm{d}s + \Phi\left(a_u, b_u, x_u^0, x_u^1\right), \tag{4}$$

and is defined in the same class as before. It is assumed that L satisfies $L(t, x, u) \geq g(u)$ in its domain, for some continuous function $g : U \to \mathbb{R}$ such that $g(u) \cdot |u|^{-1} \to +\infty$ for $u \in U$, $u \to +\infty$. This is called a Bolza problem.

As already pointed out, this approach includes the possibility that the initial and final condition vary. The point is that the assumption that S is compact determines a fixed interval $[A, B]$ containing all the domains of the admissible controls. Actually this approach works with controls that are extended to $[A, B]$ by a constant and with the corresponding states; for the latter, some Gronwall-type *a priori* estimates hold.

These estimates become boundedness estimates thanks to this common temporal domain of finite measure (in other words, thanks to the finite horizon), and even uniform boundedness estimates when U is compact. If U is not compact the coercivity assumption on L is used to obtain the same uniform boundedness estimates for a minimizing sequence of states.

This is the starting point of the proof: given a minimizing sequence $\left(x_n^0, u_n\right)_{n\in\mathbb{N}}$ one considers the associated states $(x_n)_{n\in\mathbb{N}}$, and finds that $(x_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{AC}\left([A, B]; \mathbb{R}^N\right)$. As a consequence and with the same technique, it is proven that the minimizing states satisfy:

$$\sum_{i=1}^{k} |x_n(t_i) - x_n(s_i)| \quad \leq \quad C_\epsilon \sum_{i=1}^{k} |t_i - s_i| + \epsilon C \tag{5}$$

(for every $\epsilon > 0$ and $s_i, t_i \in [A, B]$), or even

$$|x_n(t) - x_n(s)| \quad \leq \quad C |t - s|$$

if U is compact. Anyway, these are uniform estimates in n allowing to apply the Ascoli-Arzelà theorem and obtain a subsequence converging, uniformly in $[A, B]$, to an absolutely continuous function x^*.

The second step consists in defining a control u^* companion to the candidate optimal state x^*, namely such that $\dot{x}^*(t) = f(t, x^*(t), u^*(t))$ for almost every $t \in [A, B]$. In other words, one has to prove that $\dot{x}^*(t)$ belongs to the image set $f(t, x^*(t), \cdot)(U)$ whenever the derivative exists, and that the preimage can be chosen in a measurable way with respect to t. The admissibility of the control u^* thus obtained is guaranteed by standard compactness or coercivity arguments.

The control u^* will be optimal for the Mayer problem, since the functional in (3) depends, continuously, only on the state. In the case of a functional of type (4), the construction is similar but more complicated from a technical viewpoint, since it is necessary to ensure that, besides the terminal term $\Phi(a_n, b_n, x_n^0, x_n^1)$, also the integral term $\int_{a_n}^{b_n} L(s, x_n(s), u_n(s))\, \mathrm{d}s$ converges to a quantity of the desired form.

Here we outline the proof of this second step for the Mayer problem, for the sake of simplicity. Assume that the dynamics has the form

$$f(t, x, u) = F(t, x) + u,$$

and that U is convex.

Fix $\epsilon > 0$. By the equi-continuity and the uniform convergence of $(x_n)_n$, joined with the uniform continuity of f in its (compact) domain, the quantities $f(s, x_n(s), u_n(s))$ belong to a ϵ-neighbourhood of $f(t, x^*(t), \cdot)(U)$, for $n \geq n_\epsilon$ and $|t - s| < \delta_\epsilon$. Such neighbourhood turns out to be convex and closed by the assumptions on U and f. Thus, the integral mean

$$⨍_t^{t+h} f(s, x_n(s), u_n(s))\, \mathrm{d}s = \frac{x_n(t+h) - x_n(t)}{h}$$

remains in the same ϵ-neighbourhood of $f(t, x^*(t), \cdot)(U)$ for small $|h|$ and big n. Passing to the limit for $n \to \infty$, then for $h \to 0$, we see that also $\dot{x}^*(t)$ is in the set. This is enough to complete the argument since ϵ is generic and the image $f(t, x^*(t), \cdot)(U)$ is closed.[1]

As a third and final step, a measurable selection lemma - based on Lusin's Theorem - is used to ensure that the vector $u^*(t)$ satisfying $\dot{x^*}(t) = f(t, x^*(t), u^*(t))$ can be chosen with measurable dependence on t.

Summing up, the classical control theoretic approach to the existence problem starts with the convergence of a sequence of states and related values of the functional, and ends up with a control function giving the two limits the desired form; in particular no direct semi-continuity argument for the functional is used.

The approach to the existence problem that we propose goes, in a sense, in the opposite direction. We deal with scalar state equations of the form

$$\dot{x}(t) = F(x(t)) + u(t) \quad \text{or} \quad \dot{x}(t) = F(x(t)) - u(t),$$

where F has sub-linear convex-concave behaviour. In particular, a relation analogous to (2) holds.

The equation is assumed to hold almost everywhere in $[0, +\infty)$, and the functional to be maximized has the form

$$J(u) = \int_0^{+\infty} e^{-\rho t} L(x(t), u(t))\, \mathrm{d}t,$$

where L can be unbounded both from above and from below, and u is a locally integrable, positive function whose discounted global integral is finite. Specifically, we have $L(x, u) = \log(u) - cx^2$ for Shallow Lake models, and $L(x, u) = \alpha(u)\, \chi_{[0,+\infty)}(x) + \min \alpha \chi_{(-\infty,0)}(x)$ in the case of Ramsey-Skiba models, where α is a concave, bounded below increasing function satisfying the Inada's conditions. Clearly, the state constraint $x \geq 0$ can be embedded in the domain of the objective functional of the latter problem in order to deal with the simpler Lagrangian $L(x, u) = \alpha(u)$; further, the condition about the finiteness of the discounted integral of u can be dropped in favour of a suitable assumption about the behaviour of α at infinity.

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[1] Cesari's idea to treat the Bolza problem is to study the couple of integral means $\left(\fint_t^{t+h} f(s, x_n(s), u_n(s))\, \mathrm{d}s, \fint_t^{t+h} L(s, x_n(s), u_n(s))\, \mathrm{d}s\right)$ and prove with an analogous technique that for n, $1/h \to \infty$ it is arbitrarily near to the closed set of the couples $(f(t, x^*(t), u), w)$ such that $u \in U$ and $w \geq L(t, x^*(t), u)$ - thus obtaining:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x^*(t) = f(t, x^*(t), u^*(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t} \lim_{n\to\infty} \int_A^t L(s, x_n(s), u_n(s))\, \mathrm{d}s \geq L(t, x^*(t), u^*(t)). \end{cases}$$

Then one passes to integrals in the inequality. The main technical complication here is in proving that the pointwise limit exists up to a subsequence. The complete, and non-trivial, proof of the Filippov-Cesari's theorem can be found in [20], Chapter III, § 5.

We begin by considering a maximizing sequence of locally integrable controls $(u_n)_{n\in\mathbb{N}}$. In order to bypass the absence of *a priori* estimates, we need to prove some "localization" result. For every fixed compact interval $[0,T]$, we construct a new sequence $\left(u_n^T\right)_n$ upon $(u_n)_n$, such that $\left(u_n^T\right)_n$ is still maximizing and also uniformly bounded in $[0,T]$ by a quantity $N(T)$ increasing in T. We stress that $N(T)$ does not depend on the original sequence nor on the index $n \in \mathbb{N}$. By weak (relative) compactness we can extract a sequence $\left(\bar{u}_n^T\right)_n$, weakly converging in $L^1([0,T])$. We repeat the process for bigger and bigger intervals, each time starting from the maximizing sequence we ended up with in the previous step. This localization procedure allows to have the new states x_n^T under control, by means of a pointwise relation that is proven to hold between x_n and x_n^T. Such relation ensures that u_n^T is admissible also in the state-constrained case of the Ramsey-Skiba model. In this case, the localization procedure ends here, essentially due to the fact that we deal with a Lagrangian that is bounded below.

In the case of a Shallow Lake model, we need to ensure that the upper bound function N satisfies certain additional growth conditions at infinity that will be exploited at the end of the proof. Anyway, the main difference with the case of a partially bounded Lagrangian, is that we also need a "lower localization" procedure.

Accordingly, we prove that for every $T > 0$ the sequence $\left(u_n^T\right)_n$ already obtained can be considered - up to substituting a term with a more favourable one - to be bounded below in $[0,T]$, by a quantity depending decreasingly on time only, which we denote by $\eta(T)$. This can be done by a direct, constructive technique (similar to the one used in the upper localization procedure), if the dynamics is a monotone function of the state. If the function F is not globally monotone, we can reach an analogous conclusion, but the path is more difficult.

Let us proceed, for the moment, with the analysis of the monotone dynamics case. For every $T > 0$, we have obtained subsequences $\left(\bar{u}_n^T\right)_n$ such that $\eta(T) \le \bar{u}_n^T \le N(T)$ almost everywhere in $[0,T]$, and $\bar{u}_n^T \rightharpoonup u^T$ for some $u^T \in L^1([0,T])$. In order to merge properly the local weak limits, the standard diagonal argument does not work, since we are in presence of two families of sequences which *a priori* are not extracted one from the other. Precisely: at step T, the converging sequence $\left(\bar{u}_n^T\right)_n$ is defined as a subsequence of $\left(u_n^T\right)_n$, but the sequence $\left(u_n^{T+1}\right)_n$ - that follows in the construction - is obtained by applying the uniform localization results to $\left(\bar{u}_n^T\right)_n$.

Here the monotonicity of the bound functions N and η provided by the localization lemmas plays a crucial role. Thanks to this property, we are able to define a unique locally bounded (in the L^∞ norm) maximizing sequence $(v_n)_n$ together with a "pre-optimal" function $v \in L^1_{loc}([0,+\infty))$, such that v_n converges weakly to v in $L^1([0,T])$ for every $T > 0$.

As regards the case of non monotone dynamics, we need to introduce another tool in order to prove the lower localization lemma, and then adapt the interpolation process (between the families $\left\{\left(u_n^T\right)_n \,|\, T > 0\right\}$ and $\left\{\left(\bar{u}_n^T\right)_n \,|\, T > 0\right\}$) to the new operators involved in the construction. The technical issue that emerges as a result of the non-monotonicity of the dynamics can be

loosely described in the following way: the difference between two trajectories (for instance, x_n and x_n^T) may be not controllable by a satisfactory estimate, since both trajectories could stay in the zone where the dynamics has the most disadvantageous behaviour (in the case of a Shallow Lake model, an increasing behaviour). Therefore, we need to guarantee that the admissible controls can be chosen - without penalty - in a smaller class of functions that bear, in the global integral, a "heavier" exponential discount factor, thus making the above mentioned estimate - which is exponential, at any rate - acceptable. This procedure, that we call "discount reduction", is time dependent; hence its implementation will change the entire cycle of the interpolation that has to be performed successively.
Thus, we start by considering a maximizing sequence $(u_n)_n$ and we prove that for every fixed $T > 0$ it can be substituted by another maximizing sequence, say $\left(\widetilde{u}_n^T\right)_n$, whose components stay in the right L^1 space and, consequently, in the domain of the localization operator (clearly, the proof of the localization lemmas becomes more complicated in this case). The operator can be thereby evaluated in the controls $\widetilde{u}_n^T$ to produce a sequence $\left(u_n^T\right)_n$ uniformly bounded in $[0, T]$ by two quantities depending only on T. Again, the lower bound function η is proven to be decreasing and the upper bound function N is proven to be increasing.
The point is that both the discount reduction procedure and the localization procedure preserve the values of the controls at smaller intervals: this is the stepping stone that makes the diagonalization effective.
Therefore, we end up again with a sequence $(v_n)_n$ and an admissible $v \in L^1_{loc}([0, +\infty))$ such that $\eta(T) \leq v_n \leq N(T)$ a.e. in $[0, T]$ and $v_n \rightharpoonup v$ in $L^1([0, T])$ for every $T > 0$.

Now, another - standard, in this case - diagonal procedure is needed in order obtain a function $u_* \in L^1_{loc}([0, +\infty))$ and to extract from $(v_n)_n$ a sequence $(v_{n,n})_n$ such that $\tilde{L}(v_{n,n}) \rightharpoonup \tilde{L}(u_*)$, in every $L^1([0, T])$, where $\tilde{L} = \alpha$ or $\tilde{L} = \log$. Clearly, dealing with controls that are bounded also from below is necessary in order to perform this diagonalization in the case $\tilde{L} = \log$.

The following step is the proof of the pointwise convergence of the states associated with $(v_n)_n$ to the state associated with v.
The control u_* is eventually proven to be admissible and optimal, using a dominated convergence argument combined with a pointwise relation between u_* and v, which is obtained, essentially, by the concavity of $\tilde{L}$. Such relation serves to combine the two convergences $\tilde{L}(v_{n,n}) \rightharpoonup \tilde{L}(u_*)$ in $L^1([0, T])$ for every $T > 0$ and $x(\cdot; v_n) \to x(\cdot; v)$ pointwise in $[0, +\infty)$. These considerations are similar to a semi-continuity argument and allow to conclude the proof.

Our method is thereby presented in three successively enhanced versions, applied to as many increasingly difficult existence problems: the Ramsey-Skiba problem, the monotone Shallow Lake problem and the non-monotone Shallow Lake problem.

The proof schemes used in the three analyses can thus be resumed in the following way:

$$
\begin{tikzcd}
\text{uniform localization} \arrow[r, hook] & \text{local compactness} \arrow[l, bend right, "T+1\leftarrow T"']
\end{tikzcd}
$$

$$
\dashrightarrow \ \text{two families diagonalization} \ \longrightarrow \ \text{one family diagonalization}
$$

and

$$
\begin{tikzcd}
\text{discount reduction} \arrow[r, tail] & \text{uniform localization} \arrow[r, hook] & \text{local compactness} \arrow[ll, bend right, "T+1\leftarrow T"']
\end{tikzcd}
$$

$$
\dashrightarrow \ \text{two families diagonalization} \ \longrightarrow \ \text{one family diagonalization.}
$$

These patterns may be hopefully generalized to a scheme for obtaining existence proofs, applicable to a broader class of infinite horizon optimal control problems with non compact control space.

We conclude the introduction with a remark about the original Ramsey model.
A complete (and, in our knowledge, first) mathematical treatment of the problem has been given by Ekeland ([16]) and by Asheim and Ekeland ([2]). In these papers, the question of existence and uniqueness of optimal paths is accurately investigated with regard both to the classical problem and the so-called restricted problem (in which it is assumed that a trajectory is constrained to remain within a fixed interval). The classical problem is addressed to as a problem of the Calculus of Variations. The objective functional is considered as a function of the state, and the optimum is proven to exist in the class of twice differentiable trajectories whose companion control is non-negative. Clearly the latter requirement reduces to a condition on the state and its derivative.
The candidate optimal trajectory, in this class of $\mathcal{C}^2$ functions, is the solution of a certain ordinary differential equation that is written in terms of a solution V of the Hamilton-Jacobi-Bellman-Equation (HJB). The ordinary differential equation is well-posed if V is $\mathcal{C}^2$. Furthermore, to make sure that this trajectory is indeed optimal, it has to be proven that it converges at infinity. Note that the limit is identified by analysing the Euler-Lagrange optimality conditions. The optimal control is then obtained via a feedback relation.
The point is that both the construction of a $\mathcal{C}^2$ solution to HJB, that allows to define the candidate optimal trajectory, and the proof of the necessary convergence property of the latter, are based on the assumption that the dynamics F is concave.
On the other hand, our method (as far as existence of *optima* is concerned) exploits only the

Lipschitz continuity or sub-linearity property of the dynamics and not the specific hypothesis that there exists an interval where F is convex[2]. As such, the method covers also the purely concave case of the Ramsey model, and provides a proof - alternative to the one just discussed - of the existence of an optimal control, in a much broader class.

[2]For further details about the properties of the dynamics that are actually exploited in the analysis refer to the remark at the end of subsection 1.1.2

Chapter 1

The Ramsey-Skiba model

Utility maximization problems constitute a fundamental part of modern economic growth theory, since the works by Ramsey ([30]), Skiba ([34]), Romer ([31]), Lucas ([27]), Barro and Sala-i-Martin ([4]).

These models aim to formalize the dynamics of an economy throughout the quantitative description of the consumers' behaviour. Consumers are seen as homogeneous entities, as far as their operative decisions are concerned; hence the time series of their consuming choices, or consumption path, is represented by a single function, and they as a collective are named *social planner*, or simply *agent*. The agent's purpose is to maximize the utility as a function of the consumption path in a fixed time interval; this can be finite or more often (as far as economic growth literature is concerned) infinite.

The model we take into consideration has three main features that enlarge its applicability range, and, on the other hand, imply additional technical difficulties.

First, the dynamics contains a convex-concave function representing production. This non-concavity of the production function aims to include in the description factors that can show increasing returns, such as human capital investments. It is well known that the presence of non-concavity in an optimization problem may complicate the study of the regularity properties of the value function, and, consequently, the possibility of obtaining the optimum via a feedback relation. Indeed, such relation is usually obtained with a variational approach which deals with a $\mathcal{C}^2$ value function, as discussed in the main introduction.

Secondly, a state constraint is present. As we shall explain later, this is linked to the non-concavity assumption about the dynamics; furthermore, it is needed in order that value function is finite. This feature has, among others, the effect that any new control that one may introduce as a technical tool must be chosen carefully since its admissibility is not guaranteed *a priori*.

Third, we require that the admissible controls are not more than locally integrable in the positive half-line: this is the maximal class if one wants the control strategy to be a function and the state equation to have a solution. On the other side, such choice makes the application of any classical compactness result not straightforward.

From the applications viewpoint, the target of the analysis is the study of the optimal trajectories: regularity, monotonicity, asymptotic behaviour properties and similar are expected to be investigated. These properties are still not characterized in recent literature, at least in the above described case.

Therefore, the program is quite complex and has to be dealt with in many phases. Here we undertake the work, providing the existence result, some properties of the *optima*, and a deep study of the value function based mainly, but not exclusively, on Dynamic Programming and the necessary conditions linked to the Bellman Functional Equation (BFE) and to the Hamilton-Jacobi-Bellman equation (HJB). Such analysis relates to Skiba ([34]) and Askenazy - Le Van ([3]), and develops part of the work by Fiaschi and Gozzi ([18]).

We can summarize the main criticalities as follows:

1. With regard to the existence of an optimal control, we did not find a general result covering our case, either in the classical literature, e.g. Fleming and Rishel ([20]), Cesari ([13]), Zabczyk ([39]), Yong-Zhou ([38]), or in more recent publications, such as the book by Carlson, Haurie and Leizarowitz ([8]) and Zaslavski's books ([40], [41]). So we give a direct proof of the existence of an optimal control for every fixed initial state, which is the first, and the simplest, application of the method described in the main introduction.

2. Certain questions arise, that in other bounded-control models are not even present. Two such questions are the finiteness of the value function and the meaningfulness of the Hamiltonian problem, consisting in the question whether the value function is a viscosity solution to the HJB equation. The notion of viscosity solution can be characterized both in terms of super- and sub-differentials and of test functions; in any case these auxiliary tools must match the necessary restrictions to the domain of the Hamiltonian function, at least for the solutions we are interested in verifying. We are able to prove certain regularity properties of the value function ensuring that this is the case.

3. The regularity property stated in Theorem 32.ii), which is necessary in order that the HJB problem is meaningful for the value function, requires an existence result. Using optimal controls, such property can be proven by a standard argument under the hypothesis that the admissible controls are locally bounded. In our case it is necessary to come back again to the preliminary tools (Lemmas 9 and 10) in order to prove the result for the larger class of admissibility we have chosen, since those preliminary results guarantee that, in a way, we can reduce to considering locally bounded control functions.

4. The proof of the existence of a viscosity solution of HJB features a methodological specificity. Indeed, we make use again of the basic result involved in the construction of the optimum, the localization Lemma 9.

The last point in this list deserves a short deepening. The problem of the well-posedness (namely the existence and uniqueness of a viscosity solution) of the HJB equation has been examined in works such as [35], [7] and [22] in the specific case of an equation resulted from a state constrained optimal control problem.
Essentially, the aim of these works is to identify the general form assumed by the HJB equation when it derives from such kind of problems, in order to study the equation independently using techniques from the fields of PDEs and viscosity solutions.
It is not difficult to realize that the link with a constrained control problem induces some conditions at the boundary of the state space, i.e. the closed subset $\overline{\Omega}$ of $\mathbb{R}^N$ where the admissible trajectories are constrained to remain and that coincides with the domain of the HJB equation. Such boundary conditions must be implemented in the new definition of "constrained" viscosity solution (in the double sense that it derives from a constrained OC problem and includes some boundary constraint) in terms, say, of test functions.
The simplest and, in a way, most natural of these boundary conditions can be obtained from the observation that the global dynamics - namely the function b such that the state equation can be written $\dot{x} = b(x, u)$ - forms a dull corner with the exterior normal vector at (optimal) boundary points, meaning that $\langle b(x_0, u^*(0)), \nu(x_0)\rangle \leq 0$ where $x_0 \in \partial\Omega$ and u^* is optimal at x_0. Starting from this and assuming that the value function is $\mathcal{C}^1$, one discovers that the latter still verifies, in x_0, the subsolution condition in the case of a maximum value problem, or the supersolution condition in the case of a minimum value problem. It is to be noted that the reasoning cannot be replicated in order to ensure that the regular value function is a viscosity solution at $\partial\Omega$.
Ultimately, the additional boundary condition simply consists in requiring that the definition of viscosity subsolution (or supersolution for a minimum value problem) holds also at boundary points.
Soner ([35]) identifies this condition and consequently defines the notion of constrained viscosity solution: v is such if and only if v is a subsolution on Ω and a supersolution on $\overline{\Omega}$ (in his setting, the problem is that of minimizing the objective functional). Then he proves the well-posedness of HJB in this "weak-constrained" sense in the class of bounded uniformly continuous functions of $\overline{\Omega}$.
In [7], Capuzzo-Dolcetta and Lions maintain the same notion of constrained viscosity solution, and prove the well-posedness in the class $\mathcal{C}\left(\overline{\Omega}\right)$ under the assumption that Ω is bounded (hence they drop uniform continuity in favour of the mere continuity). Moreover, they provide further convergence and regularity results for the solution under additional hypotheses about the dy-

namics and the running cost.
Ishii and Koike ([22]) also assume that Ω is bounded, but consider a different notion of viscosity solution involving the upper and lower semicontinuous envelopes of the candidate solution, plus boundary conditions for both the concept of sub- and supersolution. Basically, for them v is a constrained viscosity solution of HJB if and only if v^* is a subsolution and v_* is a supersolution (with a slight difference in the equations considered in the two cases). They prove the well-posedness result among the bounded functions of $\overline{\Omega}$, thus waiving the continuity.
The point is that all of these authors assume the boundedness of the running cost and of the global dynamics as well as (directly or indirectly) the compactness of the control space - which follows for the present model from the boundedness of the global dynamics $(x, c) \mapsto F(x) - c$, in any case. In our problem, the control space, the state space, the dynamics and the instantaneous utility are unbounded; hence, the aforementioned results are not applicable at all here. Indeed, as a consequence of the assumptions, the value function is easily proven to be unbounded and therefore to be outside the class where well-posedness is proven in each of the examined papers.
In this thesis we are concerned only about the existence of a solution to HJB. It is to be noted that, even though the result is proven for the open set $\Omega = (0, +\infty)$ (Theorem 42), it is not difficult to show that the value function is indeed a viscosity subsolution on $\overline{\Omega}$, thus being a constrained viscosity solution in the sense of Soner.
Having considered the foregoing, the technical peculiarity at point 4 above appears to be justified and in fact natural, since the localization lemma must be understood as a sort of compactness generator.

The contents of the chapter are consequently arranged. The first section intends to clear up the genesis of the model and the economic motivations for the assumptions.
Then comes a technical section where some preliminary results that are crucial for the development of the theory, such as Lemmas 9 and 10, are proven. In the same section the reader will find other results like the characterization of admissible constant controls for every initial state. We also show some basic techniques for manipulating admissible controls that preserve admissibility, and how these techniques can be used to prove two other "localization lemmas" from below. These lemmas are not necessary in proving the main results of the chapter, but have meaning from the methodological viewpoint, and may reveal themselves useful in extending the existence result to other monotone, state constrained problems with completely unbounded running cost or instantaneous utility.
Afterwards, we provide some basic properties of the value function, such as finiteness and behaviour at the boundary of the domain. These results require careful handling of the data and some standard results about ordinary differential equations, but do not require optimal controls.
In the subsequent section we prove the existence of an optimal strategy for every initial state: this is the first application of the method described in the main introduction of the thesis.

Then we prove some boundedness properties of optimal controls and optimal states; these properties, besides having a meaning at application level, seem to be needed for the study of the necessary optimality conditions for the controls. The proof exploits the manipulation techniques from section 1.2.
Next we are able to prove other important regularity properties of the value function, including local Lipschitz continuity and lower boundedness of the difference quotients, using optimal controls. This section can also be considered preparatory for the subsequent one.
Eventually we give an application of the methods of Dynamic Programming. As mentioned before, the proof of the admissibility of the value function as a viscosity solution of HJB is made more complicated by the use of the preliminary lemmas; in return, it allows to obtain the result independently of the regularity of the Hamiltonian function. A "backward" Dynamic Programming analysis is also performed, reaching the conclusion that the value function is, at least at optimal points, a bilateral viscosity solution of the Hamilton-Jacobi-Bellman equation.

As already mentioned, due to the unboundedness of the instantaneous utility, the domain of the Hamiltonian of our problem has some restrictions that are reflected in restrictions to the class of the functions for which the HJB equation is meaningful from the viscosity viewpoint. This leads us to introduce the class $\mathcal{C}^+$ (Definition 36), whose members are such that their companion test functions (involved in the definition of viscosity solution) are in the domain of the Hamiltonian together with their derivatives. The value function turns out to be in $\mathcal{C}^+$ thanks to property ii), Theorem 32. We want to stress that the proof of the latter property deeply exploits the localization lemma: both the monotonicity in time and space of the bounding function N introduced in the lemma is used. In particular, monotonicity in space is needed to prove the inequality for $h < 0$.

1.1 Construction of the model

1.1.1 Economic justification

Utility functional

We assume the existence of a representative dynasty in which all members share the same endowments and consume the same amount of a certain good. Our goal is to describe the dynamics of the capital accumulated by each member of the dynasty in an infinite-horizon period and to maximize its intertemporal utility (considered as a function of the quantity of good c that has been consumed). Clearly, consuming is seen as the agent's control strategy, and the set of consumption functions (over time) will be a superset of the set of the admissible control strategies.

First, we need a notion of instantaneous utility, depending on the consumptions, in order to define the inter-temporal utility functional. We will assume that instantaneous utility, which we denote by u, is a strictly increasing and strictly concave function of the consumptions, and that it is twice continuously differentiable. Moreover, we will assume the usual Inada's conditions, that is to say:

$$\lim_{c \to 0^+} u'(c) = +\infty, \quad \lim_{c \to +\infty} u'(c) = 0.$$

We will also use the following assumptions on u:

$$u(0) = 0, \quad \lim_{c \to +\infty} u(c) = +\infty.$$

With this material, we can define the inter-temporal utility functional, which, as usual, must include a (exponential) discount factor expressing time preference for consumption:

$$U(c(\cdot)) := \int_0^{+\infty} e^{-\hat{\rho} t} e^{nt} u(c(t))\, \mathrm{d}t \tag{1.1}$$

where $\hat{\rho} \in \mathbb{R}$ is the rate of time preference and $n \in \mathbb{R}$ is the growth rate of population. The number of members of the dynasty at time zero is normalized to 1.

Production function and constraints

We consider the production or output, denoted by F, as a function of the average capital of the representative dynasty, which we denote by k. First, we assume the usual hypothesis of monotonicity, regularity and unboundedness about the production, that is to say: F is strictly increasing and continuously differentiable from $\mathbb{R}$ to $\mathbb{R}$, and

$$F(0) = 0, \quad \lim_{k \to +\infty} F(k) = +\infty$$

where we may assume $F(x) < 0$ for every $x \in (-\infty, 0)$, since the assumption that F is defined in $(-\infty, 0)$ is merely technical, as we will see later; this way we distinguish the not.

Next, we make some specific requirements. As we want to deal with a non-monotonic marginal product of capital, we assume that, in $[0, +\infty)$, F is first strictly concave, then strictly convex and then again strictly concave up to $+\infty$. This means that in the first phase of capital accumulation, the production shows decreasing returns to scale, which become increasing from a certain level of *pro capite* capital $\underline{k}$. Then, when *pro capite* endowment exceed a threshold $\overline{k} > \underline{k}$, decreasing returns to scale characterize the production anew.

Moreover, we ask that the marginal product in $+\infty$ is strictly positive, so that we can deal with endogenous growth. Observe that this limit surely exists, as F' is (strictly) decreasing in a neighbourhood of $+\infty$. Of course the assumption is equivalent to the fact that the average product of capital tends to a strictly positive quantity for large values of the average stock of capital. Moreover, requiring that the marginal product has a strictly positive lower bound is necessary to ensure a positive long-run growth rate.

As far as the agent's behaviour is concerned, the following constraints must be satisfied, for every time $t \geq 0$:

$$\begin{aligned} &k(t) \geq 0,\ c(t) \geq 0 \\ &i(t) + c(t) \leq F(k(t)),\ \dot{k}(t) = i(t) \end{aligned}$$

where $i(t)$ is the per capita investment at time t. Observe that the first assumption is needed in order to make the agent's optimal strategy possibly different from the case of monotonic marginal product. In fact if condition $\forall t \geq 0 : k(t) \geq 0$ was not present, then heuristically the convex range of production function would be not relevant to establish the long-run behaviour of economy, since every agent would have the possibility to get an amount of resources such that he can fully exploit the increasing return; therefore only the form of production function for large k would be relevant.

Another heuristic remark turns out to be crucial: the monotonicity of u respect to c implies that, if c is an optimal consumption path, then the production is completely allocated between investment and consumption, that is to say $i(t) + c(t) = F(k(t))$ for every $t \geq 0$. This remark, combined with the last of the above conditions implies that the dynamics of capital allocation, for an initial endowment $k_0 \geq 0$, is described by the following Cauchy's problem:

$$\begin{cases} \dot{k}(t) = F(k(t)) - c(t) & \text{for } t \geq 0 \\ k(0) = k_0 \end{cases} \tag{1.2}$$

Considering the first two constraints, the agent's target can be expressed the following way: given an initial endowment of capital $k_0 \geq 0$, maximize the functional in (1.1), when $c(\cdot)$ varies among measurable functions which are everywhere positive in $[0, +\infty)$ and such that the unique solution to problem (1.2) is also everywhere positive in $[0, +\infty)$; the latter requirement is usually called a *state constraint*.

A few reflections are still necessary in order to begin the analytic work. First, we will consider only the case when the time discount rate $\hat{\rho}$ and the population growth rate n satisfy

$$\hat{\rho} - n > 0,$$

which is the most interesting from the economic point of view. Second, we weaken the requirement that c is integrable and positive in $[0, +\infty)$ (in order that c is admissible) to the requirement that c is locally integrable and almost everywhere positive in $[0,+\infty)$.

Finally, we need another assumption about instantaneous utility u so that the functional in (1.1) is finite. To identify the best hypothesis, we temporarily restrict our attention to the particular but significant case in which u is a concave power function and F is linear; namely:

$$u\left(c\right)=c^{1-\sigma},\quad c\geq 0$$
$$F\left(k\right)=Lk,\quad k\geq 0$$

for some $\sigma\in(0,1)$ and $L>0$ (of course in this case F does not satisfy all of the previous assumptions). Using Gronwall's Lemma, it is easy to verify that for any admissible control c (starting from an initial state k_0) and for every time $t\geq 0$, $\int_0^t c\left(s\right)\mathrm{d}s\leq k_0e^{Lt}$. Hence, setting $\rho=\hat{\rho}-n$:

$$\begin{aligned} U\left(c\left(\cdot\right)\right) &= \lim_{T\to+\infty}\int_0^T e^{-\rho t}u\left(c\left(t\right)\right)\mathrm{d}t \\ &= \lim_{T\to+\infty} e^{-\rho T}\int_0^T u\left(c\left(s\right)\right)\mathrm{d}s+\lim_{T\to+\infty}\rho\int_0^T e^{-\rho t}\int_0^t u\left(c\left(s\right)\right)\mathrm{d}s\mathrm{d}t. \end{aligned}$$

Hence using Jensen inequality, we reduce the problem of the convergence of $U\left(c\left(\cdot\right)\right)$ to the problem of the convergence of

$$\int_1^{+\infty} te^{-\rho t}e^{L(1-\sigma)t}\mathrm{d}t$$

which is equivalent to the condition $L\left(1-\sigma\right)<\rho$. Perturbing this clause by the addition of a positive quantity ϵ_0 we get $\left(L+\epsilon_0\right)\left(1-\sigma\right)<\rho-\epsilon_0$ which is in its turn equivalent to the requirement that the function $e^{\epsilon_0 t}e^{-\rho t}\left(e^{(L+\epsilon_0)t}\right)^{1-\sigma}=e^{\epsilon_0 t}e^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)$ tends to 0 as $t\to+\infty$.

Turning back to the general case, we are suggested to assume precisely the same condition, taking care of defining the constant L as $\lim_{k\to+\infty}F'\left(k\right)$ (which has already been assumed to be strictly positive).

1.1.2 Formal definition

Hence the mathematical frame of the economic problem can be defined precisely as follows:

Definition 1. For every $k_0\geq 0$ and for every $c\in L^1_{loc}\left([0,+\infty),\mathbb{R}\right)$:

$k\left(\cdot;k_0,c\right)$ is the only solution to the Cauchy's problem

$$\begin{cases} k\left(0\right)=k_0 \\ \dot{k}\left(t\right)=F\left(k\left(t\right)\right)-c\left(t\right) \quad t\geq 0 \end{cases} \tag{1.3}$$

in the unknown k, where $F:\mathbb{R}\to\mathbb{R}$ has the following properties:

$$F\in\mathcal{C}^1(\mathbb{R},\mathbb{R}),\ F'>0 \text{ in } \mathbb{R},\ F(0)=0,\ \lim_{x\to+\infty}F(x)=+\infty,\ \lim_{x\to+\infty}F'(x)>0,$$
$$F \text{ is concave in } [0,\underline{k}]\cup\left[\overline{k},+\infty\right) \text{ for some } 0<\underline{k}<\overline{k} \text{ and } F \text{ is convex in } \left[\underline{k},\overline{k}\right]$$

Moreover, we set $L:=\lim_{x\to+\infty}F'(x)$.

Definition 2. Let $k_0\geq 0$.

The set of *admissible consumption strategies* with initial capital k_0 is

$$\Lambda(k_0):=\left\{c\in L^1_{loc}([0,+\infty),\mathbb{R})\,/c\geq 0 \text{ almost everywhere, } k(\cdot;k_0,c)\geq 0\right\}$$

The *intertemporal utility functional* $U(\cdot;k_0){:}\Lambda(k_0)\to\mathbb{R}$ is

$$U(c;k_0):=\int_0^{+\infty}e^{-\rho t}u(c(t))\,\mathrm{d}t\ \ \forall c\in\Lambda(k_0)$$

where $\rho>0$, and the function $u:[0,+\infty)\to\mathbb{R}$, representing instantaneous utility, is strictly increasing and strictly concave and satisfies:

$$\begin{aligned}&u\in\mathcal{C}^2((0,+\infty),\mathbb{R})\cap\mathcal{C}^0([0,+\infty),\mathbb{R}),\ u(0)=0,\ \lim_{x\to+\infty}u(x)=+\infty\\&\lim_{x\to 0^+}u'(x)=+\infty,\ \lim_{x\to+\infty}u'(x)=0\\&\exists\epsilon_0>0:\lim_{t\to+\infty}e^{\epsilon_0 t}e^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)=0\end{aligned}\tag{1.4}$$

The *value function* $V:[0,+\infty)\to\mathbb{R}$ is

$$V(k_0):=\sup_{c\in\Lambda(k_0)}U(c;k_0)\ \ \forall k_0\geq 0$$

Remark 3. The last condition in (1.4) implies:

$$\int_0^{+\infty}e^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t<+\infty,\ \int_0^{+\infty}te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t<+\infty.$$

Important remark. The structural assumptions about the dynamics of the problem (Definition 1) are such as to include the setting of Skiba's well-known generalization of the Ramsey problem, introduced in [34]. In particular, it is assumed that production, represented by the space-dynamics function F, has concave-convex-concave behavior in its domain (i.e., with respect to capital).
It is here essential to observe that this particular qualitative property could indeed be dropped in favour of a much weaker sub-linearity assumption, which is actually sufficient to extend our results to the concave dynamics case of the original Ramsey model (as partially specified in the introduction) as well as to the convex-concave case of the problem in [34], and has direct affinity with the usual minimal layout required for a general optimal control problem.

- In fact, global Lipschitzianity and monotonicity are the only properties of F that, through Remarks 7 and 8, are used in the proof of the existence of an optimal control (see Section 1.4 and Lemma 9). Remark 7 essentialy states that the derivative of F is non-negative and bounded above, while Remark 8 establishes comparative estimates for admissible paths, exploiting a sub-linearity property of F of the form

$$F(x) \leq \overline{M}x \quad \forall x \geq 0,$$

  which is stated in Remark 4 as an immediate consequence of the boundedness of F'.

- On the other hand, the qualitative properties of the value function in Sections 1.3 and 1.6 depend on sub-linearity estimates for F other than that used in the existence proof (namely the estimate in Remark 4); such additional estimates are the relation in Remark 18 and relation (1.29) in the proof of Theorem 22, point iii), and follow essentially from the fact the F has an asymptote.

- Eventually, the whole treatment of the necessary conditions for a function of the initial state to be the value function of the problem - contained in the subsequent Section 1.7 - prescinds from the particular convex-concave behaviour of the dynamics.

In particular, none of the above mentioned qualitative characteristics depend on the assumption that there exists an interval where F in convex. Indeed, all such proofs are based only - as far as the dynamics is concerned - on sub-linearity properties of the same nature as the assumptions usually invoked in a multidimensional problem (in other words, when the dynamics F takes vectorial values).
We stress that the choice to base the analysis on the assumptions that define the Ramsey-Skiba model is made to enlighten one of the most substantial applications of the mathematical methods developed in this thesis.

The same philosophy undergoes the layout chosen for the study of the problems in Chapters 2 and 3.

1.2 Technical tools

In this section we provide technical result which will reveal themselves fundamental to prove the existence of an optimum for this problem and to make the Dynamic Programming approach rigorous. Some of these tools will be developed to form the basis for a more general method for proving existence results in infinite horizon settings with non-compact control space. Such method will be empowered to apply to the more difficult case of an "instantaneous utility" which is unbounded both from below and from above - as in the case of Shallow Lake type models.

Remark 4. Set

$$\overline{M} := \max_{[0,+\infty)} F' = \max \left\{ F'(0), F'\left(\bar{k}\right), L \right\}.$$

Recalling that F is strictly increasing with $F(0) = 0$, we see that, for any $x, y \in [0, +\infty)$:

$$\begin{aligned} &|F(x) - F(y)| \leq \overline{M} |x - y| \\ &F(x) \leq \overline{M} x \end{aligned}$$

In particular F is Lipschitz-continuous.

This implies that the Cauchy's problem (1.3) admits a unique global solution (that is to say, defined on $[0, +\infty)$) - even if the dynamics is not continuous with respect to the time variable. Indeed the mapping

$$\mathcal{F}(k)(t) := k_0 + \int_0^t F(k(s))\, \mathrm{d}s - \int_0^t c(s)\, \mathrm{d}s$$

is a contraction on the space $X := \left(\mathcal{C}^0 \left(\left[0, \frac{1}{1+\overline{M}} \right] \right), \|\cdot\|_\infty \right)$, and so admits a unique fixed point $k(\cdot; k_0, c)$. Considering the mapping

$$\mathcal{F}(k)(t) := k\left(\frac{1}{1+\overline{M}}; k_0, c \right) + \int_{\frac{1}{1+\overline{M}}}^t F(k(s))\, \mathrm{d}s - \int_{\frac{1}{1+\overline{M}}}^t c(s)\, \mathrm{d}s$$

on the space $X' := \left(\mathcal{C}^0 \left(\left[\frac{1}{1+\overline{M}}, \frac{2}{1+\overline{M}} \right] \right), \|\cdot\|_\infty \right)$, one can extend the function $k(\cdot; k_0, c)$ to the interval $\left[\frac{1}{1+\overline{M}}, \frac{2}{1+\overline{M}} \right]$, and so on.

In a few words, the existence and uniqueness of the solution depends on the fact that the dynamics in equation (1.3) is defined for *every* state and is *globally* Lipschitz-continuous.

Remark 5. We recall that if k_1 and k_2 are two solutions of (1.3), then the function

$$h(t) := \begin{cases} \dfrac{F(k_1(t)) - F(k_2(t))}{k_1(t) - k_2(t)} & \text{if } k_1(t) \neq k_2(t) \\ F'(k_1(t)) & \text{if } k_1(t) = k_2(t) \end{cases}$$

is continuous in $[0, +\infty)$.

As a consequence, we have a well known comparison result, which in our case can be stated as follows:

Let $k_1, k_2 \geq 0$, $c_1, c_2 \in L^1_{loc}([0, +\infty), \mathbb{R})$, $T_0 \geq 0$ and $T_1 \in (T_0, +\infty]$ such that $c_1 \leq c_2$ almost everywhere in $[T_0, T_1]$. Then the following implications hold:

$$k(T_0; k_1, c_1) = k(T_0; k_2, c_2) \implies \forall t \in [T_0, T_1] : k(t; k_1, c_1) \geq k(t; k_2, c_2) \tag{1.5}$$

$$k(T_0; k_1, c_1) > k(T_0; k_2, c_2) \implies \forall t \in [T_0, T_1] : k(t; k_1, c_1) > k(t; k_2, c_2). \tag{1.6}$$

Lemma 6. *There exists a function $g : (0, +\infty) \to (0, +\infty)$ which is convex, strictly decreasing and such that*

$$g(x) \leq u'(x) \quad \forall x > 0.$$

Proof. Let

$$\Sigma_{u'} := \left\{ (x, y) \in (0, +\infty)^2 / y \geq u'(x) \right\}$$
$$K_{u'} := \bigcap \left\{ K \in \mathcal{P}\left(\mathbb{R}^2\right) / K \text{ is closed and convex, } K \supseteq \Sigma_{u'} \right\}.$$

Thus $K_{u'}$ is the convex closure of $\Sigma_{u'}$. Observe that, for any $x > 0$, the function $H_x(y) := (x, y)$ belongs to $\mathcal{C}^0\left(\mathbb{R}, \mathbb{R}^2\right)$, so any set of the form

$$\{y \geq 0 / (x, y) \in K_{u'}\} = H_x^{-1}(K_{u'}) \bigcap [0, +\infty)$$

is closed in $\mathbb{R}$ (since $K_{u'}$ is closed), and consequently has a minimum element. Now define

$$\forall x > 0 : g(x) := \min\{y \geq 0 / (x, y) \in K_{u'}\}.$$

i) This definition implies that for every $(x, y) \in K_{u'}$, $g(x) \leq y$; hence

$$g(x) \leq u'(x) \quad \forall x > 0$$

because for any $x > 0$, $(x, u'(x)) \in \Sigma_{u'} \subseteq K_{u'}$.

ii) Secondly, g is convex in $(0,+\infty)$. Let $x_0, x_1 > 0$ and $\lambda \in (0,1)$. By definition of g, $(x_0, g(x_0)), (x_1, g(x_1)) \in K_{u'}$, which is a convex set. Hence

$$(1-\lambda)(x_0, g(x_0)) + \lambda(x_1, g(x_1)) \in K_{u'}.$$

By the first property in i), this implies

$$g((1-\lambda)x_0 + \lambda x_1) \leq (1-\lambda)g(x_0) + \lambda g(x_1).$$

iii) Observe that the definition of g does not exclude that $g(x) = 0$ for some $x > 0$. Indeed we show that $g > 0$ in $(0,+\infty)$.

Fix $x > 0$, and consider the closed-convex approximation of $\Sigma_{u'}$

$$K_x := \left\{(t,y) \in [0,x] \times [0,+\infty) / y \geq \frac{u'(x)}{x}(x-t)\right\} \bigcup [x,+\infty) \times [0,+\infty).$$

By construction $K_{u'} \subseteq K_x$ which implies $(t, g(t)) \in K_x$ for any $t > 0$. In particular, for every $t \in (0,x)$:

$$g(t) \geq \frac{u'(x)}{x}(x-t) > 0$$

because $u' > 0$. Hence $g > 0$ in $(0,x)$. Since $x > 0$ is generic, we obtain $g > 0$ in $(0,+\infty)$.

iv) Finally we show that$g$ is strictly decreasing. Take $0 < x_0 < x_1$. By ii) and by definition of convexity, for every $n \in \mathbb{N}$:

$$g(n(x_1 - x_0) + x_0) \quad \geq \quad n[g(x_1) - g(x_0)] + g(x_0).$$

Hence by the assumptions on u and by i):

$$\begin{aligned} 0 &= \lim_{n\to+\infty} u'(n(x_1 - x_0) + x_0) \geq \limsup_{n\to+\infty} g(n(x_1 - x_0) + x_0) \\ &\geq \lim_{n\to+\infty} n[g(x_1) - g(x_0)] + g(x_0) \end{aligned}$$

which implies $g(x_1) < g(x_0)$, remembering that $g > 0$ by iii). $\square$

Remark 7. The function h defined in Remark 5 satisfies

$$0 \leq h \leq \overline{M} \quad \text{in } [0,+\infty).$$

where $\overline{M}$ is defined as in Remark 4.

Remark 8. Let $k_0 \geq 0$ and $c \in \Lambda(k_0)$. Then, for every $t \geq 0$:

$$\begin{aligned} k(t; k_0, c) &\leq k_0 e^{\overline{M}t} \\ \int_0^t c(s)\, \mathrm{d}s &\leq k_0 e^{\overline{M}t} \end{aligned}$$

Indeed, by Remark 4 and remembering that $c \geq 0$, we have, for every $t \geq 0$, $\dot{k}(t; k_0, c) \leq \overline{M} k(t; k_0, c)$ - which implies by (1.5):

$$k(t; k_0, c) \leq k_0 e^{\overline{M}t} \quad \forall t \geq 0.$$

Now integrating both sides of the state equation, again by Remark 4 and by the fact that $k(\cdot; k_0, c) \geq 0$ we see that, for every $t \geq 0$:

$$\begin{aligned} \int_0^t c(s)\, \mathrm{d}s &= k_0 - k(t; k_0, c) + \int_0^t F(k(s; k_0, c))\, \mathrm{d}s \\ &\leq k_0 + \overline{M} \int_0^t k(s; k_0, c)\, \mathrm{d}s \\ &\leq k_0 + \overline{M} k_0 \int_0^t e^{\overline{M}s} \mathrm{d}s = k_0 e^{\overline{M}t}. \end{aligned}$$

Lemma 9. (Localization Lemma). *There exists a function $N : (0, +\infty)^2 \to (0, +\infty)$, increasing in both variables, such that:*

for every $(k_0, T) \in (0, +\infty)^2$ and every $c \in \Lambda(k_0)$, there exists a control function $c^T \in \Lambda(k_0)$ satisfying

$$\begin{gathered} U\left(c^T; k_0\right) \geq U(c; k_0) \\ c^T = c \wedge N(k_0, T) \ \textit{almost everywhere in } [0, T] \end{gathered}$$

In particular, c^T is bounded above, in $[0, T]$, by a quantity which does not depend on the original control c, but only on T and on the initial status k_0.

Proof. Let g be the function defined in Lemma 6 and $\beta := \frac{\log\left(1+\overline{M}\right)}{\overline{M}}$. Define, for every $(k_0, T) \in (0, +\infty)^2$:

$$\begin{aligned} \alpha(k_0, T) &:= \beta e^{-\rho(T+\beta)} g\left[k_0 \left(\frac{e^{\overline{M}(T+\beta)}}{\beta} + e^{\overline{M}T}\right)\right] \\ N(k_0, T) &:= \inf\left\{\tilde{N} > 0 / \forall N \geq \tilde{N} : u'(N) < \alpha(k_0, T)\right\} \\ &= \inf\left\{\tilde{N} > 0 / u'\left(\tilde{N}\right) < \alpha(k_0, T)\right\}. \end{aligned}$$

In the first place, $N(k_0,T) \neq +\infty$, because $\alpha(k_0,T) > 0$ for every $k_0 > 0$, $T > 0$ and $\lim_{N\to+\infty} u'(N) = 0$.

In the second place, $u'((0,+\infty)) = (0,+\infty)$, which implies $N(k_0,T) > 0$: otherwise, since $(u')^{-1}(\alpha(k_0,T)) > 0$, there would exist $N > 0$ such that

$$
\begin{aligned}
&N < (u')^{-1}(\alpha(k_0,T)) \\
&u'(N) < \alpha(k_0,T)
\end{aligned}
$$

which is absurd because u' is decreasing; hence the quantity $u'(N(k_0,T))$ is well defined. Moreover by the continuity of u',

$$
u'(N(k_0,T)) = \alpha(k_0,T). \tag{1.7}
$$

The function $N(\cdot,\cdot)$ is also increasing in both variables, because $\alpha(\cdot,\cdot)$ is decreasing in both variables and u' is decreasing.

Indeed, for $k_0 \leq k_1$ and for a fixed $T > 0$, suppose that $N(k_1,T) < N(k_0,T)$. Then by definition of infimum we could choose $\tilde{N} \in [N(k_1,T), N(k_0,T))$ such that $u'\left(\tilde{N}\right) < \alpha(k_1,T)$, which implies

$$
u'\left(\tilde{N}\right) < \alpha(k_0,T)
$$

by the monotonicity of α. Since $\tilde{N} > 0$, this implies $N(k_0,T) \leq \tilde{N}$, a contradiction. With an analogous argument we prove that $N(\cdot,\cdot)$ is increasing in the second variable.

Now let $k_0, T > 0$ and $c \in \Lambda(k_0)$ as in the hypothesis. If $c \leq N(k_0,T)$ almost everywhere in $[0,T]$, then define $c^T := c$. If, on the contrary, $c > N(k_0,T)$ in a non-negligible subset of $[0,T]$, then define:

$$
c^T(t) := \begin{cases} c(t) \wedge N(k_0,T) & \text{if } t \in [0,T] \\ c(t) + I_T & \text{if } t \in (T, T+\beta] \\ c(t) & \text{if } t > T+\beta \end{cases}
$$

where $I_T := \int_0^T e^{-\rho t}(c(t) - c(t) \wedge N(k_0,T))\,\mathrm{d}t$. Observe that by Remark 8:

$$
\begin{aligned}
0 < \ I_T &\leq \int_0^T (c(t) - c(t) \wedge N(k_0,T))\,\mathrm{d}t \\
&\leq \int_0^T c(t)\,\mathrm{d}t \\
&\leq k_0 e^{\overline{M}T}
\end{aligned} \tag{1.8}
$$

In order to prove the admissibility of such control function, we compare the orbit $k := k(\cdot; k_0, c)$

to the orbit $k^T := k\left(\cdot; k_0, c^T\right)$. In the first place, observe that by (1.5) and by definition of c^T:

$$k^T(t) \geq k(t) \quad \forall t \in [0,T] \tag{1.9}$$

Now by the state equation, we have:

$$\dot{k^T} - \dot{k} = F\left(k^T\right) - F(k) + c - c^T. \tag{1.10}$$

Set for every $t \geq 0$:

$$h(t) := \begin{cases} \frac{F\left(k^T(t)\right) - F(k(t))}{k^T(t) - k(t)} & \text{if } k^T(t) \neq k(t) \\ F'(k(t)) & \text{if } k^T(t) = k(t) \end{cases}$$

as in Remark 5. Hence by (1.10)

$$\dot{k^T}(t) - \dot{k}(t) = h(t)\left[k^T(t) - k(t)\right] + c(t) - c^T(t) \quad \forall t \geq 0.$$

By Remark 5, the function h is continuous in $[0, +\infty)$, so this is a typical linear equation with measurable coefficient of degree one, satisfied by $k^T - k$. Hence, multiplying both sides by the continuous function $t \to \exp\left(-\int_0^t h(s)\,\mathrm{d}s\right)$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left[k^T(t) - k(t)\right] e^{-\int_0^t h(s)\mathrm{d}s}\right\} = \left[c(t) - c^T(t)\right] e^{-\int_0^t h(s)\mathrm{d}s} \quad \forall t \geq 0$$

which implies, integrating between 0 and any $t \geq 0$:

$$k^T(t) - k(t) \quad = \quad \int_0^t \left[c(s) - c^T(s)\right] e^{\int_s^t h}\mathrm{d}s \tag{1.11}$$

Now observe that

$$h \leq \overline{M} \text{ in } [0, +\infty) \text{ and } h \geq 0 \text{ in } [0,T] \tag{1.12}$$

by (1.9) and the monotonicity of F (or, directly, by Remark 7). Set $t \in (T, T+\beta]$; then by (1.11) and (1.12):

$$\begin{aligned} k^T(t) - k(t) &= \int_0^T \left[c(s) - c(s) \wedge N(k_0, T)\right] e^{\int_s^t h}\mathrm{d}s - I_T \cdot \int_T^t e^{\int_s^t h}\mathrm{d}s \\ &\geq \int_0^T \left[c(s) - c(s) \wedge N(k_0, T)\right] \mathrm{d}s - I_T \cdot \int_T^t e^{\overline{M}(t-s)}\mathrm{d}s \\ &\geq \int_0^T e^{-\rho s}\left[c(s) - c(s) \wedge N(k_0, T)\right] \mathrm{d}s - I_T \cdot \int_T^{T+\beta} e^{\overline{M}(T+\beta-s)}\mathrm{d}s \\ &= I_T\left(1 - \frac{e^{\overline{M}\beta} - 1}{\overline{M}}\right) = 0 \end{aligned} \tag{1.13}$$

This also implies, by (1.5) and by definition of c^T,

$$k^T(t) \geq k(t) \quad \forall t \geq T + \beta$$

Such inequality, together with (1.9) and (1.13), gives us the general inequality

$$k^T(t) \geq k(t) \geq 0 \quad \forall t \geq 0.$$

This implies, associated with the obvious fact that $c^T \geq 0$ almost everywhere in $[0, +\infty)$, that $c^T \in \Lambda(k_0)$.

Now we prove the "optimality" property of c^T respect to c. By the concavity of u, and setting $N := N(k_0, T)$ for simplicity of notation, we have:

$$\begin{aligned}
U(c; k_0) - U\left(c^T; k_0\right) &= \int_0^{+\infty} e^{-\rho t}\left[u(c(t)) - u\left(c^T(t)\right)\right] \mathrm{d}t \\
&= \int_{[0,T]\cap\{c\geq N\}} e^{-\rho t}\left[u(c(t)) - u(c(t)\wedge N)\right] \mathrm{d}t \\
&\quad + \int_T^{T+\beta} e^{-\rho t}\left[u(c(t)) - u(c(t) + I_T)\right] \mathrm{d}t \\
&\leq \int_{[0,T]\cap\{c\geq N\}} e^{-\rho t} u'(c(t)\wedge N)\left[c(t) - c(t)\wedge N\right] \mathrm{d}t \\
&\quad - I_T \int_T^{T+\beta} e^{-\rho t} u'(c(t) + I_T)\, \mathrm{d}t \\
&= u'(N) \int_0^T e^{-\rho t}\left[c(t) - c(t)\wedge N\right] \mathrm{d}t \\
&\quad - I_T \int_T^{T+\beta} e^{-\rho t} u'(c(t) + I_T)\, \mathrm{d}t \\
&= I_T\left[u'(N) - \int_T^{T+\beta} e^{-\rho t} u'(c(t) + I_T)\, \mathrm{d}t\right]. \qquad (1.14)
\end{aligned}$$

Now we exhibit a certain lower bound which is independent on the particular control function *c*. By Jensen inequality, by Lemma 6 and by (1.8), we have:

$$
\begin{aligned}
\int_T^{T+\beta} e^{-\rho t} u'\left(c\left(t\right)+I_T\right) \mathrm{d}t &\geq \int_T^{T+\beta} e^{-\rho t} g\left(c\left(t\right)+I_T\right) \mathrm{d}t \\
&\geq e^{-\rho(T+\beta)} \int_T^{T+\beta} g\left(c\left(t\right)+I_T\right) \mathrm{d}t \\
&\geq \beta e^{-\rho(T+\beta)} g\left(\frac{1}{\beta}\int_T^{T+\beta}\left[c\left(t\right)+I_T\right] \mathrm{d}t\right) \\
&\geq \beta e^{-\rho(T+\beta)} g\left(\frac{1}{\beta}\int_0^{T+\beta} c\left(t\right) \mathrm{d}t + I_T\right) \\
&\geq \beta e^{-\rho(T+\beta)} g\left[k_0\left(\frac{e^{\overline{M}(T+\beta)}}{\beta}+e^{\overline{M}T}\right)\right] \\
&= \alpha\left(k_0, T\right).
\end{aligned}
$$

Hence by (1.7) and (1.14):

$$
\begin{aligned}
U\left(c;k_0\right)-U\left(c^T;k_0\right) &\leq I_T\left[u'\left(N\left(k_0,T\right)\right)-\int_T^{T+\beta} e^{-\rho t} u'\left(c\left(t\right)+I_T\right) \mathrm{d}t\right] \\
&\leq I_T\left[u'\left(N\left(k_0,T\right)\right)-\alpha\left(k_0,T\right)\right]=0.
\end{aligned}
$$

□

Lemma 10. *Let $0<k_0<k_1$ and $c\in\Lambda\left(k_0\right)$. Then there exists a control function $\underline{c}^{k_1-k_0}\in\Lambda\left(k_1\right)$ such that*

$$
U\left(\underline{c}^{k_1-k_0};k_1\right)-U\left(c;k_0\right)\geq u'\left(N\left(k_0,k_1-k_0\right)+1\right)\int_0^{k_1-k_0} e^{-\rho t}\, dt
$$

where N is the function defined in Lemma 9.

Proof. Fix k_0, k_1 and c as in the hypothesis and take $c^{k_1-k_0}$ as in Lemma 9 (where it is understood that $T=k_1-k_0$).Then define:

$$
\underline{c}^{k_1-k_0}\left(t\right):=\begin{cases} c^{k_1-k_0}\left(t\right)+1 & \text{if } t\in\left[0,k_1-k_0\right) \\ c^{k_1-k_0}\left(t\right) & \text{if } t\geq k_1-k_0 \end{cases}
$$

First we prove that $\underline{c}^{k_1-k_0}\in\Lambda\left(k_1\right)$, showing that

$$
\underline{k}:=k\left(\cdot;k_1;\underline{c}^{k_1-k_0}\right)>k\left(\cdot;k_0,c^{k_1-k_0}\right)=:k \tag{1.15}
$$

over $(0,+\infty)$. Suppose by contradiction that this is not true, and take $\tau := \inf\{t > 0 / \underline{k}(t) \leq k(t)\}$. Then by the continuity of the orbits, $\underline{k}(\tau) \leq k(\tau)$, which implies $\tau > 0$. Considering the orbits as solutions to an integral equation we have:

$$k(\tau) = k_0 + \int_0^\tau F(k(t))\,\mathrm{d}t - \int_0^\tau c^{k_1-k_0}(t)\,\mathrm{d}t$$

$$\underline{k}(\tau) = k_1 + \int_0^\tau F(\underline{k}(t))\,\mathrm{d}t - \int_0^\tau c^{k_1-k_0}(t)\,\mathrm{d}t - \min\{\tau, k_1 - k_0\}.$$

Hence

$$\begin{aligned} 0 \geq \underline{k}(\tau) - k(\tau) &= k_1 - k_0 + \int_0^\tau [F(\underline{k}(t)) - F(k(t))]\,\mathrm{d}t - \min\{\tau, k_1 - k_0\} \\ &\geq \int_0^\tau [F(\underline{k}(t)) - F(k(t))]\,\mathrm{d}t \end{aligned}$$

By the definition of τ and the strict monotonicity of F, this quantity must be strictly positive, which is absurd. Hence

$$\begin{aligned} &k\left(\cdot; k_1; \underline{c}^{k_1-k_0}\right) > k\left(\cdot; k_0, c^{k_1-k_0}\right) \geq 0 \text{ in } [0,+\infty) \\ &\underline{c}^{k_1-k_0} \geq c^{k_1-k_0} \geq 0 \text{ a.e. in } [0,+\infty) \end{aligned}$$

which implies $\underline{c}^{k_1-k_0} \in \Lambda(k_0)$.

Secondly, remembering the properties of $c^{k_1-k_0}$ given by Lemma 9, we have

$$\begin{aligned} U\left(\underline{c}^{k_1-k_0}; k_1\right) - U(c; k_0) &\geq U\left(\underline{c}^{k_1-k_0}; k_1\right) - U\left(c^{k_1-k_0}; k_0\right) \\ &= \int_0^{k_1-k_0} e^{-\rho t}\left[u\left(c^{k_1-k_0}(t) + 1\right) - u\left(c^{k_1-k_0}(t)\right)\right]\mathrm{d}t \\ &\geq \int_0^{k_1-k_0} e^{-\rho t} u'\left(c^{k_1-k_0}(t) + 1\right)\mathrm{d}t \\ &\geq u'(N(k_0, k_1 - k_0) + 1) \int_0^{k_1-k_0} e^{-\rho t}\mathrm{d}t \end{aligned}$$

which concludes the proof. □

Here is a simple characterisation of the admissible constant controls.

Proposition 11. *Let $k_0, c \geq 0$. Then*

i) $k(\cdot; k_0, F(k_0)) \equiv k_0$

ii) the function constantly equal to c is admissible at k_0 (which we write $c \in \Lambda(k_0)$) if, and only if

$$c \in [0, F(k_0)].$$

In particular the null function is admissible at any initial state $k_0 \geq 0$.

Proof. i) By the uniqueness of the orbit.

ii)($\Longleftarrow$) First observe that $F(k_0) \in \Lambda(k_0)$, by i). Assume $c \in [0, F(k_0))$ and set $k := k(\cdot; k_0, c)$. Hence

$$\dot{k}(0) = F(k_0) - c > 0$$

which implies, by the continuity of $\dot{k}$, that we can find $\delta > 0$ such that k is strictly increasing in $[0, \delta]$. In particular $\dot{k}(\delta) = F(k(\delta)) - c > F(k_0) - c$ because F is strictly increasing too. By the fact that $\dot{k}(\delta) > 0$ we see that there exists $\hat{\delta} > \delta$ such that k is strictly increasing in $\left[0, \hat{\delta}\right]$ - and so on. Hence k is strictly increasing in $[0, +\infty)$ with

$$\dot{k} \geq F(k_0) - c \quad \text{in } [0, +\infty).$$

Hence k tends to $+\infty$ for $t \to +\infty$. This shows that $c \in \Lambda(k_0)$.

($\Longrightarrow$) Suppose that $c > F(k_0)$ and set again $k := k(\cdot; k_0, c)$. Then

$$\dot{k}(0) = F(k_0) - c < 0$$

so that we can find $\delta > 0$ such that k is strictly decreasing in $[0, \delta]$, and $\dot{k}(\delta) = F(k(\delta)) - c < F(k_0) - c < 0$. Hence one can arbitrarily extend the neighbourhood of 0 in which $\dot{k}$ is strictly less than the strictly negative constant $F(k_0) - c$, which implies that

$$\lim_{t \to +\infty} k(t) = -\infty.$$

Hence k cannot be everywhere-positive and $c \notin \Lambda(k_0)$. $\square$

Corollary 12. *The set sequence $(\Lambda(k))_{k \geq 0}$ is strictly increasing, that is:*

$$\Lambda(k_0) \subsetneq \Lambda(k_1)$$

for every $0 \leq k_0 < k_1$.

Proof. For every $c \in \Lambda(k_0)$, $k(\cdot; k_0, c) \leq k(\cdot; k_1, c)$ by 1.5, which implies the second orbit being positive, and so $c \in \Lambda(k_1)$.

On the other hand, by Proposition 11 and by the strict monotonicity of F, the constant control $\hat{c} \equiv F\left(\hat{k}\right)$ belongs to $\Lambda(k_1) \setminus \Lambda(k_0)$ for any $\hat{k} \in (k_0, k_1]$. $\square$

We now describe two basic methods for constructing a control admissible at point k_1 given a control which is admissible at a different point k_0. The first remark generalises the construction in Lemma 10.

Remark 13. (Admissibility of a control obtained by addition of a constant)

Let $0 \le k_0 < k_1$, $c \in \Lambda(k_0)$, $0 \le t_1 < t_2$ and $H > 0$. Define

$$\tilde{c} := c + H \cdot \chi_{(t_1,t_2)}.$$

Then, if H is sufficiently small, say $H(t_2 - t_1) \le k_1 - k_0$, then

$$k(\cdot; k_1, \tilde{c}) - k(\cdot; k_0, c) \ge 0. \tag{1.16}$$

In particular, $\tilde{c} \in \Lambda(k_1)$.

Proof. For simplicity of notation set $\mathrm{k}_1 := k(\cdot; k_1, \tilde{c})$ and $\mathrm{k}_0 := k(\cdot; k_0, c)$. It is a fact that

$$\mathrm{k}_1(t) - \mathrm{k}_0(t) = e^{\int_0^t q(\tau)\mathrm{d}\tau}(k_1 - k_0) \quad \forall t \in [0, t_1] \tag{1.17}$$

$$\mathrm{k}_1(t) - \mathrm{k}_0(t) = e^{\int_{t_1}^t q(\tau)\mathrm{d}\tau}\left\{\mathrm{k}_1(t_1) - \mathrm{k}_0(t_1) + \int_{t_1}^t e^{-\int_{t_1}^s q(\tau)\mathrm{d}\tau}\left[c(s) - \tilde{c}(s)\right]\mathrm{d}s\right\} \quad \forall t > t_1, \tag{1.18}$$

where q is the (continuous) function

$$q(t) = \begin{cases} \frac{F(\mathrm{k}_1(t)) - F(\mathrm{k}_0(t))}{\mathrm{k}_1(t) - \mathrm{k}_0(t)} & \text{if } \mathrm{k}_1(t) \ne \mathrm{k}_0(t) \\ F'(\mathrm{k}_1(t)) & \text{if } \mathrm{k}_1(t) = \mathrm{k}_0(t). \end{cases}$$

Hence by (1.17) we have

$$\mathrm{k}_1(t) - \mathrm{k}_0(t) \ge k_1 - k_0 > 0 \quad \forall t \in [0, t_1], \tag{1.19}$$

since $q \ge 0$ everywhere because $F' \ge 0$. Assume by contradiction that (1.16) is not true; hence

$$t^* := \inf\{t \ge 0 / \mathrm{k}_1(t) - \mathrm{k}_0(t) < 0\} < +\infty.$$

By continuity we have $\mathrm{k}_1(t^*) - \mathrm{k}_0(t^*) \le 0$ (indeed, since $t^* \ne 0$, we also have $\mathrm{k}_1(t^*) - \mathrm{k}_0(t^*) \ge 0$). Thus relation (1.19) implies $t_1 < t^*$.

Moreover: if $t^* \ge t_2$ then $\mathrm{k}_1 - \mathrm{k}_0 \ge 0$ in $[0, t_2]$ and consequently by comparison $\mathrm{k}_1 - \mathrm{k}_0 \ge 0$ in

$[0,+\infty)$, and the proof is over. If $t^* \in (t_1, t_2)$ then:

$$
\begin{aligned}
0 &= \mathrm{k}_1(t^*) - \mathrm{k}_0(t^*) \overset{(1.18)}{=} \mathrm{k}_1(t_1) - \mathrm{k}_0(t_1) - H \int_{t_1}^{t^*} e^{-\int_{t_1}^{s} q(\tau)\mathrm{d}\tau} \mathrm{d}s \\
&\geq \mathrm{k}_1(t_1) - \mathrm{k}_0(t_1) - H(t^* - t_1) \overset{(1.19)}{\geq} k_1 - k_0 - H(t^* - t_1) \\
&> k_1 - k_0 - H(t_2 - t_1) \geq 0,
\end{aligned}
$$

a contradiction. □

Remark 14. Relation (1.16) can be made *strict* if either $F' > 0$ (as in the present case) or $H(t_2 - t_1) < k_1 - k_0$. Indeed, define

$$
t^* := \inf\{t \geq 0 / \mathrm{k}_1(t) - \mathrm{k}_0(t) \leq 0\}
$$

and assume $t^* < +\infty$. Then $\mathrm{k}_1(t^*) - \mathrm{k}_0(t^*) = 0$ and $t_1 < t^*$. If $t^* > t_2$ then $\mathrm{k}_1 - \mathrm{k}_0 > 0$ in $[0, t_2]$ and consequently by comparison $\mathrm{k}_1 - \mathrm{k}_0 > 0$ in $[0,+\infty)$, which contradicts $t^* < +\infty$. Then we use $t^* \in (t_1, t_2]$:

$$
\begin{aligned}
0 &= \mathrm{k}_1(t^*) - \mathrm{k}_0(t^*) \overset{(1.18)}{=} \mathrm{k}_1(t_1) - \mathrm{k}_0(t_1) - H \int_{t_1}^{t^*} e^{-\int_{t_1}^{s} q(\tau)\mathrm{d}\tau} \mathrm{d}s \\
&\geq k_1 - k_0 - H \int_{t_1}^{t^*} e^{-\int_{t_1}^{s} q(\tau)\mathrm{d}\tau} \mathrm{d}s;
\end{aligned}
$$

then either $e^{-\int_{t_1}^{s} q(\tau)\mathrm{d}\tau} H < H$ for every $s \in (t_1, t^*)$ (if $F' > 0$) or $k_1 - k_0 - (t_2 - t_1) H > 0$, and we still reach a contradiction.

Remark 15. (Admissibility of a control whose trajectory imitates a given higher trajectory) Let $0 < k_0 < k_1$, $c \in \Lambda(k_1)$, and $\gamma \in [0, F(k_0)]$. There is a natural way to construct a control which is admissible at k_0 using c. First observe that the trajectory $k(\cdot; k_0, \gamma)$ is strictly increasing and tends to $+\infty$ at $+\infty$, with

$$
\dot{k} \geq F(k_0) - \gamma \quad \text{in } [0,+\infty), \tag{1.20}
$$

as shown in the proof of Proposition 11. Define:

$$
T = T(k_0, k_1, \gamma) := (k(\cdot; k_0, \gamma))^{-1}(k_1).
$$

Hence, the control which is equal to γ in $[0, T]$ and for $t > T$ behaves like c does in $[0,+\infty)$ must be admissible at k_0. Namely:

$$
\underset{\sim}{c} := \gamma \chi_{[0,T]} + c(\cdot - T)\chi_{(T,+\infty)} \in \Lambda(k_0).
$$

Indeed, $k\left(t;k_0,\underset{\sim}{c}\right)=k\left(t;k_0,\gamma\right)>0$ for $t\in[0,T]$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}k\left(t+T;k_0,\underset{\sim}{c}\right)&=\dot{k}\left(t+T;k_0,\underset{\sim}{c}\right)=F\left(k\left(t+T;k_0,\underset{\sim}{c}\right)\right)-c\left(t\right);\\ k\left(0+T;k_0,\underset{\sim}{c}\right)&=k\left(T;k_0,\gamma\right)=k_1\end{aligned}$$

which implies $k\left(\cdot+T;k_0,\underset{\sim}{c}\right)=k\left(\cdot;k_1,c\right)\geq 0$ by uniqueness.
Additionally, a simple change of variable shows that

$$U\left(k_0;\underset{\sim}{c}\right)=e^{-\rho T}U\left(k_1;c\right).$$

Using Remarks 13 and 15, we prove two simple lower localization lemmas .

Lemma 16 (Lower localization lemma)**.** *Let $k_0>0$, $c\in\Lambda\left(k_0\right)$ and $\epsilon>0$ sufficiently small and such that:*

$$U\left(k_0;c\right)>V\left(k_0\right)-\epsilon.$$

Then, for every $T>0$, there exists a control $c_T\in\Lambda\left(k_0\right)$ and a constant $m\left(c,\epsilon,k_0,T\right)>0$ such that

$$\begin{aligned}&i)\ U\left(k_0;c_T\right)>V\left(k_0\right)-\epsilon\\ &ii)\ c_T\geq m\left(c,\epsilon,k_0,T\right)\ \text{ a.e. in } [0,T].\end{aligned}$$

Proof. Take k_0, ϵ and c as in the statement of the theorem. Since c is ϵ-optimal, we can choose $\eta=\eta\left(c,\epsilon,k_0\right)$ such that

$$e^{-\frac{2\rho\eta}{F\left(k_0\right)}}U\left(k_0;c\right)>V\left(k_0\right)-\epsilon.$$

Fix $T>0$; Remark 13 implies:

$$\tilde{c}_T:=c+\frac{\eta}{T}\chi_{[0,T]}\in\Lambda\left(k_0+\eta\right).$$

We know by (1.20) that for every constant $\xi\in\left(0,F\left(k_0\right)\right)$, the trajectory $k\left(\cdot;k_0,\xi\right)$ satisfies

$$\dot{k}\left(\cdot;k_0,\xi\right)\geq F\left(k_0\right)-\xi$$

and consequently tends to $+\infty$ at $+\infty$.
Define $\mathcal{T}\left(k_0,\eta\right)$ by the formula

$$k\left(\mathcal{T}\left(k_0,\eta\right);k_0,\frac{F\left(k_0\right)}{2}\right)=k_0+\eta;$$

this implies, by Lagrange's Theorem:

$$0 < \mathcal{T}(k_0, \eta) \leq \frac{2\eta}{F(k_0)}. \tag{1.21}$$

Indeed, for some $\xi \in (0, \mathcal{T}(k_0, k_1))$:

$$\begin{aligned}
\eta &= k\left(\mathcal{T}(k_0, \eta); k_0, \frac{F(k_0)}{2}\right) - k\left(0; k_0, \frac{F(k_0)}{2}\right) \\
&= \dot{k}\left(\xi; k_0, \frac{F(k_0)}{2}\right)\mathcal{T}(k_0, \eta) \\
&\geq \frac{F(k_0)}{2}\mathcal{T}(k_0, \eta).
\end{aligned}$$

Now define:

$$\begin{aligned}
c_T &:= \frac{F(k_0)}{2}\chi_{[0,\mathcal{T}(k_0,\eta)]} + \tilde{c}_T(\cdot - \mathcal{T}(k_0, \eta))\,\chi_{(\mathcal{T}(k_0,\eta),+\infty)}, \\
m(c, \epsilon, k_0, T) &:= \frac{F(k_0)}{2} \wedge \frac{\eta(c, \epsilon, k_0)}{T}.
\end{aligned}$$

Observe that in general $T > \mathcal{T}(k_0, \eta)$, and anyway we could ensure that it is so by choosing η sufficiently small (without loss of generality), since η does not depend on T. It is a fact that:

$$\begin{aligned}
&c_T \in \Lambda(k_0) \\
&c_T \geq m(c, \epsilon, k_0, T) \text{ a.e. in } [0, T],
\end{aligned}$$

by Remark 15, since $\tilde{c}_T \in \Lambda(k_0 + \eta)$. Furthermore:

$$\begin{aligned}
U(k_0; c_T) &= u\left(\frac{F(k_0)}{2}\right)\int_0^{\mathcal{T}(k_0,\eta)} e^{-\rho t}\mathrm{d}t + \int_{\mathcal{T}(k_0,\eta)}^{+\infty} e^{-\rho t} u(\tilde{c}_T(t - \mathcal{T}(k_0, \eta)))\,\mathrm{d}t \\
&\geq e^{-\rho\mathcal{T}(k_0,\eta)}\int_0^{+\infty} e^{-\rho s} u(\tilde{c}_T(s))\,\mathrm{d}s \\
&= e^{-\rho\mathcal{T}(k_0,\eta)}\left\{\int_0^T e^{-\rho s} u\left(c(s) + \frac{\eta}{T}\right)\mathrm{d}s + \int_T^{+\infty} e^{-\rho s} u(c(s))\,\mathrm{d}s\right\} \\
&\geq e^{-\rho\mathcal{T}(k_0,\eta)} U(k_0; c) \\
&\geq e^{-\frac{2\rho\eta}{F(k_0)}} U(k_0; c) > V(k_0) - \epsilon
\end{aligned}$$

by (1.21) and the choice of η. ◻

Lemma 17 (Lower uniform localization lemma)**.** *Let $k_0 > 0$, $c \in \Lambda(k_0)$ and $\epsilon > 0$ such that:*

$$U(k_0; c) > V(k_0) - \epsilon,$$

with ϵ sufficiently small. Then, for every $T > 0$, there exists a control $c_T \in \Lambda(k_0)$ and a constant $m(\epsilon, k_0, T) > 0$ such that

$$\begin{aligned} &i)\ U(k_0; c_T) > V(k_0) - 2\epsilon \\ &ii)\ c_T \geq m(\epsilon, k_0, T)\ \text{ a.e. in } [0, T]. \end{aligned}$$

Proof. Choose $\eta = \eta(\epsilon, k_0)$ such that

$$e^{-\frac{2\rho\eta}{F(k_0)}} > \frac{V(k_0) - 2\epsilon}{V(k_0) - \epsilon} > 0.$$

Fix $T > 0$ and define, like in Lemma 16:

$$\begin{aligned} \tilde{c}_T &:= c + \frac{\eta}{T}\chi_{[0,T]} \\ \mathcal{T}(k_0, \eta) &:= k\left(\cdot; k_0, \frac{F(k_0)}{2}\right)^{-1}(k_0 + \eta) \\ c_T &:= \frac{F(k_0)}{2}\chi_{[0,\mathcal{T}(k_0,\eta)]} + \tilde{c}_T(\cdot - \mathcal{T}(k_0, \eta))\,\chi_{(\mathcal{T}(k_0,\eta),+\infty)} \\ m(\epsilon, k_0, T) &:= \frac{F(k_0)}{2} \wedge \frac{\eta(\epsilon, k_0)}{T}. \end{aligned}$$

Obviously we still have:

$$\begin{aligned} &c_T \in \Lambda(k_0) \\ &c_T \geq m(\epsilon, k_0, T)\ \text{ a.e. in } [0, T], \end{aligned}$$

plus (1.21). The advantage here is that the constant $m(\epsilon, k_0, T)$ does not depend on the intial control c. In return, we have to weaken the estimate for the value:

$$\begin{aligned} U(k_0; c_T) &\geq e^{-\rho\mathcal{T}(k_0,\eta)}U(k_0; c) \\ &\geq e^{-\frac{2\rho\eta}{F(k_0)}}U(k_0; c) \\ &> e^{-\frac{2\rho\eta}{F(k_0)}}\left[V(k_0) - \epsilon\right] > V(k_0) - 2\epsilon. \end{aligned}$$

□

1.3 First properties of the value function

We begin by proving some properties of the value function whose proofs do not require the existence of an optimal control. Due to the complexity - and the novelty - of the existence result it seems reasonable to separate these properties from others relying on optimal controls. The latter are outlined in Section 1.6.

The first thing to do in the analysis of an optimization problem is to prove the finiteness of the extremum we aim to reach. In other words, we need to prove that the value function takes only finite values. The asymptotic properties of F' make F sub-linear: this allows us to prove certain uniform estimates (Lemma 20) leading to the desired result. These estimates will also reveal themselves useful both in the construction of the optimal control (as they ensure the dominated convergence in a crucial step of the approximation) and in its the characterization given in the Dynamic Programming section.

Remark 18. Set $M_0, \hat{M} \geq 0$ such that:

$$\begin{aligned} &\forall x \geq M_0 : F(x) \leq (L + \epsilon_0)\, x \\ &\hat{M} := \max_{[0, M_0]} F. \end{aligned}$$

(which is possible because $\lim_{x \to +\infty} \frac{F(x)}{x} = L$). Hence, for every $x \geq 0$:

$$F(x) \leq (L + \epsilon_0)\, x + \hat{M}$$

Remark 19. Since u is a concave function satisfying $u(0) = 0$, u is sub-additive in $[0, +\infty)$ and satisfies:

$$\forall x > 0 : \forall K > 1 : u(Kx) \leq K u(x)$$

Lemma 20. *Let $k_0 \geq 0$. There exists a number $M(k_0) > 1$ and a continuous, strictly positive function $\psi_{k_0} : (0, +\infty) \to \mathbb{R}$ such that, for any $c \in \Lambda(k_0)$:*

$$\begin{aligned} &i) \quad \forall t \geq 0 : \int_0^t c(s)\, ds \leq tM(k_0) \left[1 + e^{(L+\epsilon_0)t}\right] + \frac{M(k_0)}{L + \epsilon_0} \\ &ii) \quad \forall t > 0 : e^{-\rho t} \int_0^t u(c(s))\, ds \leq \psi_{k_0}(t) \\ &iii) \quad U(c; k_0) = \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c(s))\, ds dt. \end{aligned}$$

Both $M(k_0)$ and ψ_{k_0} depend only on k_0 and the problem's data (in particular they don't depend *on c). Moreover ψ_{k_0} satisfies*

$$\lim_{t\to+\infty}\psi_{k_0}(t)=0,\qquad \int_0^{+\infty}\psi_{k_0}(t)\,dt<+\infty$$

Proof. i) Set $\kappa:=k(\cdot;k_0,c)$ and $M(k_0):=1+\max\left\{(L+\epsilon_0)k_0,\hat{M}\right\}$, where $\hat{M}$ is the quantity defined in Remark 18. Observe that, by Remark 18, for every $x\geq 0$:

$$F(x)\leq (L+\epsilon_0)x+M(k_0).$$

Fix $t\geq 0$; by the state equation, we have for any $s\in[0,t]$

$$\kappa(s)\leq k_0+sM(k_0)+(L+\epsilon_0)\int_0^s\kappa(\tau)\,\mathrm{d}\tau$$

which implies by Gronwall's inequality:

$$\kappa(s)\leq[k_0+sM(k_0)]\,e^{(L+\epsilon_0)s}\quad\forall s\in[0,t],$$

as $s\to k_0+sM(k_0)$ is increasing. So

$$\begin{aligned}\int_0^t(L+\epsilon_0)\kappa(s)\,\mathrm{d}s &\leq k_0(L+\epsilon_0)\int_0^t e^{(L+\epsilon_0)s}\mathrm{d}s+M(k_0)(L+\epsilon_0)\int_0^t s\cdot e^{(L+\epsilon_0)s}\mathrm{d}s\\ &= k_0e^{(L+\epsilon_0)t}-k_0+tM(k_0)e^{(L+\epsilon_0)t}-\frac{M(k_0)}{(L+\epsilon_0)}e^{(L+\epsilon_0)t}+\frac{M(k_0)}{(L+\epsilon_0)}\\ &= tM(k_0)e^{(L+\epsilon_0)t}+\left[k_0-\frac{M(k_0)}{(L+\epsilon_0)}\right]e^{(L+\epsilon_0)t}+\frac{M(k_0)}{(L+\epsilon_0)}-k_0\\ &\leq tM(k_0)e^{(L+\epsilon_0)t}+\frac{M(k_0)}{(L+\epsilon_0)}-k_0\end{aligned}$$

Hence, again by the state equation, for every $t\geq 0$:

$$\begin{aligned}\int_0^t c(s)\,\mathrm{d}s &= k_0-\kappa(t)+\int_0^t F(\kappa(s))\,\mathrm{d}s\\ &\leq k_0+tM(k_0)+\int_0^t(L+\epsilon_0)\kappa(s)\,\mathrm{d}s\;\leq\; tM(k_0)\left[1+e^{(L+\epsilon_0)t}\right]+\frac{M(k_0)}{(L+\epsilon_0)}.\end{aligned}$$

which proves the first assertion.

ii) In the second place, it follows by Jensen inequality, the monotonicity of u and Remark 19, that for every $t \geq 0$:

$$0 \leq e^{-\rho t} \int_0^t u(c(s))\,\mathrm{d}s \leq te^{-\rho t} u\left(\frac{\int_0^t c(s)\,\mathrm{d}s}{t}\right) \leq te^{-\rho t} u\left(M(k_0)\left[1 + e^{(L+\epsilon_0)t}\right] + \frac{M(k_0)}{t(L+\epsilon_0)}\right)$$
$$\leq te^{-\rho t}\left\{u(M(k_0)) + M(k_0)\,u\left(e^{(L+\epsilon_0)t}\right) + u\left(\frac{M(k_0)}{t(L+\epsilon_0)}\right)\right\} =: \psi_{k_0}(t).$$

This proves the inequality in ii); from the last assumption on u in (1.4) we deduce that

$$\lim_{t \to +\infty} \psi_{k_0}(t) = 0.$$

Hence $\lim_{T\to\infty} e^{-\rho T} \int_0^T u(c(s))\,\mathrm{d}s = 0$ and this implies the identity in iii) by a simple integration by parts.

It remains to be proven that $\psi_{k_0} \in L^1([0, +\infty))$. We have:

$$\begin{aligned}
\int_0^{+\infty} \psi_{k_0}(t)\,\mathrm{d}t &= \int_0^{+\infty} te^{-\rho t}\left\{u(M(k_0)) + M(k_0)\,u\left(e^{(L+\epsilon_0)t}\right) + u\left(\frac{M(k_0)}{t(L+\epsilon_0)}\right)\right\}\mathrm{d}t \\
&\leq u(M(k_0)) \int_0^{+\infty} te^{-\rho t}\mathrm{d}t + M(k_0) \int_0^{+\infty} te^{-\rho t} u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t \\
&\quad + u\left(\frac{M(k_0)}{L+\epsilon_0}\right)\left\{\int_0^1 e^{-\rho t}\mathrm{d}t + \int_1^{+\infty} te^{-\rho t}\mathrm{d}t\right\}.
\end{aligned}$$

This estimate follows again by the monotonicity of u and the concavity properties stated in Remark 19 . By Remark 3 the upper bound is finite. □

So we have established the starting point of the theory.

Corollary 21. *The value function $V : [0, +\infty) \to \mathbb{R}$ is well-definite; that is, for every $k_0 \geq 0$, $V(k_0) < +\infty$.*

Proof. By Lemma 20 we have:

$$V(k_0) = \sup_{c \in \Lambda(k_0)} U(c; k_0) \leq \rho \int_0^{+\infty} \psi_{k_0}(t)\,\mathrm{d}t < +\infty.$$

□

The next result describes the behaviour of the value function at the boundary of the domain.

Theorem 22. *The value function* $V : [0, +\infty) \to \mathbb{R}$ *satisfies:*

$$\begin{aligned} i) \qquad & \lim_{k\to+\infty} V(k) = +\infty \\ ii) \qquad & \lim_{k\to+\infty} \frac{V(k)}{k} = 0 \\ iii) \qquad & \lim_{k\to 0} V(k) = V(0) = 0 \end{aligned}$$

Proof. **i)** For every $k_0 \geq 0$ the constant control $F(k_0)$ is admissible at k_0 by Proposition 11; hence

$$V(k_0) \geq U(F(k_0); k_0) = \frac{u(F(k_0))}{\rho} \to +\infty$$

as $k_0 \to +\infty$, by the assumptions on u and F.

ii) Set $\hat{M} > 0$ as in Remark 18 and $k_0 > 0$ such that:

$$k_0 > \frac{1}{L+\epsilon_0}\hat{M} \tag{1.22}$$

Hence, for every $x > 0$:

$$F(x) \leq (L+\epsilon_0)(x+k_0) \tag{1.23}$$

By reasons that will be clear later, suppose also that:

$$k_0 > \frac{1}{L+\epsilon_0} \tag{1.24}$$

Observe that the proof of Lemma 20, i) only requires $M(k_0) \geq \hat{M}, k_0(L+\epsilon_0)$; hence (1.22) and (1.23) imply that this property holds for $M(k_0) = k_0(L+\epsilon_0)$ - which means that:

$$\forall t \geq 0 : \int_0^t c(s)\,\mathrm{d}s \leq k_0 + tk_0(L+\epsilon_0)\left[1 + e^{(L+\epsilon_0)t}\right]. \tag{1.25}$$

In particular

$$\forall t \geq 1 : \frac{\int_0^t c(s)\,\mathrm{d}s}{t} \leq k_0 + k_0(L+\epsilon_0) + k_0(L+\epsilon_0)e^{(L+\epsilon_0)t}. \tag{1.26}$$

Now set

$$J_c(\alpha,\beta) := \int_\alpha^\beta te^{-\rho t} u\left(\frac{\int_0^t c(s)\,\mathrm{d}s}{t}\right)\mathrm{d}t \tag{1.27}$$

and fix $N > 0$.

We provide three different estimates, over $J_c(0,1)$, $J_c(1,N)$ and $J_c(N,+\infty)$, using Remark 19.

First, we have by (1.25):

$$
\begin{aligned}
J_c(0,1) &\leq \int_0^1 te^{-\rho t}\frac{1}{t}u\left(\int_0^1 c(s)\,\mathrm{d}s\right)\mathrm{d}t \\
&\leq u\left[k_0\left(1+(L+\epsilon_0)\left(1+e^{(L+\epsilon_0)}\right)\right)\right]\frac{1-e^{-\rho}}{\rho} \\
&\leq u(k_0)\frac{1-e^{-\rho}}{\rho}\left[1+(L+\epsilon_0)\left(1+e^{(L+\epsilon_0)}\right)\right].
\end{aligned}
$$

Moreover, by (1.26):

$$
\begin{aligned}
J_c(1,N) &\leq \int_1^N te^{-\rho t}u\left(k_0+k_0(L+\epsilon_0)+k_0(L+\epsilon_0)e^{(L+\epsilon_0)t}\right)\mathrm{d}t \\
&\leq u(k_0+k_0(L+\epsilon_0))\int_1^N te^{-\rho t}\mathrm{d}t+u(k_0(L+\epsilon_0))\int_1^N te^{-\rho t}e^{(L+\epsilon_0)t}\mathrm{d}t \\
&\leq u[k_0(1+L+\epsilon_0)]\left(1+e^{(L+\epsilon_0)N}\right)\int_1^N te^{-\rho t}\mathrm{d}t
\end{aligned}
$$

Finally, remembering that $k_0(L+\epsilon_0)>1$ by (1.24),

$$
\begin{aligned}
J_c(N,+\infty) &\leq \int_N^{+\infty} te^{-\rho t}u\left(k_0+k_0(L+\epsilon_0)+k_0(L+\epsilon_0)e^{(L+\epsilon_0)t}\right)\mathrm{d}t \\
&\leq u(k_0+k_0(L+\epsilon_0))\int_N^{+\infty} te^{-\rho t}\mathrm{d}t+k_0(L+\epsilon_0)\int_N^{+\infty} te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t
\end{aligned}
$$

Now we show that

$$
\lim_{k\to+\infty}\frac{V(k)}{k}=0.
$$

Fix $\eta>0$; by Remark 3, we can chose $N_\eta>0$ such that

$$
(L+\epsilon_0)\int_{N_\eta}^{+\infty} te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t<\eta.
$$

Hence for k_0 satisfying:

$$
k_0>\max\left\{\frac{1}{L+\epsilon_0}\hat{M},\frac{1}{L+\epsilon_0}\right\}
$$

and for *every* $c \in \Lambda(k_0)$, the above estimates imply:

$$\begin{aligned}
U(c;k_0) &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c(s))\,\mathrm{d}s\mathrm{d}t \\
&\leq \rho J_c(0,1) + \rho J_c(1,N_\eta) + \rho J_c(N_\eta,+\infty) \\
&\leq u(k_0)\left(1-e^{-\rho}\right)\left[1+(L+\epsilon_0)\left(e^{(L+\epsilon_0)}+1\right)\right] + \\
&\quad +u(k_0)(1+L+\epsilon_0)\left(1+e^{(L+\epsilon_0)N_\eta}\right)\int_1^{N_\eta} te^{-\rho t}\mathrm{d}t + \\
&\quad +u(k_0)(1+L+\epsilon_0)\int_{N_\eta}^{+\infty} te^{-\rho t}\mathrm{d}t + k_0\eta
\end{aligned} \tag{1.28}$$

following Remark 19, Lemma 20, iii), (1.27) and Jensen inequality. Now observe that:

$$\lim_{k_0\to+\infty}\frac{u(k_0)}{k_0} = \lim_{k_0\to+\infty} u'(k_0) = 0.$$

Hence for k_0 sufficiently large (say $k_0 > k^*$):

$$\begin{aligned}
\frac{u(k_0)}{k_0} &< \eta\Big\{\left(1-e^{-\rho}\right)\left[1+(L+\epsilon_0)\left(e^{(L+\epsilon_0)}+1\right)\right] + \\
&\quad +(1+L+\epsilon_0)\left(1+e^{(L+\epsilon_0)N_\eta}\right)\int_1^{N_\eta} te^{-\rho t}\mathrm{d}t + (1+L+\epsilon_0)\int_{N_\eta}^{+\infty} te^{-\rho t}\mathrm{d}t\Big\}^{-1}
\end{aligned}$$

Observe that this is possible because the expression into the brackets does not depend on k_0. In fact, like N_η, it depends only on η and on the problem's data L, ϵ_0, ρ - and so does k^*.

By (1.28), this implies for *every* $c \in \Lambda(k_0)$:

$$U(c;k_0) \leq 2k_0\eta$$

which gives, taking the sup over $\Lambda(k_0)$:

$$V(k_0) \leq 2k_0\eta.$$

Hence the assertion is proven, because the previous inequality holds for every

$$k_0 > \max\left\{\frac{1}{L+\epsilon_0}\hat{M}, \frac{1}{L+\epsilon_0}, k^*\right\},$$

and the last quantity is a threshold depending only on η and on the problem's data.

iii) In the first place, we prove that

$$V(0) = 0.$$

Let $c \in \Lambda(0)$; by definition, $c \geq 0$ so that

$$\forall t \geq 0 : \dot{k}(t; 0, c) \leq F(k(t; 0, c)).$$

Observe that F is precisely the function which defines the dynamics of $k(\cdot; 0, 0)$, hence by (1.5):

$$\forall t \geq 0 : k(t; 0, c) \leq k(t; 0, 0) = 0$$

where the last equality holds by Lemma 11, i).

Hence $k(\cdot; 0, c) \equiv 0$ which together with $F(0) = 0$ implies $c \equiv 0$. So $\Lambda(0) = \{0\}$, which implies

$$V(0) = U(0; 0) = \int_0^{+\infty} e^{-\rho t} u(0)\, \mathrm{d}t = 0$$

Now we show that

$$\lim_{k \to 0} V(k) = 0.$$

In this case we have to study the behaviour of $V(k_0)$ when $k_0 \to 0$, so we use the sublinearity of $F(x)$ for $x \to +\infty$ and the concavity of F near 0.

As a first step, we construct a linear function which is always above F with these two tools. Indeed we show that there is $m > 0$ such that the function

$$G(x) := \begin{cases} mx & \text{if } x \in [0, \bar{k}] \\ (L + \epsilon_0)(x - \bar{k}) + m\bar{k} & \text{if } x \geq \bar{k} \end{cases}$$

satisfies

$$\forall x \geq 0 : F(x) \leq G(x). \tag{1.29}$$

If $F'(\bar{k}) \leq L + \epsilon_0$ then it is enough to choose $m > \max\left\{F'(0), F'(\bar{k}), \frac{F(\bar{k})}{\bar{k}}\right\}$.

If $F'(\bar{k}) > L + \epsilon_0$ then take $\bar{x} > \bar{k}$ such that $F' \leq L + \epsilon_0$ in $(\bar{x}, +\infty)$; a first-order development in $\bar{x}$ with Lagrange remainder shows that

$$\forall x > \bar{k} : \; F(x) < F(\bar{x}) + (L + \epsilon_0)(x - \bar{k}) + \max_{[\bar{k}, \bar{x}]} F.$$

Hence it is enough to choose $m > \max\left\{F'(0), F'(\bar{k}), \frac{F(\bar{x}) + M}{\bar{k}}\right\}$ (where $M = \max_{[\bar{k}, \bar{x}]} F$) in order that condition (1.29) is satisfied.

Observe that condition $m > F'\left(\bar{k}\right)$ is still necessary to ensure that $mx > F(x)$ for $x \in \left[\underline{k}, \bar{k}\right]$ (Lagrange's theorem proves that it is sufficient).

Suppose also, for reasons that will be clear later, that

$$m > 1. \tag{1.30}$$

Now take $k_0 > 0$, $c \in \Lambda(k_0)$ and consider the function $h : [0, +\infty) \to \mathbb{R}$ which is the unique solution to the Cauchy's problem

$$\begin{cases} h(0) = k_0 \\ \dot{h}(t) = G(h(t)) \qquad t \geq 0 \end{cases}$$

Hence, by (1.29) and (1.5), $k := k(\cdot; k_0, c) \leq h$. So, setting

$$\bar{t} := \frac{1}{m} \log\left(\frac{\bar{k}}{k_0}\right) \text{ and } \hat{k} := \bar{k}\left(m - L - \epsilon_0\right)$$

we get, for every $t \in [0, \bar{t}]$:

$$h(t) = k_0 e^{mt}$$

and, for every $t \geq \bar{t}$:

$$\begin{aligned} h(t) &= e^{(L+\epsilon_0)t} \int_{\bar{t}}^{t} e^{-(L+\epsilon_0)s} \hat{k} \mathrm{d}s + \bar{k} e^{-(L+\epsilon_0)\bar{t}} = \frac{\hat{k} e^{-(L+\epsilon_0)\bar{t}}}{L+\epsilon_0} e^{(L+\epsilon_0)t} + \bar{k} e^{-(L+\epsilon_0)\bar{t}} - \frac{\hat{k}}{L+\epsilon_0} \\ &=: \omega_0(k_0) e^{(L+\epsilon_0)t} + \omega_1(k_0) - \frac{\hat{k}}{L+\epsilon_0} \end{aligned}$$

where by definition of $\bar{t}$ the functions ω_i satisfy:

$$\begin{aligned} \omega_0(k_0) &= \frac{\hat{k}}{L+\epsilon_0} e^{-(L+\epsilon_0)\bar{t}} = \frac{\hat{k}}{L+\epsilon_0} \left(\frac{k_0}{\bar{k}}\right)^{\frac{L+\epsilon_0}{m}} \\ \omega_1(k_0) &= \bar{k} e^{-(L+\epsilon_0)\bar{t}} = \bar{k} \left(\frac{k_0}{\bar{k}}\right)^{\frac{L+\epsilon_0}{m}}. \end{aligned}$$

Using the state equation, we deduce by the above computations of h two estimates for the integrals of c.

For every $t \in [0, \bar{t}]$ (remembering that h is increasing so that $\forall s \leq t : \, h(s) \leq \bar{k}$):

$$\begin{aligned}\int_0^t c(s)\,\mathrm{d}s &\leq k_0 + \int_0^t F(k(s))\,\mathrm{d}s \leq k_0 + \int_0^t G(h(s))\,\mathrm{d}s \\ &= k_0 + \int_0^t m k_0 e^{ms}\mathrm{d}s = k_0 e^{mt}. \end{aligned} \tag{1.31}$$

Instead, for every $t > \bar{t}$:

$$\begin{aligned}\int_0^t c(s)\,\mathrm{d}s &\leq k_0 + \int_0^{\bar{t}} G(h(s))\,\mathrm{d}s + \int_{\bar{t}}^t G(h(s))\,\mathrm{d}s \\ &\leq k_0 e^{m\bar{t}} + \int_{\bar{t}}^t \left\{(L+\epsilon_0)\,h(s) + \hat{k}\right\}\mathrm{d}s \\ &\leq \bar{k} + (t-\bar{t})\,\hat{k} + (L+\epsilon_0)\int_{\bar{t}}^t \left\{\omega_0(k_0)\,e^{(L+\epsilon_0)s} + \omega_1(k_0) - \frac{\hat{k}}{L+\epsilon_0}\right\}\mathrm{d}s \\ &\leq \bar{k} + \omega_0(k_0)\left[e^{(L+\epsilon_0)t} - e^{(L+\epsilon_0)\bar{t}}\right] + (L+\epsilon_0)(t-\bar{t})\,\omega_1(k_0) \\ &\leq \bar{k} + \omega_0(k_0)\,e^{(L+\epsilon_0)t} - \frac{\hat{k}}{L+\epsilon_0} + (L+\epsilon_0)(t-\bar{t})\,\omega_1(k_0) \end{aligned} \tag{1.32}$$

where we have used $h(s) \geq \bar{k}$ for $s \in (\bar{t}, t)$ and the fact that $k_0 e^{m\bar{t}} = \bar{k}$.

Now observe that

$$\begin{aligned} &\lim_{k_0 \to 0} \omega_0(k_0) = \lim_{k_0 \to 0} \omega_1(k_0) = 0 \\ &\lim_{k_0 \to 0} \bar{t} = \lim_{k_0 \to 0} \frac{1}{m}\log\left(\frac{\bar{k}}{k_0}\right) = +\infty. \end{aligned} \tag{1.33}$$

Hence if k_0 is small enough (say $k_0 < k^*$), we may assume $\bar{t} > 1$ and $\omega_i(k_0) \leq 1$ for $i = 0, 1$, so that (1.32) implies, for every $t > \bar{t}$:

$$\frac{\int_0^t c(s)\,\mathrm{d}s}{t} \leq \bar{k} + e^{(L+\epsilon_0)t} + (L+\epsilon_0)\frac{(t-\bar{t})}{t} \leq \bar{k} + e^{(L+\epsilon_0)t} + (L+\epsilon_0) \tag{1.34}$$

Hence, by Lemma 20, iii) , by Remark 19, and by (1.31), (1.34), the following inequality holds for every $k_0 < k^*$ and every $c \in \Lambda(k_0)$:

$$
\begin{aligned}
0 \;\leq\; U(c;k_0) \;&=\; \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c(s))\, \mathrm{d}s\mathrm{d}t \;\leq\; \rho \int_0^{+\infty} t e^{-\rho t} u\left(\frac{\int_0^t c(s)\,\mathrm{d}s}{t}\right) \mathrm{d}t \\
&\leq\; \rho \int_0^1 e^{-\rho t} u\left(\int_0^t c(s)\,\mathrm{d}s\right) \mathrm{d}t + \rho \int_1^{\bar{t}} t e^{-\rho t} u\left(\frac{k_0 e^{mt}}{t}\right) \mathrm{d}t + \\
&\quad + \rho \int_{\bar{t}}^{+\infty} t e^{-\rho t} u\left(\bar{k} + e^{(L+\epsilon_0)t} + (L+\epsilon_0)\right) \mathrm{d}t \\
&\leq\; \rho \int_0^1 e^{-\rho t} u\left(k_0 e^{mt}\right) \mathrm{d}t + \rho u\left(\frac{k_0 e^{m\bar{t}}}{\bar{t}}\right) \int_1^{\bar{t}} t e^{-\rho t} \mathrm{d}t + \\
&\quad + \rho u\left(\bar{k} + (L+\epsilon_0)\right) \int_{\bar{t}}^{+\infty} t e^{-\rho t} \mathrm{d}t + \rho \int_{\bar{t}}^{+\infty} t e^{-\rho t} u\left(e^{(L+\epsilon_0)t}\right) \mathrm{d}t \\
&\leq\; \rho u\left(k_0 e^{m}\right) \int_0^1 e^{-\rho t} \mathrm{d}t + \rho u\left(\frac{\bar{k}}{\bar{t}}\right) \frac{e^{-\rho}(1+\rho)}{\rho^2} + \\
&\quad + \rho u\left(\bar{k} + (L+\epsilon_0)\right) \int_{\bar{t}}^{+\infty} t e^{-\rho t} \mathrm{d}t + \rho \int_{\bar{t}}^{+\infty} t e^{-\rho t} u\left(e^{(L+\epsilon_0)t}\right) \mathrm{d}t
\end{aligned}
$$

where we used also the fact that the function $t \to \frac{e^{mt}}{t}$ is increasing for $t > 1$, by condition (1.30).

It follows from (1.33) and the fact that $\lim_{x \to 0} u(x) = 0$, together with Remark 3, that the above quantity tends to 0 as $k_0 \to 0$; moreover, that quantity does not depend on c.

Hence, noticing that k^* depends only on the data and m, we see that for any $\epsilon > 0$ there exists $\delta \in (0, k^*]$ such that for every $k_0 \in (0, \delta)$ and for *every* $c \in \Lambda(k_0)$:

$$
U(c;k_0) \leq \epsilon,
$$

which implies, taking the sup over $\Lambda(k_0)$, that $V(k_0) \leq \epsilon$ - and the assertion follows. $\square$

1.4 Existence of the optimal control

In this section we deal with the fundamental topic of any optimization problem: the existence of an optimal control. For any fixed $k_0 \geq 0$, we look for a control $c^* \in \Lambda(k_0)$ satisfying

$$
U(c^*;k_0) = \sup_{c \in \Lambda(k_0)} U(c;k_0) = V(k_0).
$$

We preliminary observe that the peculiar features of our problem, particularly the absence of any boundedness conditions on the admissible controls, force us to make use of this result in proving certain regularity and monotonicity properties of the value function which usually do not require such a settlement - and which we postpone for this reason.

First observe that by Theorem 22, iii) if we set $c_0 :\equiv 0$, then $U(c_0, 0) = 0 = V(0)$; hence c_0 is optimal at 0.

Let $k_0 > 0$; this will be the initial state which we will refer to during the whole section - hence the meaning of this symbol will not change in this context.

We split the construction in various steps; first we make a simple but important

Remark 23. Suppose that $(f_n)_{n\in\mathbb{N}}, f$ are functions in $L^1_{loc}([0,+\infty),\mathbb{R})$ such that for *every* $N \in \mathbb{N}$, $f_n \rightharpoonup f$ in $L^1([0,N],\mathbb{R})$. If $T > 0$, $T \in \mathbb{R}$, then it follows from the definition of weak convergence that, for $g \in L^\infty([0,T],\mathbb{R})$:

$$\int_0^T g(s) f_n(s)\,\mathrm{d}s = \int_0^{[T]+1} \chi_{[0,T]} g(s) f_n(s)\,\mathrm{d}s \to \int_0^{[T]+1} \chi_{[0,T]} g(s) f(s)\,\mathrm{d}s = \int_0^T g(s) f(s)\,\mathrm{d}s.$$

Hence $f_n \rightharpoonup f$ in $L^1([0,T],\mathbb{R})$, for every $T > 0, T \in \mathbb{R}$.

Step 1. The first step is to find a maximizing sequence of controls which are admissible at k_0 and a function $\gamma \in L^1_{loc}([0,+\infty),\mathbb{R})$, such that the sequence weakly converges to γ in $L^1([0,T],\mathbb{R})$, for every $T > 0$.

By definition of supremum, we can find a maximizing sequence; that is to say, there exist a sequence $(c_n)_{n\in\mathbb{N}} \subseteq \Lambda(k_0)$ of admissible controls satisfying:

$$\lim_{n\to+\infty} U(c_n; k_0) = V(k_0).$$

In order to apply the tools we set up at the beginning of the chapter, we need the following result.

Lemma 24. *Let $T > 0$ and $(f_n)_{n\in\mathbb{N}} \subseteq L^1_{loc}([0,+\infty),\mathbb{R})$. Suppose that there exists a constant $M(T) > 0$ such that*

$$\|f_n\|_{\infty,[0,T]} \leq M(T) \quad \forall n \in \mathbb{N}.$$

Then there exist a subsequence $(\overline{f}_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ and a function $f \in L^1([0,T],\mathbb{R})$ such that

$$\overline{f}_n \rightharpoonup f \text{ in } L^1([0,T],\mathbb{R}).$$

Proof. For every $0 \leq t_0 < t_1 \leq T$:

$$\int_{t_0}^{t_1} |f_n(s)|\,\mathrm{d}s \leq \|f_n\|_{\infty,[0,T]} \cdot (t_1 - t_0) \leq M(T) \cdot (t_1 - t_0).$$

Hence, by the fact that the family $\{(t_0,t_1) \in \mathcal{P}([0,T]) / t_0, t_1 \in [0,T]\}$ generates the Borel σ-

algebra in $[0,T]$, and by the regularity property of the Lebesgue measure, it is easy to verify that the latter relation holds for every measurable set $E \subseteq [0,T]$; that is to say

$$\int_E |f_n(s)|\, \mathrm{d}s \leq M(T) \cdot \mu(E).$$

This easily implies that the densities $\{d_n / n \in \mathbb{N}\}$ given by $d_n(E) := \int_E f_n(s)\, \mathrm{d}s$ are absolutely equicontinuous. So the thesis follows from the Dunford-Pettis criterion. Observe that the additional requirement that for any $\epsilon > 0$ there exists a compact set $K_\epsilon \subseteq [0,T]$ such that

$$\int_{[0,T]\setminus K_\epsilon} |f_n(s)|\, \mathrm{d}s \leq \epsilon \quad \forall n \in \mathbb{N}$$

is obviously satisfied. $\square$

Now we apply Lemma 9 to $(c_n)_{n\in\mathbb{N}}$ in order to find a new sequence $\left(c_n^1\right)_{n\in\mathbb{N}} \subseteq \Lambda(k_0)$ such that, for every $n \in \mathbb{N}$:

$$\begin{aligned}
&U\left(c_n^1; k_0\right) \geq U\left(c_n; k_0\right)\\
&c_n^1 = c_n \wedge N(k_0, 1) \text{ a.e. in } [0,1].
\end{aligned}$$

In particular $\left(c_n^1\right)_{n\in\mathbb{N}} \subseteq L^1_{loc}([0,+\infty), \mathbb{R})$ and $\left\|c_n^1\right\|_{\infty,[0,1]} \leq N(k_0, 1)$ for every $n \in \mathbb{N}$. Hence by Lemma 24, there exists a sequence $\left(\overline{c}_n^1\right)_{n\in\mathbb{N}}$ extracted from $\left(c_n^1\right)_{n\in\mathbb{N}}$ and a function $c^1 \in L^1([0,1], \mathbb{R})$ such that

$$\overline{c}_n^1 \rightharpoonup c^1 \text{ in } L^1([0,1], \mathbb{R}).$$

Now define, for every $n \in \mathbb{N}$:

$$c_n^2 := \left(\overline{c}_n^1\right)^2$$

where $\left(\overline{c}_n^1\right)^2$ is understood with the notation of Lemma 9.

Hence for every $n \in \mathbb{N}$:

$$\begin{aligned}
&U\left(c_n^2; k_0\right) \geq U\left(\overline{c}_n^1; k_0\right)\\
&c_n^2 = \overline{c}_n^1 \wedge N(k_0, 2) \text{ a.e. in } [0,2].
\end{aligned}$$

Again by Lemma 24, we can exhibit a subsequence $\left(\overline{c}_n^2\right)_{n\in\mathbb{N}}$ of $\left(c_n^2\right)_{n\in\mathbb{N}}$ and a function $c^2 \in L^1([0,2], \mathbb{R})$ such that

$$\overline{c}_n^2 \rightharpoonup c^2 \text{ in } L^1([0,2], \mathbb{R}).$$

Following this pattern we are able to give a recursive definition of a family $\left\{\left(\left(c_n^T\right)_{n\in\mathbb{N}}, \left(\overline{c}_n^T\right)_{n\in\mathbb{N}}, c^T\right) / T \in \mathbb{N}\right\}$ and a family of sequences of indices $\{\sigma_T(\cdot) : \mathbb{N} \to \mathbb{N} / T \in \mathbb{N}\}$

satisfying, for every $T, n \in \mathbb{N}$:

$$\begin{aligned}
&\sigma_T(\cdot) \text{ is strictly increasing and } \sigma_T(n) \geq n \\
&c_n^T \in \Lambda(k_0), \ \bar{c}_n^T = c_{\sigma_T(n)}^T \\
&U\left(c_n^{T+1}; k_0\right) \geq U\left(\bar{c}_n^T; k_0\right) \\
&c_n^{T+1} = \bar{c}_n^T \wedge N(k_0, T+1) \text{ a.e. in } [0, T+1] \\
&\bar{c}_n^T \rightharpoonup c^T \text{ in } L^1([0,T], \mathbb{R})
\end{aligned} \tag{1.35}$$

Now fix $T \in \mathbb{N}$. The above relations clearly imply that for every $n \in \mathbb{N}$ there exist sets $U_n^T, V_n^T \subseteq [0,T]$ such that $\mu\left([0,T] \setminus U_n^T\right) = \mu\left([0,T] \setminus V_n^T\right) = 0$ and

$$\begin{aligned}
&\bar{c}_n^{T+1} = c_{\sigma_{T+1}(n)}^{T+1} = \bar{c}_{\sigma_{T+1}(n)}^T \wedge N(k_0, T+1) \quad \text{in } U_n^T \\
&\bar{c}_{\sigma_{T+1}(n)}^T = c_{\sigma_T \circ \sigma_{T+1}(n)}^T \leq N(k_0, T) \quad \text{in } V_n^T
\end{aligned}$$

By the monotonicity of the function $N(\cdot,\cdot)$ in the second variable (Lemma 9) we obtain

$$\bar{c}_n^{T+1} = \bar{c}_{\sigma_{T+1}(n)}^T \quad \text{in } W_n^T := U_n^T \cap V_n^T. \tag{1.36}$$

Hence $\left(\bar{c}_n^{T+1}\right)_n$ coincides, as a sequence, with $\left(\bar{c}_{\sigma_{T+1}(n)}^T\right)_n$ in $\bigcap_n W_n^T$ - that is to say almost everywhere in $[0,T]$. By the properties of σ_{T+1} in (1.35), the latter is a subsequence of $\left(\bar{c}_n^T\right)_n$. By the essential uniqueness of the weak limit in $L^1([0,T])$ we have:

$$c^{T+1} = c^T \text{ almost everywhere in } [0,T]. \tag{1.37}$$

It remains to be constructed a maximizing sequence $(\gamma_n)_{n\in\mathbb{N}} \subseteq \Lambda(k_0)$ and a function $\gamma \in L_{loc}^1([0,+\infty), \mathbb{R})$ such that

$$\gamma_n \rightharpoonup \gamma \text{ in } L^1([0,T], \mathbb{R}) \quad \forall T > 0.$$

Definition 25. i) $\gamma : [0,+\infty) \to \mathbb{R}$ is the function

$$\gamma(t) := c^{[t]+1}(t) \quad \forall t \geq 0$$

ii) $\forall n \in \mathbb{N}$: $\gamma_n := \bar{c}_n^n$.

Now, if we consider (for any fixed $T \in \mathbb{N}$), the restriction to $[0,T]$ of the sequence $\gamma_T, \gamma_{T+1}, \gamma_{T+2}, ..$ we see that there exists a subset of $[0,T]$, with negligible complementary, in which such sequence coincides with a subsequence of $\left(\bar{c}_n^T\right)_n$. Indeed, by computations similar to those carried out after Remark 1.36 we find that:

$$
\begin{aligned}
&\gamma_T = \overline{c}_T^T \\
&\gamma_{T+1} = \overline{c}^T_{\sigma_{T+1}(T+1)} \quad \text{in } W^T_{T+1} \\
&\gamma_{T+2} = \overline{c}^T_{\sigma_{T+1}\circ\sigma_{T+2}(T+2)} \quad \text{in } W^{T+1}_{T+2} \cap W^T_{\sigma_{T+2}(T+2)} \cap [0,T] \\
&\dots
\end{aligned}
$$

Any of these sets almost coincides with $[0,T]$ (and so does the intersection); moreover by the properties of the σ_n's we have $T < \sigma_{T+1}(T+1) < \sigma_{T+1}\circ\sigma_{T+2}(T+2)$.

Now we can state the following

Proposition 26. *Let* $(\gamma_n)_{n\in\mathbb{N}}$, γ *as in Definition 25. Then* $(\gamma_n)_{n\in\mathbb{N}} \subseteq \Lambda(k_0)$, $\gamma \in L^1_{loc}([0,+\infty),\mathbb{R})$ *and*

$$
\lim_{n\to+\infty} U(\gamma_n;k_0) = V(k_0).
$$

Moreover, for every $T\in\mathbb{N}$, $(\gamma_n)_{n\geq T}$ *, as a sequence, coincides almost everywhere in* $[0,T]$ *with a subsequence of* $\left(\overline{c}_n^T\right)_{n\in\mathbb{N}}$*. Consequently*

$$
\begin{aligned}
&\|\gamma_n\|_{\infty,[0,T]} \leq N(k_0,T) \quad \forall T,n\in\mathbb{N},\ n\geq T, \\
&\gamma_n \rightharpoonup \gamma \text{ in } L^1([0,T],\mathbb{R}) \quad \forall T>0, T\in\mathbb{R}.
\end{aligned}
$$

Proof. By definition 25 and by the second condition in (1.35), $\gamma_n = c^n_{\sigma_n(n)} \in \Lambda(k_0)$.

Moreover, for every $T\in\mathbb{N}$, $\gamma = c^T$ almost everywhere in $[0,T]$; hence $\gamma\in L^1([0,T],\mathbb{R})$, which implies $\gamma\in L^1_{loc}([0,+\infty),\mathbb{R})$ because T is generic.

Now fix $n\in\mathbb{N}$. The above equality for γ_n cannot be developed in $[0,+\infty)$, but the second and third condition in (1.35) imply that the following chain of inequalities for the functional holds:

$$
\begin{aligned}
U(\gamma_n;k_0) &\geq U\left(\overline{c}^{n-1}_{\sigma_n(n)};k_0\right) = U\left(c^{n-1}_{\sigma_{n-1}\circ\sigma_n(n)};k_0\right) \\
&\geq U\left(\overline{c}^{n-2}_{\sigma_{n-1}\circ\sigma_n(n)};k_0\right) \geq \dots \geq U\left(\overline{c}^1_{\sigma_2\circ\dots\circ\sigma_n(n)};k_0\right) \\
&= U\left(c^1_{\sigma_1\circ\sigma_2\circ\dots\circ\sigma_n(n)};k_0\right) \geq U\left(c_{\sigma_1\circ\sigma_2\circ\dots\circ\sigma_n(n)};k_0\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
|U(\gamma_n;k_0) - V(k_0)| &= V(k_0) - U(\gamma_n;k_0) \\
&\leq V(k_0) - U\left(c_{\sigma_1\circ\sigma_2\circ\dots\circ\sigma_n(n)};k_0\right) \\
&= \left|U\left(c_{\sigma_1\circ\sigma_2\circ\dots\circ\sigma_n(n)};k_0\right) - V(k_0)\right|;
\end{aligned}
$$

since $\sigma_1 \circ \cdots \circ \sigma_n (n) \geq n$, the fact that $(\gamma_n)_{n\in\mathbb{N}}$ is a maximizing sequence follows from the fact that, by assumption, $(c_n)_{n\in\mathbb{N}}$ is a maximizing sequence.

Now fix $T \in \mathbb{N}$ and observe that the argument developed after Definition 25 inductively shows that, for every $k \in \mathbb{N}$:

$$\begin{aligned} &\gamma_{T+k} = \bar{c}^T_{\nu_T(k)} \quad (\text{where } \nu_T(k) = \sigma_{T+1} \circ \cdots \circ \sigma_{T+k}(T+k)) \\ &\text{in } [0,T] \cap W^{T+k-1}_{T+k} \cap \bigcap_{p=1}^{k-1} W^{T+k-1-p}_{\bigodot_{j=0}^{p-1} \sigma_{T+k-(p-1-j)}} \end{aligned} \tag{1.38}$$

Since by construction any set of the form $[0,T] \setminus W^{T+k-1-p}_m$, $p = 0, .., k-1$ has null Lebesgue measure, the above relation imply $\|\gamma_{T+k}\|_{\infty,[0,T]} = \left\|\bar{c}^T_{\nu_T(k)}\right\|_{\infty,[0,T]}$. This quantity is bounded above by $N(k_0, T)$, by the second and fourth condition in (1.35).

Moreover, the intersection for $k \in \mathbb{N}$ of the sets in (1.38) has negligible complementary in $[0,T]$; since ν_T is strictly increasing, this implies that $(\gamma_n)_{n\geq T}$, as a sequence, coincides almost everywhere in $[0,T]$ with a subsequence of $\left(\bar{c}^T_n\right)_{n\in\mathbb{N}}$. In particular $\gamma_n \rightharpoonup \gamma$ in $L^1([0,T], \mathbb{R})$ by the last condition in (1.35) and by the fact that $\gamma = c^T$ almost everywhere in $[0,T]$.

As this holds for every $T \in \mathbb{N}$, it is a consequence of Remark 23 that it must hold for every real number $T > 0$. □

The first step is then accomplished.

Step 2. The next step is to show that γ is admissible at k_0. For this purpose, it is enough to prove the following

Proposition 27. *Let $T > 0$. Hence $\gamma \geq 0$ almost everywhere in $[0,T]$, and, for every $t \in [0,T]$, $k(t; k_0, \gamma) \geq 0$.*

Proof. It is well known that the weak convergence of $(\gamma_n)_{n\in\mathbb{N}}$ to γ in $L^1([0,T], \mathbb{R})$, ensured by Proposition 26, implies that

$$\liminf_{n\to+\infty} \gamma_n(t) \leq \gamma(t) \leq \limsup_{n\to+\infty} \gamma_n(t) \text{ for almost every } t \in [0,T]. \tag{1.39}$$

By proposition 26 we also have

$$\forall n \in \mathbb{N} : \text{for almost every } t \in [0,T] : 0 \leq \gamma_n(t) \leq N(k_0, T). \tag{1.40}$$

We can interchange the quantifiers in the previous relation, since a numerable intersection of full-measure sets is a full-measure set. Consequently, taking the intersection with the set where

(1.39) holds, we have

$$
\begin{aligned}
&0 \leq \gamma \leq N\left(k_0, T\right) \quad \text{a.e. in } [0,T] \\
&k\left(\cdot; k_0, 0\right) \geq \kappa \geq k\left(\cdot; k_0, N\left(k_0, T\right)\right) \quad \text{in } [0,T] \\
&k\left(\cdot; k_0, 0\right) \geq \kappa_n \geq k\left(\cdot; k_0, N\left(k_0, T\right)\right) \quad \text{in } [0,T], \forall n \in \mathbb{N}
\end{aligned}
$$

where $\kappa := k\left(\cdot; k_0, \gamma\right)$ and $\kappa_n := k\left(\cdot; k_0, \gamma_n\right)$; observe that the constant control $N\left(k_0, T\right)$ need not be admissible. The second relation follows from the first by Remark 5 and the third relation follows directly from (1.40). Hence:

$$
\left|\kappa - \kappa_n\right| \leq k\left(\cdot; k_0, 0\right) - k\left(\cdot; k_0, N\left(k_0, T\right)\right) \quad \text{in } [0,T], \forall n \in \mathbb{N}. \tag{1.41}
$$

Fix $n \in \mathbb{N}$. Subtracting the state equation for κ from the state equation for κ_n, we obtain, for every $t \in [0,T]$:

$$
\dot{\kappa_n}(t) - \dot{\kappa}(t) \;=\; F\left(\kappa_n(t)\right) - F\left(\kappa(t)\right) - \left[\gamma_n(t) - \gamma(t)\right] \;=\; h_n(t)\left[\kappa_n(t) - \kappa(t)\right] - \left[\gamma_n(t) - \gamma(t)\right],
$$

where h_n is the (continuous) function defined taking $k_1 = \kappa_n$ and $k_2 = \kappa$ in Remark 5.

Integrating both sides of this equation between 0 and t, then taking absolute values leads to:

$$
\left|\kappa_n(t) - \kappa(t)\right| \;\leq \int_0^t \left|h_n(s)\right| \left|\kappa_n(s) - \kappa(s)\right| \mathrm{d}s \;+ \left|\int_0^t \left[\gamma(s) - \gamma_n(s)\right] \mathrm{d}s\right|. \tag{1.42}
$$

Observe that, for every $s \in [0,t]$:

$$
\left|h_n(s)\right| \left|\kappa_n(s) - \kappa(s)\right| \;\leq\; \overline{M}\left[k\left(s; k_0, 0\right) - k\left(s; k_0, N\left(k_0, T\right)\right)\right],
$$

by Remark 7 and by (1.41).

This holds for every $n \in \mathbb{N}$ and for every fixed $t \in [0,T]$. Since the function of s on the right hand side obviously belongs to $L^1([0,t])$ we obtain from (2.44) (remembering that $\gamma_n \rightharpoonup \gamma$ in $L^1([0,t])$):

$$
\begin{aligned}
\limsup_{n\to+\infty} \left|\kappa_n(t) - \kappa(t)\right| \;&\leq\; \limsup_{n\to+\infty} \int_0^t \left|h_n(s)\right| \left|\kappa_n(s) - \kappa(s)\right| \mathrm{d}s \\
&\leq\; \int_0^t \limsup_{n\to+\infty} \left|h_n(s)\right| \left|\kappa_n(s) - \kappa(s)\right| \mathrm{d}s \\
&\leq\; \int_0^t \overline{M} \limsup_{n\to+\infty} \left|\kappa_n(s) - \kappa(s)\right| \mathrm{d}s.
\end{aligned} \tag{1.43}
$$

Hence by Gronwall's inequality:

$$\limsup_{n\to+\infty} |\kappa_n(t) - \kappa(t)| = 0,$$

for every $t \in [0, T]$. This is equivalent to

$$\lim_{n\to+\infty} \kappa_n = \kappa \quad \text{in } [0, T].$$

Since any κ_n is non-negative in $[0, T]$, the second assertion of the theorem is also proved. $\square$

Remark 28. The argument behind (1.43) goes as follows. Let

$$a_n := \sup_{j\geq n} |h_j| |\kappa_j - \kappa|.$$

Then

$$\begin{aligned} |a_n| = a_n &\leq \overline{M} \left[k(\cdot; k_0, 0) - k(\cdot; k_0, N(k_0, T))\right] \quad \forall n \in \mathbb{N} \\ a_n \downarrow_{n\to+\infty} &\limsup_{m\to+\infty} |h_m| |\kappa_m - \kappa| \quad \text{in } [0, t]. \end{aligned}$$

Then by Dominated Convergence:

$$\inf_{n\in\mathbb{N}} \int_0^t a_n(s)\,\mathrm{d}s = \int_0^t \limsup_{m\to+\infty} |h_m(s)| |\kappa_m(s) - \kappa(s)|\,\mathrm{d}s.$$

Moreover for every $n \in \mathbb{N}$ and every $i \geq n$:

$$a_n = \sup_{j\geq n} |h_j| |\kappa_j - \kappa| \geq |h_i| |\kappa_i - \kappa|,$$

which implies, passing to the integrals and then taking the sup for $i \geq n$:

$$\int_0^t a_n(s)\,\mathrm{d}s \geq \sup_{i\geq n} \int_0^t |h_i(s)| |\kappa_i(s) - \kappa(s)|\,\mathrm{d}s \quad \forall n \in \mathbb{N}.$$

Hence, passing to the inf for $n \in \mathbb{N}$:

$$\inf_{n\in\mathbb{N}} \int_0^t a_n(s)\,\mathrm{d}s \geq \limsup_{m\to+\infty} \int_0^t |h_m(s)| |\kappa_m(s) - \kappa(s)|\,\mathrm{d}s.$$

As a consequence of Proposition 27, γ is almost everywhere non-negative in $[0, +\infty)$ and $k(\cdot; k_0, \gamma)$ is everywhere non-negative in $[0, +\infty)$ - which precisely means that $\gamma \in \Lambda(k_0)$. Hence the second step is also ended.

Step 3. Now we are going to define the control which is optimal at k_0. In order to do this, we need to extract a subsequence from $(\gamma_n)_{n\in\mathbb{N}}$ because the weak convergence to γ in the intervals could not be enough to ensure that $\lim_{n\to+\infty} U(\gamma_n; k_0) = U(\gamma; k_0)$; we will also need the admissibility of γ. By the penultimate assertion stated in Proposition 26, and by the monotonicity of u, we have:

$$\|u(\gamma_n)\|_{\infty,[0,1]} \leq u(N(k_0, 1)) \quad \forall n \in \mathbb{N}.$$

Hence by Lemma 24, there exists a function $f^1 \in L^1([0,1], \mathbb{R})$ and a sequence $(u(\gamma_{1,n}))_{n\in\mathbb{N}}$ extracted from $(u(\gamma_n))_{n\in\mathbb{N}}$, such that

$$u(\gamma_{1,n}) \rightharpoonup f^1 \text{ in } L^1([0,1], \mathbb{R}).$$

Again by Proposition 26 and the monotonicity of u,

$$\|u(\gamma_{1,n})\|_{\infty,[0,2]} \leq u(N(k_0, 2)) \quad \forall n \in \mathbb{N}$$

which implies by Lemma 24 the existence of $f^2 \in L^1([0,2], \mathbb{R})$ and of a sequence $(u(\gamma_{2,n}))_{n\in\mathbb{N}}$ extracted from $(u(\gamma_{1,n}))_{n\in\mathbb{N}}$ such that

$$u(\gamma_{2,n}) \rightharpoonup f^2 \text{ in } L^1([0,2], \mathbb{R});$$

in particular $f^2 = f^1$ almost everywhere in $[0,1]$ by the essential uniqueness of the weak limit. Going on this way we see that there exists a family $\left\{\left(u(\gamma_{T,n})_{n\in\mathbb{N}}, f^T\right)/T \in \mathbb{N}\right\}$ satisfying, for every $T \in \mathbb{N}$:

$$\begin{aligned}
&\|u(\gamma_{T,n})\|_{\infty,[0,T]} \leq u(N(k_0, T)) \quad \forall n \in \mathbb{N} \\
&(u(\gamma_{T+1,n}))_{n\in\mathbb{N}} \text{ is extracted from } (u(\gamma_{T,n}))_{n\in\mathbb{N}} \\
&f^{T+1} = f^T \text{ almost everywhere in } [0,T] \\
&u(\gamma_{T,n}) \rightharpoonup f^T \text{ in } L^1([0,T], \mathbb{R}).
\end{aligned}$$

Hence, for every $T \in \mathbb{N}$, the sequence $(u(\gamma_{n,n}))_{n\geq T}$ is extracted from $(u(\gamma_{T,n}))_{n\in\mathbb{N}}$. If we define $f(t) := f^{[t]+1}(t)$, then $f = f^T$ almost everywhere in $[0,T]$. So

$$u(\gamma_{n,n}) \rightharpoonup f \text{ in } L^1([0,T], \mathbb{R}) \quad \forall T > 0. \tag{1.44}$$

by construction and by Remark 23. This implies that

$$0 \leq \liminf_{n \to +\infty} u\left(\gamma_{n,n}(t)\right) \leq f(t)$$

for almost every $t \in \mathbb{R}$.

Now define $c^* : [0, +\infty) \to \mathbb{R}$ as

$$c^*(t) := \begin{cases} u^{-1}(f(t)) & \text{if } f(t) \geq 0 \\ 0 & \text{if } f(t) < 0. \end{cases}$$

Obviously $c^* \geq 0$ everywhere in $[0, +\infty)$. Moreover, again by the properties of the weak convergence, for any $T \in \mathbb{N}$ and for almost every $t \in [0, T]$:

$$f(t) \leq \limsup_{n \to +\infty} u\left(\gamma_{n,n}(t)\right) \leq u\left(N\left(k_0, T\right)\right).$$

This implies, together with the fact that u^{-1} is increasing, that c^* is bounded above by $N(k_0, T)$ almost everywhere in $[0, T]$. As this holds for every $T \in \mathbb{N}$,

$$c^* \in L^\infty_{loc}\left([0, +\infty), \mathbb{R}\right). \tag{1.45}$$

To complete the proof of the admissibility of c^*, we show that $c^* \leq \gamma$ almost everywhere in $[0, +\infty)$.

Fix $T > 0$ and let $t_0 \in [0, T]$ be a Lebesgue point for both f and γ in $[0, T]$; then take $t_1 \in (t_0, T)$. By the concavity of u and by Jensen inequality:

$$\frac{\int_{t_0}^{t_1} u\left(\gamma_{n,n}(s)\right) \mathrm{d}s}{t_1 - t_0} \leq u\left(\frac{\int_{t_0}^{t_1} \gamma_{n,n}(s)\, \mathrm{d}s}{t_1 - t_0}\right) \tag{1.46}$$

Observe that $\left(\gamma_{n,n}\right)_{n \geq 1}$ is a subsequence of $\left(\gamma_{1,n}\right)_{n \in \mathbb{N}}$, which is in its turn extracted from $\left(\gamma_n\right)_{n \in \mathbb{N}}$. Hence $\gamma_{n,n} \rightharpoonup \gamma$ in $L^1\left([0, T], \mathbb{R}\right)$, which implies $\lim_{n \to +\infty} \int_{t_0}^{t_1} \gamma_{n,n}(s)\, \mathrm{d}s = \int_{t_0}^{t_1} \gamma(s)\, \mathrm{d}s$. So taking the limit for $n \to +\infty$ in (1.46), by the continuity of u and by (1.44), we have:

$$\frac{\int_{t_0}^{t_1} f(s)\, \mathrm{d}s}{t_1 - t_0} \leq u\left(\frac{\int_{t_0}^{t_1} \gamma(s)\, \mathrm{d}s}{t_1 - t_0}\right).$$

As t_0 is a Lebesgue point for both f and γ in $[0, T]$, we can take the limit for $t_1 \to t_0$ in the previous inequality and get $f(t_0) \leq u\left(\gamma(t_0)\right)$.

By the Lebesgue Point Theorem, this argument works for almost every $t_0 \in [0, T]$. So by the monotonicity of u^{-1} we deduce

$$c^* \leq \gamma \text{ almost everywhere in } [0, T].$$

Because T is generic, we have by (1.5): $k(t; k_0, c^*) \geq k(t; k_0, \gamma)$ for every $t \in \mathbb{R}$. Hence by the admissibility of γ at k_0, $k(\cdot; k_0, c^*) \geq 0$. This implies, together with (1.45) and $c^* \geq 0$ in $[0, +\infty)$,

$$c^* \in \Lambda(k_0).$$

Finally, observe that by Lemma 20 we can apply the Dominated Convergence Theorem to the functions $t \to e^{-\rho t} \int_0^t u(\gamma_{n,n}(s)) \, \mathrm{d}s$, $n \in \mathbb{N}$.

Hence, using the functional form established in the same Lemma, part iii), by Proposition 26, by the fact that $(\gamma_{n,n})_{n \in \mathbb{N}}$ is extracted from $(\gamma_n)_{n \in \mathbb{N}}$, and by (1.44):

$$\begin{aligned}
V(k_0) &= \lim_{n \to +\infty} U(\gamma_n; k_0) = \lim_{n \to +\infty} U(\gamma_{n,n}; k_0) \\
&= \lim_{n \to +\infty} \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(\gamma_{n,n}(s)) \, \mathrm{d}s\mathrm{d}t \\
&= \rho \int_0^{+\infty} e^{-\rho t} \limsup_{n \to +\infty} \int_0^t u(\gamma_{n,n}(s)) \, \mathrm{d}s\mathrm{d}t \\
&= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t f(s) \, \mathrm{d}s\mathrm{d}t \\
&= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c^*(s)) \, \mathrm{d}s\mathrm{d}t = U(c^*; k_0).
\end{aligned}$$

So we have proved the following

Theorem 29. *For every $k_0 \geq 0$ there exists $c^* \in \Lambda(k_0)$ which is optimal at k_0 and everywhere non-negative in $[0, +\infty)$, satisfying:*

$$c^* \in L^\infty_{loc}([0, +\infty), \mathbb{R}).$$

1.5 Lower bounds for optimal strategies

Theorem 29 provides a local upper bound for optimal controls, which can be considered independent of the control itself pursuant to Lemma 9. In the present section we prove that both optimal controls and optimal trajectories are striclty positive, for the non-trivial case $k_0 > 0$.

Lemma 30. *The function $g : (0, +\infty) \to (0, +\infty)$ defined in Lemma 6 satisfies:*

$$\lim_{x \to 0} g(x) = +\infty.$$

Consequently, g is surjective.

Proof. First we remind that

$$g(x) := \min \{y \geq 0 / (x, y) \in K_{u'}\}, \text{ where}$$
$$K_{u'} = \bigcap \left\{K \subseteq \mathbb{R}^2 / K \text{ is closed and convex and } K \supseteq \mathit{SupGraph}(u')\right\}.$$

We have already proved that g is convex (hence, continuous), strictly positive and strictly decreasing. Furthermore, $g(x) \to 0$ as $x \to +\infty$ since $g \leq u'$. Hence, the property $\lim_{x \to 0} g(x) = +\infty$ will imply that g is surjective.

Let $M > 0$ and take $\epsilon > 0$ such that $u'(\epsilon) > M$ and

$$\epsilon \frac{u'(\epsilon)}{u'(\epsilon) - M} < (u')^{-1}(M).$$

Then define

$$f(x) := \begin{cases} \frac{M - u'(\epsilon)}{\epsilon}(x - \epsilon) + M & \text{if } x \in \left[0, \epsilon \frac{u'(\epsilon)}{u'(\epsilon) - M}\right] \\ 0 & \text{if } x \geq \epsilon \frac{u'(\epsilon)}{u'(\epsilon) - M}. \end{cases}$$

Then $u' \geq f$, since $x \leq \epsilon \frac{u'(\epsilon)}{u'(\epsilon) - M}$ implies $u'(x) > M$ and

$$u'(x) \geq f(x) \iff (u'(x) - M)\epsilon \geq (u'(\epsilon) - M)(\epsilon - x).$$

This means that

$$\mathit{SupGraph}(u') \subseteq \mathit{SupGraph}(f),$$

and obviously $\mathit{SupGraph}(f)$ is a convex set. Hence $K_{u'} \subseteq \mathit{SupGraph}(f)$. Since $(\epsilon, g(\epsilon)) \in K_{u'}$ we have $g(\epsilon) \geq f(\epsilon) = M$. Since g is decreasing, we have

$$g^{-1}([M, +\infty)) \supseteq (0, \epsilon).$$

$\square$

Proposition 31. *Let $k_0 > 0$ and $c^* \in \Lambda(k_0)$ optimal at k_0. Then*

$$c^*(t) \geq g^{-1}\left(\rho e^{\rho t}\frac{V(k_0+1)}{F(k_0)}\right) \qquad \text{for a.e. } t \geq 0,$$

where $g : (0,+\infty) \to (0,+\infty)$ is the strictly decreasing function in Lemmas 6 and 30.

Furthermore, the optimal trajectory $k(\cdot; k_0, c^*)$ is strictly positive in $[0,+\infty)$ and, if the minimum value for an interval $[0,T]$ is attained in $[0,T)$ then:

$$k(t; k_0, c^*) \geq (g \circ F)^{-1}\left(\rho e^{\rho T}\frac{V(k_0+1)}{F(k_0)}\right) \quad \forall t \in [0,T].$$

In particular, for every $T > 0$ there exists $\mu(T) > 0$ such that $c^* \geq \mu(T)$ almost everywhere in $[0,T]$ and $k(\cdot; k_0, c^*) \geq \mu(T)$ in $[0,T]$.

Proof. Fix $k_0 > 0$, $c^* \in \Lambda(k_0)$ and optimal, $0 \leq t_1 < t_2 \leq T$.
We prove a lower $T-$estimate for $⨍_{t_1}^{t_2} c^*(t)\,\mathrm{d}t$, using the incremental ratios of V.
Fix $k_1 > k_0$ and define

$$\tilde{c} := c^* + H \cdot \chi_{(t_1,t_2)}, \quad H = \frac{k_1 - k_0}{t_2 - t_1}.$$

Then $\tilde{c} \in \Lambda(k_1)$ by Remark (13). By the concavity of u:

$$\begin{aligned} V(k_1) - V(k_0) &\geq U(k_1; \tilde{c}) - U(k_0; c^*) \\ &= \int_{t_1}^{t_2} e^{-\rho t}\left[u(\tilde{c}(t)) - u(c^*(t))\right]\mathrm{d}t \\ &\geq H\int_{t_1}^{t_2} e^{-\rho t} u'(\tilde{c}(t))\,\mathrm{d}t \\ &\geq H e^{-\rho t_2}\int_{t_1}^{t_2} g(\tilde{c}(t))\,\mathrm{d}t. \end{aligned}$$

Hence by Jensen's inequality:

$$\begin{aligned} V(k_1) - V(k_0) &\geq H e^{-\rho t_2}(t_2 - t_1)\, g\left(⨍_{t_1}^{t_2} \tilde{c}(t)\,\mathrm{d}t\right) \\ &= H e^{-\rho t_2}(t_2 - t_1)\, g\left(⨍_{t_1}^{t_2} c^*(t)\,\mathrm{d}t + H\right) \\ &= (k_1 - k_0)\, e^{-\rho t_2} g\left(⨍_{t_1}^{t_2} c^*(t)\,\mathrm{d}t + \frac{k_1 - k_0}{t_2 - t_1}\right) \end{aligned}$$

$$\iff$$

$$\frac{V(k_1) - V(k_0)}{k_1 - k_0} e^{\rho t_2} \geq g\left(⨍_{t_1}^{t_2} c^*(t)\,\mathrm{d}t + \frac{k_1 - k_0}{t_2 - t_1}\right). \tag{1.47}$$

Now take $\gamma^* \in \Lambda(k_1)$ and *optimal*, and $\underset{\sim}{\gamma} \in \Lambda(k_0)$ like in Remark (15). Namely:

$$\underset{\sim}{\gamma} := 0\chi_{[0,T(k_0,k_1)]} + \gamma^*(\cdot - T(k_0,k_1))\,\chi_{(T(k_0,k_1),+\infty)};$$

we recall that $k(T(k_0,k_1);k_0,0) = k_1$. Hence

$$\begin{aligned} V(k_1) - V(k_0) &\le U(k_1;\gamma^*) - U\left(k_0;\underset{\sim}{\gamma}\right) \\ &= U(k_1;\gamma^*)\left(1 - e^{-\rho T(k_0,k_1)}\right) \\ &\le \rho V(k_1)\,T(k_0,k_1) \\ &\le \rho V(k_1)\,\frac{k_1-k_0}{F(k_0)}. \end{aligned} \tag{1.48}$$

(Observe that the relation $T(k_0,k_1) \le (k_1-k_0)/F(k_0)$ is a consequence of Lagrange's Theorem). Putting together (1.47) and (1.48) we obtain, for every $k_1 > k_0 > 0$:

$$g\left(\fint_{t_1}^{t_2} c^*(t)\,\mathrm{dt} + \frac{k_1-k_0}{t_2-t_1}\right) \le \frac{V(k_1)-V(k_0)}{k_1-k_0}e^{\rho t_2} \le \rho e^{\rho t_2}\frac{V(k_1)}{F(k_0)}.$$

Better, since V is (strictly) increasing:

$$g\left(\fint_{t_1}^{t_2} c^*(t)\,\mathrm{dt} + \frac{k_1-k_0}{t_2-t_1}\right) \le \rho e^{\rho t_2}\frac{V(k_0+1)}{F(k_0)} \quad \forall k_1 \in (k_0,k_0+1)\,.$$

Note that the quantity at the right hand side is *strictly* positive by the properties of F and V; moreover, the function $g:(0,+\infty)\to(0,+\infty)$ is a bijection. Hence we can invert the relation, thus obtaining:

$$\fint_{t_1}^{t_2} c^*(t)\,\mathrm{dt} + \frac{k_1-k_0}{t_2-t_1} \ge g^{-1}\left(\rho e^{\rho t_2}\frac{V(k_0+1)}{F(k_0)}\right) \quad \forall k_1 \in (k_0,k_0+1)\,,$$

which implies

$$\begin{aligned} \fint_{t_1}^{t_2} c^*(t)\,\mathrm{dt} &\ge g^{-1}\left(\rho e^{\rho t_2}\frac{V(k_0+1)}{F(k_0)}\right) \\ &\ge g^{-1}\left(\rho e^{\rho T}\frac{V(k_0+1)}{F(k_0)}\right) \\ &> 0 \end{aligned}$$

Due to Lebesgue Point Theorem, we can assume up to a zero measure set that t_1 is a Lebesgue point for c^* in $[0,T]$, and we deduce:

$$c^* \geq g^{-1}\left(\rho e^{\rho T}\frac{V(k_0+1)}{F(k_0)}\right) \quad \text{a.e. in } [0,T],$$

passing to the limit in the weaker estimate.

Since g is continuous, we actually deduce the pointwise estimate:

$$c^*(t) \geq g^{-1}\left(\rho e^{\rho t}\frac{V(k_0+1)}{F(k_0)}\right) \quad \text{for a.e. } t \geq 0.$$

We an additional simple "derivative-integral-derivative" argument, we deduce that an analogous property holds for $k^* = k(\cdot; k_0, c^*)$.

Let t_0 be a point of minimum of k^* in $[0,T]$. If $t_0 < T$ we have, for every $t \in (t_0, T]$ we have:

$$\begin{aligned} 0 \leq \frac{k^*(t) - k^*(t_0)}{t - t_0} &= ⨍_{t_0}^{t} \left[F(k^*(s)) - c^*(s)\right] \mathrm{ds} \\ &\leq ⨍_{t_0}^{t} F(k^*(s))\, \mathrm{ds} - g^{-1}\left(\rho e^{\rho T}\frac{V(k_0+1)}{F(k_0)}\right). \end{aligned}$$

Passing to the limit for $t \to t_0^+$:

$$\begin{aligned} & g^{-1}\left(\rho e^{\rho T}\frac{V(k_0+1)}{F(k_0)}\right) \leq F(k^*(t_0)) \\ & \implies k^*(t_0) \geq (g \circ F)^{-1}\left(\rho e^{\rho T}\frac{V(k_0+1)}{F(k_0)}\right) > 0. \end{aligned}$$

If $t_0 = T$ then only two cases are possible: T is a point of minimum for k^* also in $[0, T+1]$, or $k^*(T) > k^*(T+\epsilon) \geq 0$ for some $\epsilon \in (0,1)$. In the first case, we obtain

$$k^*(T) \geq (g \circ F)^{-1}\left(\rho e^{\rho(T+1)}\frac{V(k_0+1)}{F(k_0)}\right);$$

the minimum value $k^*(t_0)$ is strictly positive, anyway. □

1.6 Further properties of the value function: regularity and monotonicity

Now it is possible to establish some regularity and monotonicity properties of the value function, with the help of optimal controls. The next theorem uses the monotonicity with respect to the first variable of the function defined in Lemma 9.

Theorem 32. *The value function $V : [0, +\infty) \to \mathbb{R}$ satisfies:*

i) V is strictly increasing in $[0, +\infty)$.

ii) For every $k_0 > 0$, there exists $C(k_0), \delta > 0$ such that for every $h \in (-\delta, \delta)$:

$$\frac{V(k_0 + h) - V(k_0)}{h} \geq C(k_0)$$

iii) V is Lipschitz-continuous in every closed sub-interval of $(0, +\infty)$.

Proof. i) Let $0 < k_1$. Set $c \in (0, F(k_1)]$ and $c_1 \equiv c$ in $[0, +\infty)$; hence by Proposition 11 and by Theorem 22,

$$V(0) = 0 < \frac{u(c)}{\rho} = U(c_1; k_1) \leq V(k_1).$$

The implication $0 < k_0 < k_1 \implies V(k_0) < V(k_1)$ follows from point ii).

ii) We split the proof in two parts.

First, take $k_0, h > 0$, c optimal at k_0 and set $k_1 := k_0 + h$. Because $k_1 > k_0$ we can choose $\underline{c}^{k_1 - k_0} = \underline{c}^h \in \Lambda(k_0 + h)$ as in Lemma 10. Hence

$$V(k_0 + h) - V(k_0) \quad \geq \quad U\left(\underline{c}^h; k_0 + h\right) - U(c; k_0) \;\geq\; u'(N(k_0, h) + 1) \int_0^h e^{-\rho t} \mathrm{d}t$$

Now, by the fact that $\lim_{h \to 0} \frac{1}{h} \int_0^h e^{-\rho t} \mathrm{d}t = 1$ and that $N(k_0, \cdot)$ is increasing, there exists $\delta > 0$ such that, for any $h \in (0, \delta)$:

$$\frac{V(k_0 + h) - V(k_0)}{h} \quad \geq \quad u'(N(k_0, h) + 1) \frac{\int_0^h e^{-\rho t} \mathrm{d}t}{h} \quad \geq \quad \frac{u'(N(k_0, 1) + 1)}{2} =: C(k_0)$$

Now fix $k_0 > 0$, $h < 0$ and c optimal at $k_0 + h$.

Then again take $\underline{c}^{k_0 - (k_0 + h)} = \underline{c}^{-h} \in \Lambda(k_0)$ as in Lemma 10. Hence

$$\begin{aligned} V(k_0 + h) - V(k_0) \quad & \leq \quad U(c; k_0 + h) - U\left(\underline{c}^{-h}; k_0\right) \\ & \leq \quad -u'(N(k_0 + h, -h) + 1) \int_0^{-h} e^{-\rho t} \mathrm{d}t. \end{aligned}$$

We can assume that $-\frac{1}{h} \int_0^{-h} e^{-\rho t} \mathrm{d}t \geq \frac{1}{2}$ for $-\delta < h < 0$. Hence, by the monotonicity of $N(\cdot, \cdot)$ in both variables, for every $h \in (-\delta, 0)$:

$$\frac{V(k_0 + h) - V(k_0)}{h} \geq \frac{u'(N(k_0 + h, -h) + 1)}{2} \geq \frac{u'(N(k_0, 1) + 1)}{2} = C(k_0).$$

iii) Let $0 < k_0 < k_1$. We need a reverse inequality for $V(k_1) - V(k_0)$, so take $c_1 \in \Lambda(k_1)$

optimal at k_1. In order to define the proper $c_0 \in \Lambda(k_0)$, observe that the orbit $k = k(\cdot; k_0, 0)$ (with null control) satisfies $\dot{k} = F(k)$. With an argument similar to the one used in Proposition 11 we can see that $\dot{k}(t) > F(k_0) > 0$ for every $t > 0$, and so $\lim_{t \to +\infty} k(t) = +\infty$.

Then by Darboux's property there exists $\bar{t} > 0$ such that $k(\bar{t}) = k_1$. Observe that, since k and F are strictly increasing functions, $\dot{k}$ must also be strictly increasing. Hence applying Lagrange's theorem to k gives for some $\xi \in (0, \bar{t})$:

$$k_1 - k_0 = k(\bar{t}) - k(0) = \bar{t} \cdot \dot{k}(\xi) > \bar{t}\dot{k}(0) = \bar{t}F(k_0) \tag{1.49}$$

Now define

$$c_0(t) := \begin{cases} 0 & \text{if } t \in [0, \bar{t}] \\ c_1(t - \bar{t}) & \text{if } t > \bar{t} \end{cases}$$

It is easy to check that $c_0 \in \Lambda(k_0)$, because

$$\begin{aligned} & k(t; k_0, c_0) = k(t; k_0, 0) > 0 \quad \forall t \in [0, \bar{t}] \\ & k(t + \bar{t}; k_0, c_0) = k(t; k_1, c_1) \geq 0 \quad \forall t \geq 0 \end{aligned}$$

by the uniqueness of the orbit; as far as the second equality is concerned, observe that both orbits pass through $(0, k_1)$ and satisfy the differential equation controlled with c_1 for $t > 0$. Hence by (1.49):

$$\begin{aligned} V(k_1) - V(k_0) &\leq U(c_1; k_1) - U(c_0; k_0) = \int_0^{+\infty} e^{-\rho t} [u(c_1(t)) - u(c_0(t))] \, \mathrm{d}t \\ &= \int_0^{+\infty} e^{-\rho t} u(c_1(t)) \, \mathrm{d}t - \int_{\bar{t}}^{+\infty} e^{-\rho t} u(c_1(t - \bar{t})) \, \mathrm{d}t \\ &= \int_0^{+\infty} e^{-\rho t} u(c_1(t)) \, \mathrm{d}t - \int_0^{+\infty} e^{-\rho(s+\bar{t})} u(c_1(s)) \, \mathrm{d}s \\ &= \left(1 - e^{-\rho\bar{t}}\right) U(c_1; k_1) = \left(1 - e^{-\rho\bar{t}}\right) V(k_1) \leq \rho\bar{t}V(k_1) < \rho V(k_1) \frac{k_1 - k_0}{F(k_0)} \end{aligned}$$

So by the monotonicity of V and F we have, for $a \leq k_0 < k_1 \leq b$:

$$V(k_1) - V(k_0) \leq \rho \frac{V(b)}{F(a)} (k_1 - k_0).$$

□

1.7 Dynamic Programming

1.7.1 Dynamic Programming Principle and characterization of optimal controls

In this section we study the properties of the value function as a solution to Bellman and Hamilton-Jacobi-Bellman equations.

First observe that we can translate an orbit by translating the control, according to the next remark.

Remark 33 (Translation of the orbit). For every $k_0, \tau \geq 0$ and every $c \in L^1_{loc}[(0, +\infty), \mathbb{R})$:

$$k(\cdot; k(\tau; k_0, c), c(\cdot + \tau)) = k(\cdot + \tau; k_0, c)$$

by the uniqueness of the orbit. In particular, if $c \in \Lambda(k_0)$ then $c(\cdot + \tau) \in \Lambda(k(\tau; k_0, c))$.

The first step consists in proving a suitable version of Dynamic Programming Principle.

Theorem 34. *For every $\tau > 0$, the value function $V : [0, +\infty) \to \mathbb{R}$ satisfies the following functional equation:*

$$\forall k_0 \geq 0 : \mathrm{v}(k_0) = \sup_{c \in \Lambda(k_0)} \left\{ \int_0^\tau e^{-\rho t} u(c(t))\, dt + e^{-\rho\tau} \mathrm{v}(k(\tau; k_0, c)) \right\} \tag{1.50}$$

in the unknown $\mathrm{v} : [0, +\infty) \to \mathbb{R}$.

Proof. Fix $\tau > 0$ and $k_0 \geq 0$, and set

$$\sigma(\tau, k_0) := \sup_{c \in \Lambda(k_0)} \left\{ \int_0^\tau e^{-\rho t} u(c(t))\, \mathrm{d}t + e^{-\rho\tau} V(k(\tau; k_0, c)) \right\}.$$

We prove that

$$\sigma(\tau, k_0) = \sup_{c \in \Lambda(k_0)} U(c; k_0).$$

In the first place, we show that $\sigma(\tau, k_0)$ is an upper bound of $\{U(c; k_0) \,/\, c \in \Lambda(k_0)\}$.

Fix $c \in \Lambda(k_0)$; then by Remark 33 $c(\cdot + \tau) \in \Lambda(k(\tau; k_0, c))$; hence

$$\begin{aligned}
\sigma(\tau, k_0) &\geq \int_0^\tau e^{-\rho t} u(c(t))\,\mathrm{d}t + e^{-\rho\tau} V(k(\tau; k_0, c)) \\
&\geq \int_0^\tau e^{-\rho t} u(c(t))\,\mathrm{d}t + e^{-\rho\tau} U(c(\cdot + \tau); k(\tau; k_0, c)) \\
&= \int_0^\tau e^{-\rho t} u(c(t))\,\mathrm{d}t + \int_0^{+\infty} e^{-\rho(t+\tau)} u(c(t+\tau))\,\mathrm{d}t \\
&= \int_0^\tau e^{-\rho t} u(c(t))\,\mathrm{d}t + \int_\tau^{+\infty} e^{-\rho s} u(c(s))\,\mathrm{d}s = U(c; k_0)
\end{aligned}$$

Secondly, fix $\epsilon > 0$, and take

$$0 < \epsilon' \leq \frac{2\epsilon}{(1 + e^{-\rho\tau})}.$$

Hence there exists $\tilde{c}_\epsilon \in \Lambda(k_0)$ and $\tilde{\tilde{c}}_\epsilon \in \Lambda(k(\tau; k_0, \tilde{c}_\epsilon))$ such that

$$\begin{aligned}
\sigma(\tau, k_0) - \epsilon &\leq \sigma(\tau, k_0) - \frac{\epsilon'}{2}\left(1 + e^{-\rho\tau}\right) \\
&\leq \int_0^\tau e^{-\rho t} u(\tilde{c}_\epsilon(t))\,\mathrm{d}t + e^{-\rho\tau} V(k(\tau; k_0, \tilde{c}_\epsilon)) - e^{-\rho\tau}\frac{\epsilon'}{2} \\
&\leq \int_0^\tau e^{-\rho t} u(\tilde{c}_\epsilon(t))\,\mathrm{d}t + e^{-\rho\tau} U\left(\tilde{\tilde{c}}_\epsilon; k(\tau; k_0, \tilde{c}_\epsilon)\right) \\
&= \int_0^\tau e^{-\rho t} u(\tilde{c}_\epsilon(t))\,\mathrm{d}t + \int_0^{+\infty} e^{-\rho(t+\tau)} u\left(\tilde{\tilde{c}}_\epsilon(t)\right)\mathrm{d}t
\end{aligned}$$

Now set

$$c_\epsilon(t) := \begin{cases} \tilde{c}_\epsilon(t) & \text{if } t \in [0, \tau] \\ \tilde{\tilde{c}}_\epsilon(t - \tau) & \text{if } t > \tau \end{cases}$$

Hence $c_\epsilon \in L^1_{loc}([0, +\infty), \mathbb{R})$ and $\forall t > 0 : c_\epsilon(t + \tau) = \tilde{\tilde{c}}_\epsilon(t)$. So:

$$\sigma(\tau, k_0) - \epsilon \quad \leq \quad \int_0^{+\infty} e^{-\rho t} u(c_\epsilon(t))\,\mathrm{d}t \tag{1.51}$$

Finally, it is easy to show that $c_\epsilon \in \Lambda(k_0)$. Observe that $k(\cdot; k_0, c_\epsilon) = k(\cdot; k_0, \tilde{c}_\epsilon)$ in $[0, \tau]$ by definition of c_ϵ and by uniqueness. In particular $k(\tau; k_0, c_\epsilon) = k(\tau; k_0, \tilde{c}_\epsilon)$, so that $k(\cdot + \tau; k_0, c_\epsilon)$ and $k\left(\cdot; k(\tau; k_0, \tilde{c}_\epsilon), \tilde{\tilde{c}}_\epsilon\right)$ have the same initial value; moreover, these two orbits satisfy the same state equation (i.e. the equation associated with the control $c_\epsilon(\cdot + \tau)$) and so they coincide, again by uniqueness. Recalling that by definition $\tilde{c}_\epsilon \in \Lambda(k_0)$ and $\tilde{\tilde{c}}_\epsilon \in \Lambda(k(\tau; k_0, \tilde{c}_\epsilon))$, we have $k(t; k_0, c_\epsilon) \geq 0$ for all $t \geq 0$. Hence by (1.51) we can write

$$\sigma(\tau, k_0) - \epsilon \leq U(c_\epsilon; k_0)$$

and the assertion is proven. $\square$

Equation (1.50) is called *Bellman Functional Equation.*

A consequence of the above theorem is that every control which is optimal respect to a state, is also optimal respect to every following optimal state. But Theorem 34 also suggests and partially imply a useful characterization of optimal controls as solutions of an integral equation.

Theorem 35. *Let $k_0 \geq 0$, $c^* \in \Lambda(k_0)$. Hence the following are equivalent:*

i) c^* is optimal at k_0

ii) For every $\tau > 0$:

$$V(k_0) = \int_0^{\tau} e^{-\rho t} u(c^*(t))\, dt + e^{-\rho\tau} V(k(\tau; k_0, c^*))$$

Moreover, i) or ii) imply that for every $\tau > 0$, $c^*(\cdot + \tau)$ is admissible and optimal at $k(\tau; k_0, c^*)$.

Proof. i) $\Rightarrow$ ii) Let us assume that c^* is admissible and optimal at $k_0 \geq 0$ and fix $\tau > 0$. Observe that $c^*(\cdot + \tau)$ is admissible at $k(\tau; k_0, c^*)$ by Remark 33. Hence, by Theorem 34:

$$\begin{aligned} V(k_0) &\geq \int_0^{\tau} e^{-\rho t} u(c^*(t))\, \mathrm{d}t + e^{-\rho\tau} V(k(\tau; k_0, c^*)) \\ &\geq \int_0^{\tau} e^{-\rho t} u(c^*(t))\, \mathrm{d}t + e^{-\rho\tau} U(c^*(\cdot + \tau); k(\tau; k_0, c^*)) \\ &= \int_0^{+\infty} e^{-\rho t} u(c^*(t))\, \mathrm{d}t \;=\; U(c^*; k_0) \;=\; V(k_0). \end{aligned} \tag{1.52}$$

Hence

$$V(k_0) = \int_0^{\tau} e^{-\rho t} u(c^*(t))\, \mathrm{d}t + e^{-\rho\tau} V(k(\tau; k_0, c^*)). \tag{1.53}$$

ii) $\Rightarrow$ i) Suppose that $c^* \in \Lambda(k_0)$ and (1.53) holds for every $\tau > 0$. For every $\epsilon > 0$ pick $\hat{c}_\epsilon \in \Lambda\left(k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right)$ such that:

$$V\left(k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right) - \epsilon \leq U\left(\hat{c}_\epsilon; k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right). \tag{1.54}$$

Then define

$$c_\epsilon(t) := \begin{cases} c^*(t) & \text{if } t \in \left[0, \frac{1}{\epsilon}\right] \\ \hat{c}_\epsilon\left(t - \frac{1}{\epsilon}\right) & \text{if } t > \frac{1}{\epsilon} \end{cases}$$

By the same arguments we used in the proof of Theorem 34 , $c_\epsilon \in \Lambda(k_0)$ and, obviously, $c_\epsilon\left(t + \frac{1}{\epsilon}\right) = \hat{c}_\epsilon(t)$ for every $t > 0$.

Hence, taking $\tau = 1/\epsilon$ in (1.53), we have by (1.54):

$$\begin{aligned} V(k_0) - \epsilon e^{-\rho/\epsilon} &= \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t))\,\mathrm{d}t + e^{-\rho/\epsilon}\left[V\left(k\left(\frac{1}{\epsilon};k_0,c^*\right)\right) - \epsilon\right] \\ &\leq \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t))\,\mathrm{d}t + e^{-\rho/\epsilon} U\left(\hat{c}_\epsilon; k\left(\frac{1}{\epsilon};k_0,c^*\right)\right) \\ &= \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t))\,\mathrm{d}t + \int_0^{+\infty} e^{-\rho\left(t+\frac{1}{\epsilon}\right)} u\left(c_\epsilon\left(t+\frac{1}{\epsilon}\right)\right)\mathrm{d}t \\ &= \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t))\,\mathrm{d}t + \int_{1/\epsilon}^{+\infty} e^{-\rho s} u(c_\epsilon(s))\,\mathrm{d}s \qquad (1.55) \end{aligned}$$

Now we show that the second addend tends to 0 as $\epsilon \to 0$. First, using Jensen inequality and the properties of the function ψ_{k_0} established in Lemma 20, we see that for every $T \geq 1/\epsilon$:

$$\begin{aligned} \int_{1/\epsilon}^{T} e^{-\rho s} u(c_\epsilon(s))\,\mathrm{d}s &= \left[e^{-\rho s}\int_{1/\epsilon}^{s} u(c_\epsilon(\tau))\,\mathrm{d}\tau\right]_{s=1/\epsilon}^{s=T} + \rho\int_{1/\epsilon}^{T} e^{-\rho s}\int_{1/\epsilon}^{s} u(c_\epsilon(\tau))\,\mathrm{d}\tau\mathrm{d}s \\ &\leq e^{-\rho T}\int_0^T u(c_\epsilon(\tau))\,\mathrm{d}\tau + \rho\int_{1/\epsilon}^{T} e^{-\rho s}\int_0^s u(c_\epsilon(\tau))\,\mathrm{d}\tau\mathrm{d}s \\ &\leq \psi_{k_0}(T) + \rho\int_{1/\epsilon}^{T} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau)\,\mathrm{d}\tau}{s}\right)\mathrm{d}s \\ &\to \rho\int_{1/\epsilon}^{+\infty} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau)\,\mathrm{d}\tau}{s}\right)\mathrm{d}s \quad \text{as } T \to +\infty \qquad (1.56) \end{aligned}$$

(remembering that c_ϵ is admissible at k_0). By point i) of Lemma 20, for every $\epsilon < 1$ and every $s \geq 1/\epsilon$:

$$\begin{aligned} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau)\,\mathrm{d}\tau}{s}\right) &\leq s e^{-\rho s} u\left(M(k_0)\left[1 + e^{(L+\epsilon_0)s}\right] + \frac{M(k_0)}{s(L+\epsilon_0)}\right) \\ &\leq s e^{-\rho s}\left\{u(M(k_0)) + M(k_0)\,u\left(e^{(L+\epsilon_0)s}\right) + u\left(\frac{M(k_0)}{L+\epsilon_0}\right)\right\} \end{aligned}$$

(remembering that u is increasing and has the properties in Remark 19) which implies, together with (1.56), for every $\epsilon < 1$:

$$\begin{aligned}
0 \leq \int_{1/\epsilon}^{+\infty} e^{-\rho s} u\left(c_\epsilon(s)\right) \mathrm{d}s \quad &\leq \quad \rho \int_{1/\epsilon}^{+\infty} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau)\,\mathrm{d}\tau}{s}\right) \mathrm{d}s \\
&\leq \quad \rho \left[u\left(M\left(k_0\right)\right) + u\left(\frac{M\left(k_0\right)}{L+\epsilon_0}\right)\right] \int_{1/\epsilon}^{+\infty} s e^{-\rho s} \mathrm{d}s + \\
&+ \quad \rho M\left(k_0\right) \int_{1/\epsilon}^{+\infty} s e^{-\rho s} u\left(e^{(L+\epsilon_0)s}\right) \mathrm{d}s.
\end{aligned}$$

By Remark 3 the last integral converges, hence the upper bound tends to 0 as $\epsilon \to 0$.

Hence, letting $\epsilon \to 0$ in (1.55), we find:

$$V\left(k_0\right) \leq \int_0^{+\infty} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t = U\left(c^*; k_0\right)$$

which implies that c^* is optimal at k_0.

Finally, if i) holds, then by (1.52):

$$V\left(k\left(\tau; k_0, c^*\right)\right) = U\left(c^*\left(\cdot + \tau\right); k\left(\tau; k_0, c^*\right)\right).$$

□

1.7.2 The value function as a viscosity solution of HJB

In many interesting cases the value function V is non-differentiable. Moreover, in general it is not possible to prove the differentiability of V relying only on the fact that it solves the Bellman Functional Equation, or BFE (in our case, equation (1.50)), since the latter needs not have a unique regular solution. Of course such equation has a natural "infinitesimal version" (usually called Hamilton-Jacobi-Bellman equation, or HJB, which is in general a first order non-linear PDE), and it can be proven that any continuously differentiable solution to BFE is indeed a solution of HJB. This is of no help without information about the regularity of V; furthermore, HJB could have no classical solution (see e.g. [38]).

This is why the theory of viscosity solutions plays a key role in Dynamic Programming methods: one wonders if the value function is a solution of HJB in a weaker sense. As pointed out in the introduction, our case is a bit special meaning that the problem itself of the value function being a viscosity solution of HJB equation must be proven to be meaningful. Indeed the "right" equation involves an Hamiltonian function whose domain is not $\mathbb{R}^N$ (in our case $\mathbb{R}^2$)[1], so the

[1]This turns out to be a consequence of the unboundedness of the the running cost u and, again, of the non-compactness of the control space.

test functions involved in the definition of viscosity solution must match this restriction. This is ensured by asking that the candidate solution has a special property, stronger than monotonicity.

Definition 36. Let $f \in \mathcal{C}^0\left((0,+\infty),\mathbb{R}\right)$; we say that $f \in \mathcal{C}^+\left((0,+\infty),\mathbb{R}\right)$ if, and only if, for every $k_0 > 0$ there exist $\delta, C^+, C^- > 0$ such that

$$\frac{f(k_0+h)-f(k_0)}{h} \geq C^+ \quad \forall h \in (0,\delta)$$
$$\frac{f(k_0+h)-f(k_0)}{h} \geq C^- \quad \forall h \in (-\delta,0)$$

We note that by Theorem 32, (ii) the value function V satisfies

$$V \in \mathcal{C}^+\left((0,+\infty),\mathbb{R}\right). \tag{1.57}$$

Definition 37. The function $H : [0,+\infty) \times (0,+\infty) \to \mathbb{R}$ defined by

$$H(k,p) := -\sup\left\{[F(k)-c]\cdot p + u(c) \;/\; c \in [0,+\infty)\right\}$$

is called *Hamiltonian.*

The equation

$$\rho \mathrm{v}(k) + H\left(k, \mathrm{v}'(k)\right) = 0 \quad \forall k > 0 \tag{1.58}$$

in the unknown $\mathrm{v} \in \mathcal{C}^+\left((0,+\infty),\mathbb{R}\right) \cap \mathcal{C}^1\left((0,+\infty),\mathbb{R}\right)$ is called *Hamilton-Jacobi-Bellman equation* (HJB).

Observe that any solution of (1.58) must be strictly increasing, by Definition 36.

Remark 38. The Hamiltonian is always finite. Indeed

$$-\sup_{c\in[0,+\infty)}\left\{[F(k)-c]\cdot p + u(c)\right\} > -\infty \iff p > 0.$$

If $p > 0$, since $\lim_{c\to+\infty} u'(c) = 0$ we can choose $c_p \geq 0$ such that $u'(c_p) \leq p$; this implies by the concavity of u:

$$\forall c \geq 0 : u(c) - cp \leq u(c) - u'(c_p)\, c \leq u(c_p) - u'(c_p)\, c_p,$$

so that

$$-F(k)\, p - \sup_{c\in[0,+\infty)}\left\{u(c) - cp\right\} \geq -F(k)\, p - u(c_p) + u'(c_p)\, c_p > -\infty.$$

Otherwise, when $p \leq 0$, since $\lim_{c \to +\infty} u(c) = +\infty$ we have

$$-F(k)p - \sup_{c \in [0,+\infty)} \{u(c) - cp\} \leq -F(k)p - \sup_{c \in [0,+\infty)} u(c) = -\infty.$$

Definition 39. A function $v \in \mathcal{C}^+((0,+\infty),\mathbb{R})$ is called a *viscosity subsolution* [*supersolution*] *of* (HJB) if, and only if:
for every $\varphi \in \mathcal{C}^1((0,+\infty),\mathbb{R})$ and for every local maximum [minimum] point $k_0 > 0$ of $v - \varphi$:

$$\begin{aligned} \rho v(k_0) - \sup\{[F(k_0) - c] \cdot \varphi'(k_0) + u(c) \ / \ c \in [0,+\infty)\} &= \\ \rho v(k_0) + H(k_0, \varphi'(k_0)) &\leq 0 \\ &[\geq 0] \end{aligned}$$

If v is both a viscosity subsolution of (HJB) and a viscosity supersolution of (HJB), then we say that v is a *viscosity solution of* (HJB).

Remark 40. The latter definition is well posed. Indeed, let $v \in \mathcal{C}^+((0,+\infty),\mathbb{R})$ and $\varphi \in \mathcal{C}^1((0,+\infty),\mathbb{R})$. If k_0 is a local maximum for $v - \varphi$ in $(0,+\infty)$, then for $h < 0$ big enough we have:

$$\begin{aligned} &v(k_0) - v(k_0 + h) \geq \varphi(k_0) - \varphi(k_0 + h) \implies \\ &0 < C^- \leq \frac{v(k_0) - v(k_0 + h)}{h} \leq \frac{\varphi(k_0) - \varphi(k_0 + h)}{h}. \end{aligned}$$

If k_0 is a local minimum for $v - \varphi$ in $(0,+\infty)$, then for $h > 0$ small enough we have:

$$\begin{aligned} &v(k_0) - v(k_0 + h) \leq \varphi(k_0) - \varphi(k_0 + h) \implies \\ &0 < C^+ \leq \frac{v(k_0) - v(k_0 + h)}{h} \leq \frac{\varphi(k_0) - \varphi(k_0 + h)}{h}. \end{aligned}$$

In both cases, we have $\varphi'(k_0) > 0$, so the quantity $H(k_0, \varphi'(k_0))$ involved in the definition is well-defined.

By (1.57) we see that the value function is a good candidate to be a viscosity solution of HJB. We are now going to prove that this is indeed the case. As pointed out in the introduction, this will be done without any regularity assumption on H; nevertheless, this function can be easily shown to be continuous, since for every $k \geq 0$, $p > 0$:

$$H(k,p) = F(k)p + (-u)^*(p),$$

where $(-u)^*$ is the (convex) conjugate function of the convex function $-u$.

Lemma 41. *Let* $k_0 > 0$ *and* $(c_T)_{T>0} \subseteq \Lambda(k_0)$ *satisfying:*

$$\|c_T\|_{\infty,[0,T]} \leq N(k_0, T) \quad \forall T > 0.$$

where N *is the function defined in* Lemma 9. *Hence*

$$\forall T \in [0,1] : \forall t \in [0,T] : |k(t; k_0, c_T) - k_0| \leq T e^{\overline{M}t} [F(k_0) + N(k_0, 1)].$$

In particular $k(T; k_0, c_T) \to k_0$ *as* $T \to 0$.

Proof. Set k_0 and $(c_T)_{T>0}$ as in the hypothesis and fix $0 \leq T \leq 1$. Hence integrating both sides of the state equation we get, for every $t \in [0,T]$:

$$k(t; k_0, c_T) - k_0 \quad = \quad \int_0^t [F(k_0) - c_T(s)] \, \mathrm{d}s + \int_0^t [F(k(s; k_0, c_T)) - F(k_0)] \, \mathrm{d}s$$

which implies by Remark 7:

$$\begin{aligned} |k(t; k_0, c_T) - k_0| \quad &\leq \quad \int_0^t |F(k_0) - c_T(s)| \, \mathrm{d}s + \int_0^t |F(k(s; k_0, c_T)) - F(k_0)| \, \mathrm{d}s \\ &\leq \quad \int_0^T |F(k_0) - c_T(s)| \, \mathrm{d}s + \bar{M} \int_0^t |k(s; k_0, c_T) - k_0| \, \mathrm{d}s \end{aligned}$$

Hence by Gronwall's inequality and by the monotonicity of $N(k_0, \cdot)$, for every $T \in [0,1]$ and every $t \in [0,T]$:

$$\begin{aligned} |k(t; k_0, c_T) - k_0| \quad &\leq \quad e^{\bar{M}t} \int_0^T |F(k_0) - c_T(s)| \, \mathrm{d}s. \\ &\leq \quad T e^{\bar{M}t} [F(k_0) + N(k_0, T)] \\ &\leq \quad T e^{\bar{M}t} [F(k_0) + N(k_0, 1)]. \end{aligned}$$

□

Theorem 42. *The value function* $V : [0, +\infty) \to \mathbb{R}$ *is a viscosity solution of* (HJB).
Consequently, if $V \in \mathcal{C}^1([0, +\infty), \mathbb{R})$, *then* V *is strictly increasing and is a solution of* (HJB) - (1.58) *in the classical sense.*

Proof. In the first place, we show that V is a viscosity supersolution of (HJB).
Let $\varphi \in \mathcal{C}^1((0, +\infty), \mathbb{R})$ and $k_0 > 0$ be a local minumum point of $V - \varphi$, so that

$$V(k_0) - V \leq \varphi(k_0) - \varphi \tag{1.59}$$

in a proper neighbourhood of k_0. Now fix $c \in [0, +\infty)$ and set $k := k(\cdot; k_0, c)$. As $k_0 > 0$, there exists $T_c > 0$ such that $k > 0$ in $[0, T_c]$. Hence the control

$$\tilde{c}(t) := \begin{cases} c & \text{if } t \in [0, T_c] \\ 0 & \text{if } t > T_c \end{cases}$$

is admissible at k_0. Then by Theorem 34, for every $\tau \in [0, T_c]$:

$$\begin{aligned} V(k_0) - V(k(\tau)) &\geq \int_0^\tau e^{-\rho t} u(\tilde{c}(t))\, \mathrm{d}t + V(k(\tau)) \left[e^{-\rho\tau} - 1\right] \\ &= u(c) \int_0^\tau e^{-\rho t} \mathrm{d}t + V(k(\tau)) \left[e^{-\rho\tau} - 1\right]. \end{aligned}$$

Hence by (1.59) and by the continuity of k, we have for every $\tau > 0$ sufficiently small:

$$\frac{\varphi(k(0)) - \varphi(k(\tau))}{\tau} \quad \geq \quad u(c) \frac{\int_0^\tau e^{-\rho t} \mathrm{d}t}{\tau} + V(k(\tau)) \frac{[e^{-\rho\tau} - 1]}{\tau}.$$

Letting $\tau \to 0$ and using the continuity of V and k:

$$-\varphi'(k_0)[F(k_0) - c] \geq u(c) - \rho V(k_0)$$

which implies, taking the sup for $c \geq 0$:

$$\rho V(k_0) + H(k_0, \varphi'(k_0)) \geq 0$$

Secondly we show that V is a viscosity subsolution of (HJB).

Let $\varphi \in \mathcal{C}^1((0, +\infty), \mathbb{R})$ and $k_0 > 0$ be a local maximum point of $V - \varphi$, so that

$$V(k_0) - V \geq \varphi(k_0) - \varphi \tag{1.60}$$

in a proper neighborhood $\mathcal{N}(k_0)$ of k_0.

Fix $\epsilon > 0$ and, using the definition of V, define a family of controls $(c_{T,\epsilon})_{T>0} \subseteq \Lambda(k_0)$ such that for every $T > 0$:

$$V(k_0) - T\epsilon \leq U(c_{T,\epsilon}; k_0). \tag{1.61}$$

Now take $(c_{T,\epsilon})^T$ as in Lemma 9 and set $\bar{c}_{T,\epsilon} := (c_{T,\epsilon})^T$ for simplicity of notation (so that

$\bar{c}_{T,\epsilon} \in \Lambda(k_0)$). We have:

$$\begin{aligned} V(k_0) - T\epsilon &\leq U(c_{T,\epsilon}; k_0) \leq U(\bar{c}_{T,\epsilon}; k_0) \\ &= \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t))\,\mathrm{d}t + e^{-\rho T} \int_T^{+\infty} e^{-\rho(s-T)} u(\bar{c}_{T,\epsilon}(s-T+T))\,\mathrm{d}s \\ &= \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t))\,\mathrm{d}t + e^{-\rho T} U(\bar{c}_{T,\epsilon}(\cdot + T); k(T; k_0, \bar{c}_{T,\epsilon})) \\ &\leq \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t))\,\mathrm{d}t + e^{-\rho T} V(k(T; k_0, \bar{c}_{T,\epsilon})) \end{aligned}$$

where we have used Remark 33.

By Lemma 41 we have for $T > 0$ sufficiently small (say $T < \hat{T}$),

$$k(T; k_0, \bar{c}_{T,\epsilon}) \in \mathcal{N}(k_0).$$

Hence, setting $\bar{k}_{T,\epsilon} := k(\cdot; k_0, \bar{c}_{T,\epsilon})$, for every $T < \hat{T}$, we have by (1.60):

$$\begin{aligned} \varphi(k_0) - \varphi\left(\bar{k}_{T,\epsilon}(T)\right) - e^{-\rho T} V\left(\bar{k}_{T,\epsilon}(T)\right) &\leq V(k_0) - V\left(\bar{k}_{T,\epsilon}(T)\right) - e^{-\rho T} V\left(\bar{k}_{T,\epsilon}(T)\right) \\ &\leq \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t))\,\mathrm{d}t - V\left(\bar{k}_{T,\epsilon}(T)\right) + T\epsilon \end{aligned}$$

which implies

$$\begin{aligned} &\int_0^T - \left\{\varphi'\left(\bar{k}_{T,\epsilon}(t)\right)\left[F\left(\bar{k}_{T,\epsilon}(t)\right) - \bar{c}_{T,\epsilon}(t)\right] + e^{-\rho t} u(\bar{c}_{T,\epsilon}(t))\right\}\mathrm{d}t \\ &\leq V\left(\bar{k}_{T,\epsilon}(T)\right)\left[e^{-\rho T} - 1\right] + T\epsilon. \end{aligned} \tag{1.62}$$

Observe that the integral at the left hand member bigger than:

$$\begin{aligned} &\int_0^T - \{[\varphi'(k_0) + \omega_1(t)][F(k_0) - \bar{c}_{T,\epsilon}(t) + \omega_2(t)] + u(\bar{c}_{T,\epsilon}(t))\}\,\mathrm{d}t = \\ &\int_0^T - \{\varphi'(k_0)[F(k_0) - \bar{c}_{T,\epsilon}(t)] + u(\bar{c}_{T,\epsilon}(t))\}\,\mathrm{d}t + \\ &+ \int_0^T - \{\varphi'(k_0)\,\omega_2(t)\,\mathrm{d}t + \omega_1(t)[\omega_2(t) + F(k_0) - \bar{c}_{T,\epsilon}(t)]\}\,\mathrm{d}t \end{aligned} \tag{1.63}$$

where ω_1, ω_2 are functions which are continuous in a neighborhood of 0 and satisfy:

$$\omega_1(0) = \omega_2(0) = 0.$$

This implies, for $T<1$:

$$\begin{aligned}
&\left|\int_0^T \varphi'(k_0)\,\omega_2(t)\,\mathrm{d}t+\int_0^T \omega_1(t)\left[\omega_2(t)+F(k_0)-\bar{c}_{T,\epsilon}(t)\right]\mathrm{d}t\right| \\
&\leq\ |\varphi'(k_0)|\,o_1(T)+o_2(T)+\int_0^T |\omega_1(t)|\left[F(k_0)+\bar{c}_{T,\epsilon}(t)\right]\mathrm{d}t \\
&\leq\ |\varphi'(k_0)|\,o_1(T)+o_2(T)+\left[F(k_0)+N(k_0,T)\right]o_3(T) \\
&\leq\ |\varphi'(k_0)|\,o_1(T)+o_2(T)+\left[F(k_0)+N(k_0,1)\right]o_3(T)
\end{aligned}$$

where

$$\lim_{T\to 0}\frac{o_i(T)}{T}=0$$

for $i=1,2,3$. Observe that this is true even if the o_is depend on T, by Lemma 41. For instance,

$$\begin{aligned}
|o_1(T)| &= \left|\int_0^T \omega_2(t)\,\mathrm{d}t\right|\leq T\max_{[0,T]}|\omega_2|=T\,|\omega_2(\tau_T)| \\
&= T\left|F\left(\bar{k}_{T,\epsilon}(\tau_T)\right)-F(k_0)\right| \\
&\leq \overline{M}T\left|\bar{k}_{T,\epsilon}(\tau_T)-k_0\right|\leq\overline{M}T^2e^{\bar{M}\tau_T}\left[F(k_0)+N(k_0,1)\right]
\end{aligned}$$

Moreover, by the fact that $V\in\mathcal{C}^+([0,+\infty),\mathbb{R})$ and by Remark 40, we have for any $t\in[0,T]$:

$$\begin{aligned}
-\left\{\varphi'(k_0)\left[F(k_0)-\bar{c}_{T,\epsilon}(t)\right]+u\left(\bar{c}_{T,\epsilon}(t)\right)\right\} &\geq -\sup_{c\geq 0}\left\{\varphi'(k_0)\left[F(k_0)-c\right]+u(c)\right\} \\
&= H\left(k_0,\varphi'(k_0)\right)>-\infty,
\end{aligned}$$

by which we can write:

$$\int_0^T -\left\{\varphi'(k_0)\left[F(k_0)-\bar{c}_{T,\epsilon}(t)\right]+u\left(\bar{c}_{T,\epsilon}(t)\right)\right\}\mathrm{d}t\ \geq\ T\cdot H\left(k_0,\varphi'(k_0)\right).$$

Hence, by (1.62) and (1.63):

$$\begin{aligned}
&V\left(\bar{k}_{T,\epsilon}(T)\right)\left[e^{-\rho T}-1\right]+T\epsilon \\
&\geq\ -\int_0^T\left\{\varphi'(k_0)\left[F(k_0)-\bar{c}_{T,\epsilon}(t)\right]+u\left(\bar{c}_{T,\epsilon}(t)\right)\right\}\mathrm{d}t+ \\
&\ +\int_0^T-\left\{\varphi'(k_0)\,\omega_2(t)\,\mathrm{d}t+\omega_1(t)\left[\omega_2(t)+F(k_0)-\bar{c}_{T,\epsilon}(t)\right]\mathrm{d}t\right\} \\
&\geq\ T\cdot H\left(k_0,\varphi'(k_0)\right)+o_{T\to 0}(T)
\end{aligned}$$

for any $0<T<1,\hat{T}$. Hence dividing by T, and then letting $T\to 0$, again by Lemma 41 and the continuity of V we obtain:

$$-\rho V(k_0) + \epsilon \geq H(k_0, \varphi'(k_0))$$

which proves the assertion since ϵ is arbitrary. □

1.7.3 Reverse Dynamic Programming

The set of the viscosity solutions to a differential equation is, in general, not stable under the change of sign of the equation. This particular phenomenon makes it interesting to ask wether the value function is also a viscosity solution to the "reverse" HJB equation

$$-\rho \mathrm{v}(k) - H(k, \mathrm{v}'(k)) = 0, \quad k > 0. \tag{1.64}$$

We will prove that this is true at points k that can be reached by an optimal trajectory.

We preliminary observe that the present section does not require a change of setting for the problem, like for instance the introduction of a new set of admissible controls. We only need to consider "backward" trajectories associated with standard "backward translated" admissible controls. To this scope, we introduce the following notation.

Definition 43. Let $k_0 \geq 0$, $T > 0$. If c is a non negative constant, the function $\theta(\cdot; k_0, c)$ is the unique solution to the following Cauchy Problem in the unknown θ:

$$\begin{cases} \theta(0) = k_0 \\ \dot{\theta}(t) = F(\theta(t)) - c \quad \forall t \leq 0. \end{cases}$$

If $c \in L^1_{loc}([0, +\infty))$, the function $\theta(\cdot; k_0, c(\cdot + T))$ is the unique solution to the following Cauchy Problem in the unknown θ:

$$\begin{cases} \theta(0) = k_0 \\ \dot{\theta}(t) = F(\theta(t)) - c(t + T) \quad \forall t \in [-T, 0]. \end{cases}$$

Lemma 44. *Let $k_0 > 0$, $c \geq 0$. Then there exists $\mathcal{T}_1(k_0, c) > 0$ such that, for every $T \in (0, \mathcal{T}_1(k_0, c))$ there exists $y_0(T) = y(k_0, c, T)$ such that*

$$\begin{aligned} &y_0(T) > 0 \\ &k(\cdot; y_0(T), c) > 0 \quad \textit{in } [0, T] \\ &k(T; y_0(T), c) = k_0. \end{aligned}$$

Furthermore, $y_0(T) \to k_0$ *as* $T \to 0$.

Proof. Fix $k_0 > 0$ and $c \geq 0$, and set $\theta = \theta(\cdot; k_0, c)$ as in Definition 43. Choose $T(k_0, c) > 0$ such that $\theta > 0$ in $[-T(k_0, c), 0]$, and take $T \in (0, T(k_0, c))$. Set $y_0(T) = \theta(-T)$ (which depends also on k_0 and c since θ does). Observe that the function $k(\cdot + T; y_0(T), c)$ is well defined in $[-T, 0]$ (even if the constant control c may not be admissible at $y_0(T)$), takes value $y_0(T)$ in $-T$ and satisfies the same differential equation as θ in $[-T, 0]$ (given by the dynamics $F(\cdot) - c$). Hence

$$k(\cdot + T; y_0(T), c) = \theta \quad \text{in } [-T, 0].$$

In particular

$$\begin{gathered} k(\cdot; y_0(T), c) > 0 \quad \text{in } [0, T] \\ k(T; y_0(T), c) = \theta(0) = k_0. \end{gathered}$$

Obviously $y_0(T) \to k_0$ as $T \to 0$ since the continuous function θ is not defined upon T. $\square$

Corollary 45. *The value function V is a viscosity subsolution of the "reverse" HJB equation*

$$-\rho \mathrm{v}(k) - H(k, \mathrm{v}'(k)) = 0 \quad \forall k > 0 \tag{1.65}$$

in the unknown v, *where*

$$H(k, p) := -\sup_{c \geq 0} \{p[F(k) - c] + u(c)\} \quad \forall k \geq 0, p > 0.$$

Proof. Let $k_0 > 0$ be a local maximum point of $V - \varphi$, where φ is a $\mathcal{C}^1$ function defined in a neighborhood of k_0. We know by Theorem 34 that V satisfies the Bellman Functional Equation:

$$\begin{aligned} &\forall y_0 > 0 : \forall T > 0 : \\ &V(y_0) = \sup_{c \in \Lambda(y_0)} \left\{ \int_0^T e^{-\rho t} u(c(t))\, \mathrm{d}t + e^{-\rho T} V(k(T; y_0, c)) \right\}. \end{aligned} \tag{1.66}$$

The idea is to interchange the roles of the two states appearing as arguments of V in the latter relation: $k(T; y_0, c)$ "becomes k_0" thanks to Lemma (44).

First we deduce the appropriate relation with constant controls. Fix a constant $c \geq 0$; any trajectory $k(\cdot; y_0, c)$ - for $y_0 > 0$ - is strictly positive in an interval $[0, \mathcal{T}_2(y_0, c)]$; hence the control $\gamma := c\chi_{[0, \mathcal{T}_2(y_0, c)]}$ belongs to $\Lambda(y_0)$. Applying the above relation to γ leads to:

$$\forall y_0 > 0 : \forall T \in (0, \mathcal{T}_2(y_0, c)) : V(y_0) \geq u(c) \int_0^T e^{-\rho t} \mathrm{d}t + e^{-\rho T} V(k(T; y_0, c)).$$

Now take $\mathcal{T}_1(k_0, c)$ and, for $T \in (0, \mathcal{T}_1(k_0, c))$, $y_0(T)$ like in Lemma (44). Hence, for every $T \in (0, \mathcal{T}_1(k_0, c) \wedge \mathcal{T}_2(y_0(T), c))$:

$$V(y_0(T)) \geq u(c) \int_0^T e^{-\rho t} \mathrm{d}t + e^{-\rho T} V(k_0)$$
$$\Longleftrightarrow$$
$$V(y_0(T)) - V(k_0) + V(k_0)\left(1 - e^{-\rho T}\right) \geq u(c) \int_0^T e^{-\rho t} \mathrm{d}t$$

which implies, since $y_0(T) \to k_0$ as $T \to 0$,

$$\varphi(y_0(T)) - \varphi(k_0) + V(k_0)\left(1 - e^{-\rho T}\right) \geq u(c) \int_0^T e^{-\rho t} \mathrm{d}t,$$

for $T > 0$ sufficiently small. Since $y_0(T) = \theta(-T; k_0, c) = \theta(-T)$ (as shown in the proof of Lemma 44), dividing by T and passing to the limit for $T \to 0$ we obtain:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}T}_{|T=0} \varphi(\theta(-T)) + \rho V(k_0) &= -\varphi'(k_0)[F(k_0) - c] + \rho V(k_0) \\ &\geq u(c). \end{aligned}$$

Thus, passing to the sup for $c \geq 0$:

$$-\rho V(k_0) - H(k_0, \varphi'(k_0)) \leq 0,$$

which proves the subsolution condition. □

We now prove the analogous of Lemma 44 for optimal controls.

Lemma 46. *Let $k_0 > 0$, $c^* \in \Lambda(k_0)$* optimal*. Then there exists $\mathcal{T}(k_0) > 0$ such that, for every $T \in (0, \mathcal{T}(k_0))$ there exists $y_0(T) = y(k_0, c^*, T)$ such that*

$$\begin{aligned} &y_0(T) > 0 \\ &k(\cdot; y_0(T), c^*) > 0 \quad \textit{in } [0, T] \\ &k(T; y_0(T), c^*) = k_0. \end{aligned}$$

Also in this case, $k_0(T) \to 0$ as $T \to 0$.

Proof. Fix $k_0 > 0$, $c^* \in \Lambda(k_0)$ optimal at k_0. We know that, by construction of the optimum, $c^*(\cdot + T) \leq N(k_0, T)$ almost everywhere in $[-T, 0]$. Consequently, for $T \in (0, 1)$:

$$c^*(\cdot + T) \leq N(k_0, 1) \quad \text{a.e. in } [-T, 0],$$

since $N(k_0,\cdot)$ is non-decreasing. Hence, by comparison,

$$\theta\left(\cdot;k_0,c^*\left(\cdot+T\right)\right)\geq\theta\left(\cdot;k_0,N\left(k_0,1\right)\right)\quad\text{in }\left[-T,0\right]. \tag{1.67}$$

Clearly $\theta\left(\cdot;k_0,N\left(k_0,1\right)\right)>0$ in $\left[-\mathcal{T}\left(k_0\right),0\right]$ for some $\mathcal{T}_0\left(k_0\right)>0$.
Thus, for $0<T<\mathcal{T}_0\left(k_0\right)\wedge 1=:\mathcal{T}\left(k_0\right)$:

$$\theta\left(\cdot;k_0,c^*\left(\cdot+T\right)\right)>0\quad\text{in }\left[-T,0\right] \tag{1.68}$$

and in particular $y_0\left(T\right):=\theta\left(-T;k_0,c^*\left(\cdot+T\right)\right)>0$.
Furthermore, setting $k:=k\left(\cdot+T;y_0\left(T\right),c^*\right)$ we have:

$$\begin{aligned}\dot{k}\left(t\right)&=F\left(k\left(t\right)\right)-c^*\left(t+T\right)\quad\forall t\in\left[-T,0\right]\\ k\left(-T\right)&=y_0\left(T\right),\end{aligned}$$

which implies $k=\theta\left(\cdot;k_0,c^*\left(\cdot+T\right)\right)$ in $[-T,0]$. Hence, by (1.68):

$$\begin{aligned}k\left(\cdot;y_0\left(T\right),c^*\right)&>0\quad\text{in }\left[0,T\right]\\ k\left(T;y_0\left(T\right),c^*\right)&=k_0.\end{aligned}$$

Eventually observe that, by Proposition 31, for $T\in(0,1)$:

$$c^*\left(\cdot+T\right)\geq g^{-1}\left(\rho e^{\rho}\frac{V\left(k_0+1\right)}{F\left(k_0\right)}\right)=:C^*\left(k_0\right)\quad\text{a.e. in }\left[-T,0\right],$$

so that $\theta\left(\cdot;k_0,c^*\left(\cdot+T\right)\right)\leq\theta\left(\cdot;k_0,C^*\left(k_0\right)\right)$. This implies, remembering (1.67):

$$y_0\left(T\right)=\theta\left(-T;k_0,c^*\left(\cdot+T\right)\right)\to k_0\text{ as }T\to 0.$$

□

Remark 47. The only property of optimal controls used in the proof of the previous lemma is the existence of local bounds that are independent on the choice of the optimum. Anyway, the Lemma may not guarantee that V is also a viscosity supersolution of equation (1.65). The starting point for proving this property should still be the Dynamic Programming Principle (1.66), considered in the "$\leq$" direction - that is to say - as the assertion that $V(y_0)$ is the least upper bound of the set at the right hand side. Or better, the following version of the principle can be used:

$$\begin{aligned}
&\forall y_0>0:\forall c^*\in\Lambda\left(y_0\right)\text{ optimal at }y_0:\forall T>0:\\
&V\left(y_0\right)=\int_0^T e^{-\rho t}u\left(c^*\left(t\right)\right)\mathrm{d}t+e^{-\rho T}V\left(k\left(T;y_0,c^*\right)\right),
\end{aligned}\tag{1.69}$$

provided by Theorem 35. A given control c^*, optimal at y_0, can be the starting point of the procedure. Consider a state $k_0>0$, a control c^* optimal at k_0 and take $y_0\left(T\right)$ as in Lemma 46: the point is that c^* need not be optimal at $y_0\left(T\right)$ - so (1.69) cannot be used for $y_0=y_0\left(T\right)$, even if it is true that $k\left(T;y_0\left(T\right),c^*\right)=k_0$.

But if it is assumed *a priori* that, given $k_0>0$, we can start from some y_0 and c^* optimal at y_0, then the proof can be accomplished.

Proposition 48. *Let $k_0>0$ be an* optimal point *for the problem - that is to say, assume that there exist y_0, $T>0$ and $c^*\in\Lambda\left(y_0\right)$ optimal at y_0 such that*

$$k_0=k\left(T;y_0,c^*\right).$$

Then we have

$$\begin{aligned}
&\forall\tau\in(0,T]:\\
&V\left(k_0\right)=-\int_0^\tau e^{\rho t}u\left(c^*\left(-t+T\right)\right)dt+e^{\rho\tau}V\left(\theta\left(-\tau;k_0,c^*\left(\cdot+T\right)\right)\right).
\end{aligned}\tag{1.70}$$

Consequently, the value function V is a viscosity supersolution of the "reverse-restricted" HJB equation:

$$-\rho\mathrm{v}\left(k\right)-H\left(k,\mathrm{v}'\left(k\right)\right)=0\quad\forall k>0\text{ optimal for the problem}$$

in the unknown v.

Proof. We first prove relation (1.70). Let $k_0>0$ be an optimal point for the problem, with $k_0=k\left(T,y_0,c^*\right)$ for some y_0, $T>0$ and $c^*\in\Lambda\left(y_0\right)$ optimal at y_0. By (1.69) we have:

$$V\left(y_0\right)=\int_0^T e^{-\rho t}u\left(c^*\left(t\right)\right)\mathrm{d}t+e^{-\rho T}V\left(k_0\right)$$
$$\overset{s=T-t}{\Longleftrightarrow}$$
$$V\left(k_0\right)=-\int_0^T e^{\rho s}u\left(c^*\left(-s+T\right)\right)\mathrm{d}s+e^{\rho T}V\left(y_0\right).$$

It is easily seen that $y_0=\theta\left(-T;k_0,c^*\left(\cdot+T\right)\right)$. Indeed, set for simplicity of notation $\theta_T=$

$\theta\left(\cdot;k_0,c^*\left(\cdot+T\right)\right)$ and $k_T := k\left(\cdot+T;y_0,c^*\right)$; both functions are defined in $[-T,0]$ and satisfy:

$$\begin{cases}\theta_T\left(0\right)=k_0\\ \dot{\theta_T}\left(t\right)=F\left(\theta_T\left(t\right)\right)-c^*\left(t+T\right) \quad \forall t\in[-T,0]\end{cases}$$

$$\begin{cases}k_T\left(0\right)=k_0\\ \dot{k_T}\left(t\right)=F\left(k_T\left(t\right)\right)-c^*\left(t+T\right) \quad \forall t\in[-T,0]\,,\end{cases}$$

where $k_T\left(0\right)=k_0$ by the optimality assumption on k_0. Hence $\theta_T=k_T$ in $[-T,0]$ and in particular $\theta_T\left(-T\right)=y_0$. Thus

$$V\left(k_0\right)=-\int_0^T e^{\rho s}u\left(c^*\left(-s+T\right)\right)\mathrm{d}s+e^{\rho T}V\left(\theta_T\left(-T\right)\right). \tag{1.71}$$

The next step is to extend (1.71) to the interval $(0,T]$ in order to obtain (1.70).

Set $\tau\in(0,T)$. We know by Theorem 35 that a "forward optimality preservation principle" holds, which implies that $c^*\left(\cdot+T-\tau\right)$ is optimal at $k\left(T-\tau;y_0,c^*\right)$.

Hence, relation (1.69) applied to these data at time τ gives:

$$\begin{aligned}&V\left(k\left(T-\tau;y_0,c^*\right)\right)=\\ &\int_0^\tau e^{-\rho t}u\left(c^*\left(t+T-\tau\right)\right)\mathrm{d}t+e^{-\rho\tau}V\left(k\left(\tau;k\left(T-\tau;y_0,c^*\right),c^*\left(\cdot+T-\tau\right)\right)\right).\end{aligned} \tag{1.72}$$

We have

$$k\left(T-\tau;y_0,c^*\right)=\theta_T\left(-\tau\right) \tag{1.73}$$

$$k\left(\cdot;k\left(T-\tau;y_0,c^*\right),c^*\left(\cdot+T-\tau\right)\right)=\theta_T\left(\cdot-\tau\right) \tag{1.74}$$

The first relation simply means that $k_T\left(-\tau\right)=\theta_T\left(-\tau\right)$, while the second is again a consequence of the uniqueness of the trajectory: both functions have the same dynamics by definition and the same initial state by (1.73). Consequently:

$$k\left(\tau;k\left(T-\tau;y_0,c^*\right),c^*\left(\cdot+T-\tau\right)\right)=k_0. \tag{1.75}$$

Plugging (1.73) and (1.75) in (1.72) we obtain:

$$V\left(\theta_T\left(-\tau\right)\right)=\int_0^\tau e^{-\rho t}u\left(c^*\left(t+T-\tau\right)\right)\mathrm{d}t+e^{-\rho\tau}V\left(k_0\right),$$

which is easily seen to be equivalent to (1.70) by the change of variable $s=\tau-t$.

We now prove that the value function V is a viscosity supersolution of the reverse HJB equation at optimal points.
Assume that $k_0 > 0$ is an optimal point for the problem and also a local minimum of $V - \varphi$ where φ is a $\mathcal{C}^1$ function defined in a neighborhood of k_0. Since $\theta_T(-\tau) \overset{\tau \to 0}{\to} k_0$ we have, for every $\tau > 0$ sufficiently small

$$V(\theta_T(-\tau)) - V(k_0) \geq \varphi(\theta_T(-\tau)) - \varphi(k_0)$$

Hence by (1.70), for every $\tau \in (0, \bar{\tau})$:

$$\begin{aligned}\int_0^\tau e^{\rho t} u(c^*(-t+T))\,\mathrm{d}t &= V(\theta_T(-\tau)) - V(k_0) + V(\theta_T(-\tau))(e^{\rho\tau} - 1) \\ &\geq \varphi(\theta_T(-\tau)) - \varphi(k_0) + V(\theta_T(-\tau))(e^{\rho\tau} - 1) \\ &= -\int_0^\tau \varphi'(\theta_T(-t))[F(\theta_T(-t)) - c^*(-t+T)]\,\mathrm{d}t \\ &+ V(\theta_T(-\tau))(e^{\rho\tau} - 1).\end{aligned}$$

Since at both sides of the latter inequality the same quantity $c^*(-t+T)$ appears, we can take the sup for $c \geq 0$ (inside the integral, thanks to the integrability of the Hamiltonian):

$$\int_0^\tau \sup_{c\geq 0} \{e^{\rho\tau} u(c) + \varphi'(\theta_T(-t))[F(\theta_T(-t)) - c]\}\,\mathrm{d}t \geq V(\theta_T(-\tau))(e^{\rho\tau} - 1).$$

Since the Hamiltonian is also continuous, dividing by τ and passing to the limit for $\tau \to 0$, we obtain

$$-\rho V(k_0) - H(k_0, \varphi'(k_0)) \geq 0.$$

Thus also the supersolution condition is satisfied by V. □

Remark 49. Relation (1.70) can be regarded to as a "Backward Dynamic Programming Principle" for optimal points.

Chapter 2

Shallow Lake models with monotone dynamics

We examine the optimal control problem related to a general monotone dynamics version of the Shallow Lake model, and we prove the existence of an optimum. In the last twenty years, a literature about this model has grown up, but, in our knowledge, no direct existence proof has been provided up to now. The optimal control problem has been introduced in [28], and has been studied mostly via dynamic programming ([26]), or from the dynamical systems viewpoint (see e.g. [24], [25] and [28]). The latter approach consists in the analysis of the adjoint system that is obtained by coupling the state equation with the adjoint equation given by the Pontryagin Maximum Principle. As it is well known, this principle provides conditions for optimality that in general are merely necessary.

As pointed out in the introduction, the main technical difficulties in order to prove the existence of an optimum arise from the fact that good *a priori* estimates for the controls and for the states are missing, because of the infinite horizon setting and the unboundedness assumption on the set of admissible controls. Indeed, the intimate nature of the model requires to be allowed to choose a (locally integrable) control function that reaches arbitrarily large values in a finite time. In addition, arbitrarily small positive controls are allowed, and this in fact reinforces the unboundedness phenomenon when optimization is taken into account, since the objective functional has logarithmic dependence on the control. As such, this is a further context where the application of any compactness result is not straightforward.

The model describes the dynamics of the accumulation of phosphorus in the ecosystem of a shallow lake, from a optimal control theory perspective. Precisely, the state equation expresses the (non-linear) relationship between the farming activities near the lake, which are responsible for the release of phosphorus, and the total amount of phosphorus in the water, which depends also on the natural production and on the natural loss consisting of sedimentation, outflow and sequestration in biomass.

Following [28], we can assert that the essential dynamics of the eutrophication process can be modelled by the differential equation:

$$\dot{P}(t) = -sP(t) + r\frac{P^2(t)}{m^2 + P^2(t)} + L(t), \tag{2.1}$$

where P is the amount of phosphorus in algae, L is the input of phosphorus (the "loading"), s is the rate of loss, r is the maximum rate of internal loading, and m is the anoxic level.

After a change of variable and of time scale, we consider the normalized equation

$$\dot{x}(\tau) = -bx(\tau) + \frac{x^2(\tau)}{1 + x^2(\tau)} + u(\tau),$$

where $x(\cdot) := P(\cdot)/m$, $u(\cdot) = L(\cdot)/r$ and $b = sm/r$. We see that the dynamics, as a function of the state, has a convex-concave behaviour.

In an economical analysis, the dynamics of pollution must be considered together with the social benefit of the different interest groups operating in the lake system. The social benefit obviously depends both on the status of the water and on the intensity of the agricultural activities near the lake; the latter, in a way, can be measured by the amount of phosphorus released in the water.

The objective functional to be maximized represents this social benefit. Mathematically it is a function of the pollution released by the farming activities, and takes into account the trade-offs between the utility of agriculture and the utility of a clear lake.

Farmers have an interest in being able to increase the loading, so that the agricultural sector can grow without the need to invest in new technology in order to reduce emissions. On the other hand, groups such as fishermen, drinking water companies and any other industry making use of the water prefer a clear lake, and the same holds for people who use to spend leisure time in relation with the lake. It is assumed that a community or country, balancing these different interests, can agree on a welfare function of the form

$$\log \mathrm{u} - c\mathrm{x}^2 \quad (c > 0),$$

in the following sense: the lake has value as a "waste sink" for agriculture $\log \mathrm{u}$, where u is the input of phosphorus due to farming; on the other hand, it provides ecological services that decrease with the total amount of phosphorus x as $-c\mathrm{x}^2$.

As a final consideration, we observe that clause iv) in Assumption 1 about the convex-concave behaviour of the internal dynamics function F could be substituted with a global lipschitz-continuity or, equivalently, a super-linearity and sub-linearity requisite. This is because a proper relation with affine functions is the only condition that F is required to satisfy in order that the results in Remarks 53 and 54 hold. Such remarks establish the basic comparative estimates for admissible trajectories used in the proof of the existence of an optimal control.
In this respect, the situation is similar to the one in Chapter 1 (see the remark at the end of subsection 1.1.2).

2.1 The optimal control problem

The present section is devoted to defining the optimal control problem and to deducing some elementary consequences of the definitions. The results include the well-posedness of the state equation and some basic comparison estimates between two admissible trajectories and between an admissible trajectory and the solution of a linear ordinary differential equation.

According to the considerations made in the last part of the latter introduction, the dynamics of the problem is described by the following evolution equation in the unknown $x(\cdot)$:

$$\begin{cases} \dot{x}(t) = F(x(t)) + u(t) & t \geq 0 \\ x(0) = x_0. \end{cases} \tag{2.2}$$

Assumption 1. *The endogenous pollution dynamics F has the following properties:*

i) $F \in \mathcal{C}^1([0,+\infty))$, $F' \leq 0$ *in* $(0,+\infty)$

ii) $F(0) = 0$, $\lim_{x \to +\infty} F(x) = -\infty$

iii) $F'_+(0) = -s_0 < 0$, $\lim_{x \to +\infty} F'(x) = -s_\infty \in (-\infty, 0)$

iv) there exists $\bar{x} > 0$ *such that* F *is convex in* $[0, \bar{x}]$ *and concave in* $[\bar{x}, +\infty)$.

Clearly, the behaviour of F for negative inputs is not relevant for modelling purposes. Nevertheless, a conventional assumption for such domain is technically needed.

Assumption 2. *Let* $-b_0 := \min\{-s_0, -s_\infty\}$. *We assume that* $F(x) = -b_0 x$ *for every* $x < 0$.

These assumptions imply that F' has discontinuity in 0 when $s_0 < s_\infty$. Anyway, this possibility is by no means harmful, as shown by the following two remarks.

Remark 50. First observe that, by Cauchy's Theorem, the dynamics F satisfies:

$$F(x) \geq -b_0 x \quad \forall x \in \mathbb{R}, \tag{2.3}$$

since $F' \geq -b_0$ in $(0, +\infty)$ while the equality holds for $x \leq 0$.

Secondly, the function F is globally Lipschitz-continuous. Indeed:

$$\left| \frac{F(x_1) - F(x_2)}{x_1 - x_2} \right| = \frac{F(x_1) - F(x_2)}{x_2 - x_1} \leq b_0 \quad \forall x_1, x_2 \in \mathbb{R}. \tag{2.4}$$

If $-s_0 \leq -s_\infty$, then F is continuously differentiable in $\mathbb{R}$ with $F' \geq -b_0$ in $\mathbb{R}$, and relation (2.4) follows, again, from Cauchy's Theorem.

In the case $-s_0 > -s_\infty$ we have $F'_+(0) > F'_-(0)$, and (2.4) is obtained in the following way. If $x_1 > x_2 \geq 0$ then use again the latter theorem; if $x_1 < x_2 \leq 0$ then the equality holds. If $x_1 \cdot x_2$ <0, we have the following direct estimate, by (2.3) :

$$\begin{aligned} F(x_1) - F(x_2) &\leq -b_0(x_1 - x_2) \quad \text{for } x_1 < 0 < x_2 \\ F(x_1) - F(x_2) &\geq -b_0(x_1 - x_2) \quad \text{for } x_2 < 0 < x_1 \end{aligned}$$

which implies (2.4) for the present case.

We stress that, as a consequence of the definitions, F' satisfies:

$$-b_0 \leq F'(x) \leq 0 \quad \forall x \geq 0.$$

Remark 51. (Solution to the state equation). For every fixed $x_0 \geq 0$ and $u \in L^1_{loc}([0, +\infty))$, the Cauchy's Problem (3.1) admits a unique solution defined in the whole temporal half-line $[0, +\infty)$, as a consequence of (2.4). This is true even if the right hand side of the state equation does not have continuous dependence on time. To prove this, we can use the following simple fixed-point argument.

First, take $\tau_0 \in \left(0, \frac{1}{b_0}\right)$ and consider the space $X_1 := \mathcal{C}^0([0, \tau_0])$ and the map $\mathcal{F}_1 : X_1 \to X_1$ such that $\mathcal{F}_1(x)(t) := x_0 + \int_0^t [F(x(s)) + u(s)]\, \mathrm{d}s$ for every $x \in X_1$, $t \in [0, \tau_0]$.

If we consider the space X_1 together with the distance induced by the L^∞ norm, then $\mathcal{F}_1$ is a contraction mapping on X_1 with Lipschitz constant $\tau_0 b_0 < 1$. Hence F admits a unique fixed point $x(\cdot; x_0, u)$. This is indeed an absolutely continuous function which solves the integral equation corresponding to (3.1) in $[0, \tau_0]$. To extend the solution to $[\tau_0, 2\tau_0]$, consider the space $X_2 := \mathcal{C}^0([\tau_0, 2\tau_0])$ with the map $\mathcal{F}_2 : X_2 \to X_2$ such that $\mathcal{F}_2(x)(t) := x(\tau_0; x_0, u) + \int_{\tau_0}^t [F(x(s)) + u(s)]\, \mathrm{d}s$ for $x \in X_2$ and $t \in [\tau_0, 2\tau_0]$; then argue in the same way.

The procedure can be repeated to obtain an essential solution of equation (3.1) defined in $[0, +\infty)$.

Notation 1. *For every $x_0 \geq 0$ and every $u \in L^1_{loc}([0, +\infty))$ the function*

$$t \to x(t; x_0, u), \quad t \geq 0$$

is the unique solution to the Cauchy's Problem (3.1) *in the unknown* $x(\cdot)$.

Now we introduce the objective functional and its domain.

For every $x_0 \geq 0$, the set of the *admissible controls* is:

$$\Lambda(x_0) \quad := \quad \left\{ u \in L^1_{loc}([0,+\infty)) \,|\, u > 0 \text{ a.e. in } [0,+\infty) \text{ and } \int_0^{+\infty} e^{-\rho t} u(t) \mathrm{d}t < +\infty \right\};$$

the *objective functional* that is to be maximized is defined by

$$\mathcal{B}(x_0; u) = \int_0^{+\infty} e^{-\rho t} \left[\log u(t) - c x^2(t; x_0, u) \right] \mathrm{d}t \quad \forall u \in \Lambda(x_0),$$

where ρ and c are fixed positive constants.
Observe that the integrability condition

$$\int_0^{+\infty} e^{-\rho t} u(t) \mathrm{d}t < +\infty$$

on admissible controls prevents the objective functional to take value $+\infty - \infty$. Indeed, if $u \in \Lambda(x_0)$, then the positive part of the above integral, i.e.

$$\int_0^{+\infty} e^{-\rho t} \left[\log u(t) - c x^2(t; x_0, u) \right]^+ \mathrm{d}t,$$

is certainly finite, since

$$e^{-\rho t} \left[\log u(t) - c x^2(t; x_0, u) \right] \chi_{\{u > \exp c x^2(t; x_0, u)\}}(t) \leq \\ e^{-\rho t} \log u(t) \chi_{\{u > 1\}}(t) \leq e^{-\rho t} u(t).$$

Furthermore, such condition is the analogue of the classical condition $u \in L^1([0,T])$, which is normally required in a finite horizon problem for a control u to be admissible.

Definition 52. The function $V : [0,+\infty) \to \mathbb{R}$ such that

$$V(x_0) = \sup_{u \in \Lambda(x_0)} \mathcal{B}(x_0; u) \quad \forall x_0 \geq 0$$

is called *value function*.
A sequence $(u_n)_{n \in \mathbb{N}} \subseteq \Lambda(x_0)$ is said to be *maximizing at* x_0 if

$$\lim_{n \to +\infty} \mathcal{B}(x_0; u_n) = V(x_0).$$

A control $u_* \in \Lambda(x_0)$ is *optimal* at x_0 if

$$\mathcal{B}(x_0; u_*) = V(x_0).$$

As a last stage of the present introductory section, we state two basic properties of the trajectories $x(\cdot; x_0, u)$ for $x_0 \geq 0$, $u \in \Lambda(x_0)$. These properties are needed throughout the paper in order to carry out the calculations.

Remark 53. (Comparison with the decreasing exponential).
Fix $x_0 \geq 0$, $u \in L^1_{loc}([0, +\infty))$ and set, for simplicity of notation, $x(\cdot) := x(\cdot; x_0, u)$ and $y(t) := e^{-b_0 t}\left(x_0 + \int_0^t e^{b_0 s} u(s)\,\mathrm{d}s\right)$. Then by (2.3) we have, for almost every $t \geq 0$:

$$\begin{aligned} \dot{x}(t) - \dot{y(t)} &= F(x(t)) + b_0 y(t) \\ &\geq -b_0 [x(t) - y(t)]; \end{aligned}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{[x(t) - y(t)] e^{b_0 t}\right\} \geq 0.$$

Since $x(0) = y(0) = x_0$, this proves the following assertion.

$$\begin{aligned} &\text{For every } x_0, t \geq 0 \text{ and every } u \in L^1_{loc}([0, +\infty)): \\ &x(t; x_0, u) \geq e^{-b_0 t}\left(x_0 + \int_0^t e^{b_0 s} u(s)\,\mathrm{d}s\right). \end{aligned} \tag{2.5}$$

Now take $\epsilon > 0$ such that $-b := -s_\infty + \epsilon < 0$. This implies that there exists a constant $M > 0$ such that

$$F(x) \leq -bx + M \quad \forall x \geq 0. \tag{2.6}$$

Fix $x_0 \geq 0$ and $u \in \Lambda(x_0)$. In particular, $u > 0$ almost everywhere in $[0, +\infty)$. Then, by (2.5), $x(\cdot; x_0, u) \geq 0$ in $[0, +\infty)$. Hence, with the same argument as before, we can exploit (2.6) in order to obtain the following estimate for $x(\cdot; x_0, u)$.

$$\begin{aligned} &\text{For every } x_0, t \geq 0 \text{ and every } u \in \Lambda(x_0): \\ &x(t; x_0, u) \leq e^{-bt}\left(x_0 + \int_0^t e^{bs}(M + u(s))\,\mathrm{d}s\right). \end{aligned} \tag{2.7}$$

Remark 54. (Formula for the difference of two trajectories).
Let $s_1, s_2 \geq 0$, $u_1, u_2 \in \Lambda(x_0)$ and $t_0 \geq 0$.

Set $x_1 = x(\cdot; s_1, u_1)$, $x_2 = x(\cdot; s_2, u_2)$ and define:

$$h(x_1, x_2)(\tau) := \begin{cases} \dfrac{F(x_1(\tau)) - F(x_2(\tau))}{x_1(\tau) - x_2(\tau)} & \text{if } x_1(\tau) \neq x_2(\tau) \\ \\ F'_+(x_1(\tau)) & \text{if } x_1(\tau) = x_2(\tau). \end{cases}$$

The function $h(x_1, x_2)$ is continuous in $[0, +\infty)$, since by (2.5) the trajectories x_1 and x_2 take non-negative values. Moreover $-b_0 \leq h \leq 0$ and the following equality holds:

$$\begin{aligned} \forall t \geq t_0 : x_1(t) - x_2(t) &= \exp\left(\int_{t_0}^{t} h(x_1, x_2)(\tau)\, \mathrm{d}\tau\right)(x_1(t_0) - x_2(t_0)) \\ &\quad + \int_{t_0}^{t} \exp\left(\int_{s}^{t} h(x_1, x_2)(\tau)\, \mathrm{d}\tau\right)(u_1(s) - u_2(s))\, \mathrm{d}s. \end{aligned} \tag{2.8}$$

In particular, taking $t_0 = 0$ and $s_1 = s_2$:

$$\forall t \geq 0 :\ x_1(t) - x_2(t) = \int_0^t \exp\left(\int_s^t h(x_1, x_2)(\tau)\, \mathrm{d}\tau\right)(u_1(s) - u_2(s))\, \mathrm{d}s \tag{2.9}$$

Indeed, for every $t \geq t_0$:

$$\begin{aligned} \dot{x}_1(t) - \dot{x}_2(t) &= F(x_1(t)) - F(x_2(t)) + u_1(t) - u_2(t) \\ &= h(x_1, x_2)(t)[x_1(t) - x_2(t)] + u_1(t) - u_2(t). \end{aligned}$$

Multiplying both sides of this equation by $\exp\left(-\int_{t_0}^{t} h(x_1, x_2)(\tau)\, \mathrm{d}\tau\right)$ we obtain:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[(x_1(t) - x_2(t)) \exp\left(-\int_{t_0}^{t} h(x_1, x_2)(\tau)\, \mathrm{d}\tau\right)\right] \\ &= \exp\left(-\int_{t_0}^{t} h(x_1, x_2)(\tau)\, \mathrm{d}\tau\right)(u_1(t) - u_2(t)) \quad \forall t \geq t_0. \end{aligned}$$

Fix $t \geq t_0$ and integrate between t_0 and t; then (2.8) is easily obtained.

Remark 55. (Comparison between trajectories) Relation (2.8) implies a well known comparison result, which in our case can be stated as follows.

Let $s_1, s_2 \geq 0$ and $u_1, u_2 \in \Lambda(x_0)$; then for every $t_0 \geq 0$ and every $t_1 \in (t_0, +\infty]$, if $u_1 \geq u_2$ almost everywhere in $[t_0, t_1]$ and $x(t_0; s_1, u_1) \geq x(t_0; s_2, u_2)$, then

$$x(t; s_1, u_1) \geq x(t; s_2, u_2) \quad \forall t \in [t_0, t_1].$$

2.2 Boundedness of the value function

In this section we show that the value function is bounded from above. That is to say, the objective functional is always less than a fixed, finite quantity not depending on the control or on the initial state.

Together with this result, we prove some estimates concerning the controls and states which are "not too far" from being optimal; such estimates will be used in the construction of the optimum.

Remark 56. The objective functional is not identically equal to $-\infty$. As a trivial example, consider the control $u \equiv 1 \in \Lambda(x_0)$ for every $x_0 \in \mathbb{R}$. Then by (2.7):

$$0 \leq x(t; x_0, u) \leq e^{-bt} x_0 + (M+1) \frac{1 - e^{-bt}}{b}$$

which implies

$$x^2(t) \leq \left(x_0^2 + \frac{(M+1)^2}{b^2} \right) e^{-2bt} + 2(M+1) \frac{x_0}{b} e^{-bt} + \frac{(M+1)^2}{b^2}.$$

Hence

$$\mathcal{B}(x_0; u) = -c \int_0^{+\infty} e^{-\rho t} x^2(t; x_0, u) \, \mathrm{d}t > -\infty.$$

Proposition 57. i) *The value function V satisfies:*

$$V(x_0) \leq \frac{1}{\rho} \log \left(\frac{\rho + b_0}{\sqrt{2ec}} \right) \quad \forall x_0 \geq 0,$$

where e is the Napier's constant.

ii) *For every $x_0 \geq 0$, there exist constants $K_1(x_0), K_2(x_0) > 0$ such that, for every $u \in \Lambda(x_0)$ with $\mathcal{B}(x_0; u)$ sufficiently close to $V(x_0)$:*

$$\int_0^{+\infty} e^{-\rho t} u(t) \, dt \leq K_1(x_0), \tag{2.10}$$

$$\int_0^{+\infty} e^{-\rho t} x(t; x_0, u)(t) \, dt \leq K_2(x_0). \tag{2.11}$$

We will also use the following weaker estimate relative to a control $u \in \Lambda(x_0)$ such that $\mathcal{B}(x_0; u)$ is sufficiently close to $V(x_0)$:

$$\int_0^t u(s) \, \mathrm{d}s < K_1(x_0) e^{\rho t} \quad \forall t \geq 0. \tag{2.12}$$

Proof. i) Let $x_0 \geq 0$, $u \in \Lambda(x_0)$, and, for simplicity of notation, $x = x(\cdot; x_0, u)$ and $\mathcal{B}(u) =$

$\mathcal{B}(x_0;u)$.

As a preliminary remark, observe that, by Tonelli's Theorem:

$$\begin{aligned}\int_0^{+\infty} e^{-(\rho+\beta)t}\int_0^t e^{\beta s}u(s)\,\mathrm{d}s\mathrm{d}t &= \int_0^{+\infty} e^{\beta s}u(s)\int_s^{+\infty} e^{-(\rho+\beta)t}\mathrm{d}t\mathrm{d}s\\ &= \frac{1}{\rho+\beta}\int_0^{+\infty} e^{\beta s}u(s)\,e^{-(\rho+\beta)s}\mathrm{d}s\\ &= \frac{1}{\rho+\beta}\int_0^{+\infty} e^{-\rho s}u(s)\,\mathrm{d}s \end{aligned} \tag{2.13}$$

for every constant $\beta > 0$. Since $t \to \rho e^{-\rho t}\mathrm{d}t$ is a probability density, by Jensen's inequality we have:

$$\begin{aligned}\int_0^{+\infty} e^{-\rho t}x^2(t)\,\mathrm{d}t &\geq \rho\left(\int_0^{+\infty} e^{-\rho t}x(t)\,\mathrm{d}t\right)^2\\ &\geq \rho\left(\int_0^{+\infty} e^{-(\rho+b_0)t}\int_0^t e^{b_0 s}u(s)\,\mathrm{d}s\mathrm{d}t\right)^2\\ &= \frac{\rho}{(\rho+b_0)^2}\left(\int_0^{+\infty} e^{-\rho t}u(t)\,\mathrm{d}t\right)^2,\end{aligned}$$

in which we have used also (2.5), and (2.13) with $\beta = b_0$.

Another application of Jensen's inequality (to the concave function log) allows us to write down the following estimate for $\mathcal{B}(u)$:

$$\begin{aligned}\mathcal{B}(u) &= \int_0^{+\infty} e^{-\rho t}\log u(t)\,\mathrm{d}t - c\int_0^{+\infty} e^{-\rho t}x^2(t)\,\mathrm{d}t\\ &\leq \frac{1}{\rho}\log\left(\rho\int_0^{+\infty} e^{-\rho t}u(t)\,\mathrm{d}t\right) - \frac{c}{\rho(\rho+b_0)^2}\left(\rho\int_0^{+\infty} e^{-\rho t}u(t)\,\mathrm{d}t\right)^2 \qquad (2.14)\\ &\leq \frac{1}{\rho}\max_{z>0}\left(\log z - \frac{c}{(\rho+b_0)^2}z^2\right) = \frac{1}{\rho}\left(\log\frac{\rho+b_0}{\sqrt{2c}} - \frac{1}{2}\right)\\ &= \frac{1}{\rho}\log\left(\frac{\rho+b_0}{\sqrt{2ec}}\right).\end{aligned}$$

ii) Assume that $\mathcal{B}(u) > V(x_0) - 1$. Fix $\tilde{K}(x_0) \geq 0$ such that

$$\frac{1}{\rho}\log z - \frac{c}{\rho(\rho+b_0)^2}z^2 \leq V(x_0) - 1 \quad \forall z > \tilde{K}(x_0).$$

Then by (2.14), the following relation holds:

$$\int_0^{+\infty} e^{-\rho t} u\left(t\right) \mathrm{d}t \leq \frac{1}{\rho} \tilde{K}\left(x_0\right) =: K_1\left(x_0\right).$$

This proves relation (2.10).

In order to prove (2.11), observe that by (2.7) and (2.13) (the latter with $\beta = b$) we have:

$$\begin{aligned}
\int_0^{+\infty} e^{-\rho t} x\left(t\right) \mathrm{d}t &\leq \int_0^{+\infty} e^{-\rho t} \left\{ e^{-bt} x_0 + \int_0^t e^{b(s-t)} \left(M + u\left(s\right)\right) \mathrm{d}s \right\} \mathrm{d}t \\
&= x_0 \int_0^{+\infty} e^{-(\rho+b)t} \mathrm{d}t + M \int_0^{+\infty} e^{-(\rho+b)t} \int_0^t e^{bs} \mathrm{d}s \mathrm{d}t \\
&\quad + \int_0^{+\infty} e^{-(\rho+b)t} \int_0^t e^{bs} u\left(s\right) \mathrm{d}s \mathrm{d}t \\
&= \frac{x_0}{\rho+b} + M \int_0^{+\infty} e^{bs} \int_s^{+\infty} e^{-(\rho+b)t} \mathrm{d}t \mathrm{d}s \\
&\quad + \frac{1}{\rho+b} \int_0^{+\infty} e^{-\rho t} u\left(t\right) \mathrm{d}t \\
&= \frac{x_0}{\rho+b} + \frac{M}{\rho\left(\rho+b\right)} + \frac{1}{\rho+b} \int_0^{+\infty} e^{-\rho t} u\left(t\right) \mathrm{d}t \\
&\leq \frac{x_0}{\rho+b} + \frac{M}{\rho\left(\rho+b\right)} + \frac{K_1\left(x_0\right)}{\rho+b} \\
&=: K_2\left(x_0\right).
\end{aligned}$$

□

2.3 Construction of an optimal control

In this section we prove our main result: the existence of a locally bounded optimal control for the maximization problem defined in Section 2.1.

Theorem 58. *For every $x_0 \geq 0$, there exists a function $u_* \in \Lambda(x_0)$ such that:*

$$\mathcal{B}\left(x_0; u_*\right) = V(x_0).$$

For every $T \in \mathbb{N}$, the function u^* satisfies:

$$\eta\left(x_0, T\right) \leq u_* \leq N\left(x_0, T\right) \;\; a.e. \;\; in \;\; [0, T]$$

for two suitable constants $\eta(x_0, T)$, $N(x_0, T) > 0$ not depending on u^*. In particular, $u_* \in L^\infty_{loc}\left([0, +\infty)\right)$.

Proof. See the end of the Chapter. □

According to the methodological description given in the introduction, we split the proof into three steps.
- First, we set up the main tool of the construction: a pair of uniform localization lemmas that allow to pass form a given maximizing sequence of controls $(u_n)_{n\in\mathbb{N}}$ (see Definition 52) to a family of maximizing sequences with certain boundedness properties.
- Secondly, with fixed initial state x_0, we interpolate between this family of sequences in order to obtain a unique locally bounded maximizing sequence $(v_{n,n})_{n\in\mathbb{N}}$ and a candidate optimal control u_*, linked by the relation: $\log v_{n,n} \rightharpoonup \log u_*$ in $L^1([0,T])$ for every $T \in \mathbb{N}$.
- Eventually, we prove that $\mathcal{B}(x_0; v_{n,n}) \to \mathcal{B}(x_0; u_*)$.

2.3.1 Step one: Uniform localization lemmas

The monotonicity assumption about the dynamics F is used only in the proof of the next two Lemmas.

Lemma 59. *There exists a function $N : [0,+\infty)^2 \to (0,+\infty)$, continuously differentiable and strictly increasing in the second variable, with the following property.*
For every $x_0 \geq 0$, $T > 0$ and for every $u \in \Lambda(x_0)$ with $\mathcal{B}(x_0;u)$ sufficiently close to $V(x_0)$, there exists a control $\tilde{u}^T \in \Lambda(x_0)$ satisfying:

$$\begin{aligned} &\mathcal{B}\left(x_0; \tilde{u}^T\right) \geq \mathcal{B}(x_0; u) \\ &\tilde{u}^T = u \wedge N(x_0, T) \quad a.\ e.\ in\ [0,T]. \end{aligned}$$

In particular, the norm $\left\|\tilde{u}^T\right\|_{L^\infty([0,T])}$ is bounded above by a quantity which does not depend on the initial control u.
Further, the state $x\left(\cdot;\tilde{u}^T, x_0\right)$ associated with the control $\tilde{u}^T$ satisfies:

$$0 \leq x\left(\cdot;\tilde{u}^T, x_0\right) \leq x(\cdot; u, x_0) \quad in\ [0,+\infty). \tag{2.15}$$

Eventually, the bound function N satisfies, for every $x_0 \geq 0$:

$$\lim_{T\to+\infty} Te^{-\rho T} \log N(x_0, T) = 0. \tag{2.16}$$

Proof. Fix $x_0,\ T \geq 0$. The equation

$$\log\beta + \beta b_0 = -Tb_0, \quad \beta > 0 \tag{2.17}$$

has a unique solution, which is strictly less than 1. Call this solution β_T, and define

$$N(x_0, T) := K(x_0)\, \beta_T^{-2} e^{2\rho(T+\beta_T)}, \tag{2.18}$$

where $K(x_0) = K_1(x_0) \vee 1$ and $K_1(x_0)$ is the constant introduced in Proposition 57.

Now fix $u \in \Lambda(x_0)$ with $\mathcal{B}(x_0; u)$ large enough, so that, in particular, relation (2.12) holds for such u, and consider the case $T > 0$.

If $u \le N(x_0, T)$ almost everywhere in $[0, T]$, then set $\tilde{u}^T := u$. In this case there is nothing to show about u, and we can pass directly to the last part of the proof, where the properties of the function $N(x_0, \cdot)$ are established.

If there exists a non-negligible subset of $[0, T]$ in which $u > N(x_0, T)$ then define

$$\begin{aligned}
\tilde{I} &:= \int_0^T [u(t) - u(t) \wedge N(x_0, T)]\, \mathrm{d}t \\
\tilde{u}^T &:= u \wedge N(x_0, T) \cdot \chi_{[0,T]} + \left(u + \tilde{I}\right) \cdot \chi_{(T, T+\beta_T]} + u \cdot \chi_{(T+\beta_T, +\infty)}.
\end{aligned}$$

As an immediate consequence of the definition of admissibility, $\tilde{u}^T \in \Lambda(x_0)$, since $u \in \Lambda(x_0)$ and $N(x_0, T) > 0$.

First we prove that relation (3.18) holds. Clearly $x\left(\cdot; \tilde{u}^T, x_0\right) \ge 0$, by (2.5) and the admissibility of $\tilde{u}^T$. For simplicity of notation we set $N = N(x_0, T)$, $\tilde{x}_T = x\left(\cdot; \tilde{u}^T, x_0\right)$ and $x = x(\cdot; u, x_0)$.

We have $\tilde{x}_T \le x$ in $[0, T]$, by Remark 55.

Fix $t \in (T, T + \beta_T]$, and set $h := h(\tilde{x}_T, x)$, like in Remark 54. Hence, by (2.9):

$$\begin{aligned}
\tilde{x}_T(t) - x(t) &= \int_0^T \exp\left(\int_s^t h(\tau)\mathrm{d}\tau\right)(u(s) \wedge N - u(s))\, \mathrm{d}s \\
&\quad + \tilde{I} \int_T^t \exp\left(\int_s^t h(\tau)\mathrm{d}\tau\right) \mathrm{d}s.
\end{aligned}$$

Since $h \ge -b_0$, the first addend is estimated in the following way:

$$\begin{aligned}
\int_0^T \exp\left(\int_s^t h(\tau)\mathrm{d}\tau\right)(u(s) \wedge N - u(s))\, \mathrm{d}s &\le \int_0^T e^{(s-t)b_0}(u(s) \wedge N - u(s))\, \mathrm{d}s \\
&\le e^{-tb_0} \int_0^T (u(s) \wedge N - u(s))\, \mathrm{d}s \\
&\le -\tilde{I} e^{-(T+\beta_T)b_0}.
\end{aligned}$$

Since $h \le 0$, the second addend is estimated from above by $\tilde{I}\beta_T$.

Thus we obtain:

$$\tilde{x}_T(t) - x(t) \le \tilde{I}\left(\beta_T - e^{-(T+\beta_T)b_0}\right),$$

and the last quantity is zero, by definition of β_T.

This implies that $\tilde{x}_T \leq x$ also in $(T+\beta_T, +\infty)$, again by Remark 55. Hence, relation (3.18) holds.

Now we estimate the "logarithmic" part of the difference between $\mathcal{B}\left(x_0; \tilde{u}^T\right)$ and $\mathcal{B}\left(x_0; u\right)$. By the concavity of the logarithm, we have:

$$
\begin{aligned}
& \int_0^{+\infty} e^{-\rho t}\left(\log \tilde{u}^T(t) - \log u(t)\right) \mathrm{d}t \\
= & \int_0^T e^{-\rho t}\left\{\log\left(u(t) \wedge N\right) - \log u(t)\right\} \mathrm{d}t \\
& + \int_T^{T+\beta_T} e^{-\rho t}\left\{\log\left(u(t) + \tilde{I}\right) - \log u(t)\right\} \mathrm{d}t \\
\geq & \int_0^T e^{-\rho t}\left(u(t) \wedge N\right)^{-1}\left\{u(t) \wedge N - u(t)\right\} \mathrm{d}t \\
& + \tilde{I}\int_T^{T+\beta_T} e^{-\rho t}\left(u(t) + \tilde{I}\right)^{-1} \mathrm{d}t \\
= & \frac{1}{N}\int_0^T e^{-\rho t}\left\{u(t) \wedge N - u(t)\right\} \mathrm{d}t \\
& + \tilde{I}\int_T^{T+\beta_T} e^{-\rho t}\left(u(t) + \tilde{I}\right)^{-1} \mathrm{d}t \\
\geq & \frac{1}{N}\int_0^T \left(u(t) \wedge N - u(t)\right) \mathrm{d}t \\
& + \tilde{I}\int_T^{T+\beta_T} e^{-\rho t}\left(u(t) + \tilde{I}\right)^{-1} \mathrm{d}t \\
= & \tilde{I}\left(\int_T^{T+\beta_T} e^{-\rho t}\left(u(t) + \tilde{I}\right)^{-1} \mathrm{d}t - \frac{1}{N}\right).
\end{aligned} \tag{2.19}
$$

Moreover, by Jensen's inequality:

$$
\begin{aligned}
\int_T^{T+\beta_T} e^{-\rho t}\left(u(t) + \tilde{I}\right)^{-1} \mathrm{d}t & \geq e^{-\rho(T+\beta_T)}\int_T^{T+\beta_T}\left(u(t) + \tilde{I}\right)^{-1} \mathrm{d}t \\
& \geq \beta_T^2 e^{-\rho(T+\beta_T)}\frac{1}{\int_T^{T+\beta_T}\left(u(t) + \tilde{I}\right) \mathrm{d}t} \\
& \geq \beta_T^2 e^{-\rho(T+\beta_T)}\frac{1}{\int_T^{T+\beta_T} u(t)\, \mathrm{d}t + \tilde{I}} \\
& \geq \beta_T^2 e^{-\rho(T+\beta_T)}\frac{1}{\int_0^{T+\beta_T} u(t)\, \mathrm{d}t}
\end{aligned}
$$

where the penultimate inequality holds since $\beta_T < 1$, while the last inequality uses the fact that

$\tilde{I} \leq \int_0^T u(t)\mathrm{d}t$.

Now by Proposition 57 (formula (2.12)) we can complete the latter estimate in the following way:

$$\begin{aligned}
\int_T^{T+\beta_T} e^{-\rho t}\left(u(t)+\tilde{I}\right)^{-1}\mathrm{d}t &\geq K(x_0)^{-1}\beta_T^2 e^{-2\rho(T+\beta_T)} \\
&=: \alpha(x_0,T).
\end{aligned} \tag{2.20}$$

Observe that, by definition, $N(x_0,T)=\alpha(x_0,T)^{-1}$. Hence, joining (3.21) with (2.20) leads to

$$\int_0^{+\infty} e^{-\rho t}\left(\log\tilde{u}^T(t)-\log u(t)\right)\mathrm{d}t \quad \geq \quad \tilde{I}\left(\alpha(x_0,T)-\frac{1}{N(x_0,T)}\right)=0. \tag{2.21}$$

This implies, by (3.18):

$$\begin{aligned}
\mathcal{B}\left(x_0;\tilde{u}^T\right)-\mathcal{B}(x_0;u) &= \int_0^{+\infty} e^{-\rho t}\left(\log\tilde{u}^T(t)-\log u(t)\right)\mathrm{d}t \\
&\quad -c\int_0^{+\infty} e^{-\rho t}\left\{\tilde{x}_T^2(t)-x^2(t)\right\}\mathrm{d}t \\
&\geq 0.
\end{aligned}$$

We now prove the regularity and monotonicity of $N(x_0,T)$ in T.

Define $f(x):=\log x+b_0x$ for every $x>0$. This is a smooth, strictly increasing function which maps $(0,+\infty)$ onto $\mathbb{R}$; hence it is - in particular - a monotone $\mathcal{C}^1$-diffeomorphism. Let ϕ be the inverse of f; then the function $T\to\phi(-Tb_0)$ belongs to $\mathcal{C}^1(\mathbb{R})$. Recall that, by definition, β_T is the unique solution of equation (3.16): this means that $\beta_T=\phi(-Tb_0)$ for every $T\geq 0$, which implies $N(x_0,\cdot)\in\mathcal{C}^1[0,+\infty)$, by (3.17).

With regard to the monotonicity of $N(x_0,\cdot)$, observe first that the function $T\to T+\beta_T$ is strictly increasing. Indeed, we have for every $T\geq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}T}(T+\beta_T)=1-b_0\phi'(-Tb_0)=1-\frac{b_0}{f'(\beta_T)}=1-\frac{b_0\beta_T}{1+b_0\beta_T}>0.$$

Moreover, β_T is a strictly decreasing function of T. This shows that $N(x_0,\cdot)$ is strictly increasing since it is the product of positive strictly increasing functions.

Finally we prove relation (3.15). Observe that:

$$\beta_T\sim e^{-Tb_0}\quad\text{for } T\to+\infty. \tag{2.22}$$

Indeed, with f defined as before, we have:

$$\lim_{x \to 0^+} \frac{f(x)}{\log x} = 1.$$

Hence $\phi(y) \sim e^y$ for $y \to -\infty$ and $\beta_T = \phi(-Tb_0) \sim e^{-Tb_0}$ for $T \to +\infty$.

It follows from (2.22) and (3.17), that:

$$\begin{aligned} Te^{-\rho T} \log N(x_0, T) &= Te^{-\rho T} \log K(x_0) + Te^{-\rho T} \log \left(\beta_T^{-2}\right) \\ &\quad + 2\rho T e^{-\rho T} (T + \beta_T) \\ &\sim Te^{-\rho T} \log \left(\beta_T^{-2}\right) \\ &\sim 2T^2 e^{-\rho T} b_0 \quad \text{for } T \to +\infty. \end{aligned}$$

This shows that (3.15) holds. $\square$

Lemma 60. *There exists a function $\eta : [0, +\infty)^2 \to (0, +\infty)$, smooth and strictly decreasing in the second variable, with the following properties:*
i) for every $x_0, T \geq 0$:

$$\eta(x_0, T) < N(x_0, T),$$

where N is the function defined in Lemma 59;
ii) for every $x_0 \geq 0$ and every $T \geq 1$, if $u \in \Lambda(x_0)$ and $\mathcal{B}(x_0; u)$ is large enough, there exists $u^T \in \Lambda(x_0)$ such that

$$\begin{aligned} &\mathcal{B}\left(x_0; u^T\right) \geq \mathcal{B}(x_0; u) \\ &u^T = (u \wedge N(x_0, T)) \vee \eta(x_0, T) \quad \textit{a. e. in } [0, T]. \end{aligned}$$

In particular the norm $\left\|\log u^T\right\|_{L^\infty([0,T])}$ is bounded above by a quantity which does not depend on u.
iii) For every $x_0 \geq 0$:

$$\lim_{T \to +\infty} Te^{-\rho T} \log \eta(x_0, T) = 0. \tag{2.23}$$

Proof. In order to define the function η, we observe beforehand that, for every $x_0 \geq 0$, there obviously exists a number $L(x_0) > \rho$ such that

$$e^{L(x_0) - \rho} - 2c\rho^{-1} e^{-L(x_0)} \geq 2cK_2(x_0), \tag{2.24}$$

where $K_2(x_0)$ is the positive constant introduced in Proposition 57.

A simple computation shows that the function $T \to e^{(L(x_0)-\rho)T} - 2c\rho^{-1} T e^{-L(x_0)T}$ is increasing if

$$L\left(x_0\right) > \rho + \frac{2c}{\rho}. \tag{2.25}$$

Now choose $L\left(x_0\right)$ satisfying (2.24) and (2.25) and define

$$\eta\left(x_0, T\right) := e^{-L(x_0)T}.$$

The properties in $i)$ and $iii)$ are immediate consequences of the definitions of η and N. For the first property, remind relations (3.16) and (3.17), from which it follows that $N\left(x_0, T\right) > 1$.
We now prove that the property in $ii)$ is satisfied by η.
By the choice of $L(x_0)$ we have:

$$e^{(L(x_0)-\rho)T} - 2c\rho^{-1}Te^{-L(x_0)T} - 2cK_2\left(x_0\right) \geq 0 \quad \forall x_0 \geq 0,\, T \geq 1. \tag{2.26}$$

Fix x_0 and u as in the hypothesis. This means in particular that relation (2.11) holds for the trajectory $x\left(\cdot; x_0, u\right)$. Fix also $T \geq 1$ and take $\tilde{u}^T$ as in Lemma 59. Define $u^T := \tilde{u}^T$ if $\tilde{u}^T \geq \eta\left(x_0, T\right)$ almost everywhere in $[0, T]$, and

$$u^T := \left(\tilde{u}^T \vee \eta\left(x_0, T\right)\right) \chi_{[0,T]} + \tilde{u}^T \chi_{(T,+\infty)}$$

if there exists a subset of $[0, T]$ of positive measure where $\tilde{u}^T < \eta\left(x_0, T\right)$. In this case define also

$$I := \int_0^T \left[\tilde{u}^T\left(s\right) \vee \eta - \tilde{u}^T\left(s\right)\right] \mathrm{d}s.$$

We show that

$$\mathcal{B}\left(x_0; u^T\right) - \mathcal{B}\left(x_0; \tilde{u}^T\right) \geq 0,$$

and the conclusion will follow from Lemma 59.
We provide two different estimates of the quantity $x\left(\cdot; x_0, u_T\right) - x\left(\cdot; x_0, \tilde{u}_T\right)$.
Set, for simplicity of notation: $x_T = x\left(\cdot; x_0, u_T\right)$, $\tilde{x}_T = x\left(\cdot; x_0, \tilde{u}_T\right)$, $h = h\left(x_T, \tilde{x}_T\right)$ (see Remark 54), and $\eta = \eta\left(x_0, T\right)$. Remembering that $h \leq 0$, we have by (2.9), for every $t \in [0, T]$:

$$\begin{aligned} x_T\left(t\right) - \tilde{x}_T\left(t\right) &= \int_0^t e^{\int_s^t h(\tau)\mathrm{d}\tau} \left[u^T\left(s\right) - \tilde{u}^T\left(s\right)\right] \mathrm{d}s \\ &\leq \int_0^T e^{\int_s^t h(\tau)\mathrm{d}\tau} \left[\tilde{u}^T\left(s\right) \vee \eta - \tilde{u}^T\left(s\right)\right] \mathrm{d}s \\ &\leq I. \end{aligned}$$

The same estimate holds for $t > T$, since $u^T = \tilde{u}^T$ in $(T, +\infty)$. Hence:

$$x_T - \tilde{x}_T \leq I \quad \text{in } [0, +\infty). \tag{2.27}$$

Moreover, since $\eta > 0$:

$$\begin{aligned} I &= \int_0^T \left[\tilde{u}^T(s) \vee \eta - \tilde{u}^T(s)\right] \mathrm{d}s \\ &= \int_{[0,T]\cap\{\tilde{u}^T \leq \eta\}} \left[\eta - \tilde{u}^T(s)\right] \mathrm{d}s \\ &\leq T\eta. \end{aligned}$$

Hence

$$x_T - \tilde{x}_T \leq T\eta \quad \text{in } [0, +\infty). \tag{2.28}$$

By (2.27) and (2.28), using the convexity relation $x^2 - y^2 \leq 2x(x - y)$, we obtain:

$$\begin{aligned} c\int_0^{+\infty} e^{-\rho t}\left[x_T^2(t) - \tilde{x}_T^2(t)\right] \mathrm{d}t &\leq 2c\int_0^{+\infty} e^{-\rho t} x_T(t)\left[x_T(t) - \tilde{x}_T(t)\right] \mathrm{d}t \\ &\leq 2cI\int_0^{+\infty} e^{-\rho t} x_T(t)\,\mathrm{d}t \\ &= 2cI\int_0^{+\infty} e^{-\rho t}\left[x_T(t) - \tilde{x}_T(t)\right] \mathrm{d}t \\ &\quad +2cI\int_0^{+\infty} e^{-\rho t}\tilde{x}_T(t)\,\mathrm{d}t \\ &\leq 2cIT\eta\int_0^{+\infty} e^{-\rho t}\mathrm{d}t + 2cI\int_0^{+\infty} e^{-\rho t} x(t; x_0, u)\,\mathrm{d}t \\ &\leq I\left(2\frac{c}{\rho}T\eta + 2cK_2(x_0)\right), \end{aligned}$$

where we also used (3.18) and (2.11) in the second-to-last and last inequality, respectively.

Moreover:

$$\begin{aligned} \int_0^{+\infty} e^{-\rho t}\left(\log u^T(t) - \log\tilde{u}^T(t)\right) \mathrm{d}t &= \int_0^T e^{-\rho t}\left(\log\left(\tilde{u}^T(t) \vee \eta\right) - \log\tilde{u}^T(t)\right) \mathrm{d}t \\ &\geq \int_0^T e^{-\rho t}\frac{1}{\tilde{u}^T(t) \vee \eta}\left(\tilde{u}^T(t) \vee \eta - \tilde{u}^T(t)\right) \mathrm{d}t \\ &= \frac{1}{\eta}\int_0^T e^{-\rho t}\left(\tilde{u}^T(t) \vee \eta - \tilde{u}^T(t)\right) \mathrm{d}t \\ &\geq \frac{e^{-\rho T}}{\eta} I. \end{aligned}$$

Joining the last two estimates leads to:

$$\begin{aligned}
\mathcal{B}\left(x_0; u^T\right) - \mathcal{B}\left(x_0; \tilde{u}^T\right) &= \int_0^{+\infty} e^{-\rho t}\left(\log u^T(t) - \log \tilde{u}^T(t)\right) \mathrm{d}t \\
&\quad -c\int_0^{+\infty} e^{-\rho t}\left[x_T^2(t) - \tilde{x}_T^2(t)\right] \mathrm{d}t \\
&\geq I\left(\frac{e^{-\rho T}}{\eta(x_0,T)} - 2\frac{c}{\rho}T\eta(x_0,T) - 2cK_2(x_0)\right) \\
&= I\left(e^{(L(x_0)-\rho)T} - 2c\rho^{-1}Te^{-L(x_0)T} - 2cK_2(x_0)\right) \\
&\geq 0,
\end{aligned}$$

where the last inequality holds by (3.13). □

2.3.2 Step two: Diagonal procedures

From this point on, the initial state $x_0 \geq 0$ is to be considered fixed.
The next Lemma applies a special diagonal procedure based on the monotonicity of the functions N and η introduced in Lemmas 59 and 60.

Lemma 61. *There exists a sequence of functions $(v_n)_{n\in\mathbb{N}} \subseteq \Lambda(x_0)$ and a function $v \in \Lambda(x_0)$ such that:*

$$\lim_{n\to+\infty} \mathcal{B}(x_0; v_n) = V(x_0) \tag{2.29}$$

$$v_n \rightharpoonup v \text{ in } L^1([0,T]) \quad \forall T > 0 \tag{2.30}$$

$$\begin{aligned}&\forall T \in \mathbb{N} : \text{almost everywhere in } [0,T] : \\ &\forall n \geq T : \eta(x_0,T) \leq v, v_n \leq N(x_0,T)\end{aligned} \tag{2.31}$$

where N, η are the functions defined in Lemmas 59 and 60 .

Proof. Set $\mathcal{B} = \mathcal{B}(x_0; \cdot)$ and fix $(u_n)_{n\in\mathbb{N}} \subseteq \Lambda(x_0)$ such that

$$\lim_{n\to+\infty} \mathcal{B}(u_n) = V(x_0).$$

Define, for every $n \in \mathbb{N}$, u_n^1 as the function obtained by applying Lemma 60 to u_n, for $T = 1$. Then

$$\begin{aligned}&u_n^1 = (u_n \wedge N(x_0,1)) \vee \eta(x_0,1) \quad \text{a.e. in } [0,1] \\ &\mathcal{B}\left(u_n^1\right) \geq \mathcal{B}(u_n).\end{aligned}$$

Hence, as a consequence of the Dunford-Pettis criterion, there exists a subsequence $\left(\overline{u}_n^1\right)_n$ of $\left(u_n^1\right)_n$ and a function $u^1 \in L^1\left([0,1]\right)$ such that

$$\overline{u}_n^1 \rightharpoonup u^1 \text{ in } L^1\left([0,1]\right).$$

Now apply Lemma 60 to the elements of the sequence $\left(\overline{u}_n^1\right)_n$ in order to obtain a sequence $\left(u_n^2\right)_n$ satisfying, for every $n \in \mathbb{N}$:

$$\begin{aligned} & u_n^2 = \left(\overline{u}_n^1 \wedge N\left(x_0, 2\right)\right) \vee \eta\left(x_0, 2\right) \quad \text{a.e. in } [0,2] \\ & \mathcal{B}\left(u_n^2\right) \geq \mathcal{B}\left(\overline{u}_n^1\right). \end{aligned}$$

Take, again by Dunford-Pettis, $\left(\overline{u}_n^2\right)_n$ extracted from $\left(u_n^2\right)_n$ and a function $u^2 \in L^1\left([0,2]\right)$ such that

$$\overline{u}_n^2 \rightharpoonup u^2 \text{ in } L^1\left([0,2]\right).$$

Iterating this process we define two families of sequences of functions:

$$\left\{\left(u_n^T\right)_n \mid T \in \mathbb{N}\right\}, \quad \left\{\left(\overline{u}_n^T\right)_n \mid T \in \mathbb{N}\right\}$$

and a family of integer-valued functions:

$$\left\{\sigma_T \mid T \in \mathbb{N}\right\}.$$

The functions $\sigma_T : \mathbb{N} \to \mathbb{N}$ are strictly increasing with $\sigma_T \geq Id$, and the following relations hold for every $T, n \in \mathbb{N}$:

$$\overline{u}_n^T = u_{\sigma_T(n)}^T \tag{2.32}$$

$$u_n^T = \left(\overline{u}_n^{T-1} \wedge N\left(x_0, T\right)\right) \vee \eta\left(x_0, T\right) \quad \text{a.e. in } [0,T] \tag{2.33}$$

$$\mathcal{B}\left(u_n^T\right) \geq \mathcal{B}\left(\overline{u}_n^{T-1}\right) \tag{2.34}$$

$$\overline{u}_n^T \rightharpoonup u^T \text{ in } L^1\left([0,T]\right) \text{ as } n \to \infty. \tag{2.35}$$

Fix $T, n \in \mathbb{N}$. We have:

$$\begin{aligned} \overline{u}_n^T \quad & = \quad u_{\sigma_T(n)}^T \overset{a.e.\ in\ [0,T]}{=} \left(\overline{u}_{\sigma_T(n)}^{T-1} \wedge N\left(x_0, T\right)\right) \vee \eta\left(x_0, T\right) \\ & \overset{a.e.\ in\ [0,T-1]}{=} \quad \overline{u}_{\sigma_T(n)}^{T-1}. \end{aligned}$$

The last equality holds since relations (2.32) and (2.33) applied to the function $\overline{u}_{\sigma_T(n)}^{T-1}$ imply $\overline{u}_{\sigma_T(n)}^{T-1} \in \left[\eta\left(x_0, T-1\right), N\left(x_0, T-1\right)\right]$ almost everywhere in $[0, T-1]$, and by Lemmas 59 and 60 the function $\eta\left(x_0, \cdot\right)$ is decreasing and the function $N\left(x_0, \cdot\right)$ is increasing.

Thus, for every $n \in \mathbb{N}$, there exists a full measure subset of $[0, T-1]$ in which $\bar{u}_n^T$ coincides with an element of the sequence $(\bar{u}_m^{T-1})_{m\in\mathbb{N}}$. Since a numerable intersection of full measure sets is a full measure set, and remembering the properties of the function σ_T, we deduce that the sequence $\left(\overline{u}_n^T\right)_n$ coincides - almost everywhere in $[0, T-1]$ - with a sequence that is extracted from $\left(\overline{u}_n^{T-1}\right)_n$. This implies $u^{T-1} = u^T$ almost everywhere in $[0, T-1]$, by the essential uniqueness of the weak limit.

Hence, defining

$$\forall t \geq 0 : v\,(t) := u^{\lfloor t \rfloor + 1}\,(t)$$

we obtain $v = u^T$ almost everywhere in $[0, T]$, for every $T \in \mathbb{N}$. Consequently:

$$\forall T \in \mathbb{N} : \overline{u}_n^T \rightharpoonup v \quad \text{in } L^1\,[0, T] \ \text{ as } n \to \infty, \tag{2.36}$$

by (2.35). Repeating the previous argument, we see that, for every $T, n \in \mathbb{N}$ and $1 \leq j \leq T-2$:

$$\begin{aligned} \bar{u}_n^T &\overset{a.e.\ in\ [0,T-1]}{=} \overline{u}_{\sigma_T(n)}^{T-1} \\ &\overset{a.e.\ in\ [0,T-2]}{=} \overline{u}_{\sigma_{T-1}\circ\sigma_T(n)}^{T-2} \\ &\dots \\ &\overset{a.e.\ in\ [0,T-j]}{=} \overline{u}_{\sigma_{T-j+1}\circ\cdots\circ\sigma_T(n)}^{T-j}. \end{aligned}$$

Observe that $\left(\overline{u}_{\sigma_{T-j+1}\circ\cdots\circ\sigma_T(n)}^{T-j}\right)_n$ is a subsequence of $\left(\overline{u}_n^{T-j}\right)_n$ since the composition $\sigma_{T-j+1} \circ \cdots \circ \sigma_T$ is strictly increasing and satisfies

$$\sigma_{T-j+1} \circ \cdots \circ \sigma_T\,(n) \geq n \quad \forall n \in \mathbb{N}.$$

Hence, inverting the quantifiers "$\forall n \in \mathbb{N}$" and "a.e. in $[0, T-j]$", we observe that $\left(\overline{u}_n^T\right)_n$ coincides, almost everywhere in $[0, T-j]$, with a subsequence of $\left(\overline{u}_n^{T-j}\right)_n$, for every $T \in \mathbb{N}$ and $j = 1, \dots, T-1$.

Define $v_n := \overline{u}_n^n$ for every $n \in \mathbb{N}$. The latter assertion implies that, for every fixed $T \in \mathbb{N}$, the sequence $(v_n)_{n\geq T}$ coincides with a subsequence of $\left(\overline{u}_n^T\right)_{n\geq 1}$, almost everywhere in $[0, T]$. Hence, by relations (2.32) and (2.33), $(v_n)_n$ satisfies:

$$\begin{aligned} &\forall T \in \mathbb{N} : \text{almost everywhere in } [0, T] : \\ &\forall n \geq T : \eta\,(x_0, T) \leq v_n \leq N\,(x_0, T)\,. \end{aligned} \tag{2.37}$$

Moreover:

$$v_n \rightharpoonup v \text{ in } L^1\,([0, T]) \quad \forall T \in \mathbb{N},$$

by (2.36).

The extension to every $T > 0$ is straightforward, and we obtain (2.30). Now fix $T > 0$; a well known property of weak convergence implies

$$\liminf_{n\to+\infty} v_n(t) \leq v(t) \leq \limsup_{n\to+\infty} v_n(t) \text{ for almost every } t \in [0,T]. \tag{2.38}$$

Considering the intersection between the two subsets of $[0,T]$ where relations (2.37) and (2.38) hold respectively, we obtain (2.31) also for v.

In order to prove (2.29), observe beforehand that $(v_n)_n \subseteq \Lambda(x_0)$ by construction: every function in the sequence is actually obtained by applying Lemma 60. Moreover:

$$\begin{aligned}
\mathcal{B}(v_n) &= \mathcal{B}\left(u^n_{\sigma_n(n)}\right) \geq \mathcal{B}\left(\overline{u}^{n-1}_{\sigma_n(n)}\right) \\
&= \mathcal{B}\left(u^{n-1}_{\sigma_{n-1}\circ\sigma_n(n)}\right) \geq \cdots \geq \mathcal{B}\left(u^{n-2}_{\sigma_{n-2}\circ\sigma_{n-1}\circ\sigma_n(n)}\right) \\
&\geq \cdots \geq \mathcal{B}\left(u^1_{\sigma_1\circ\cdots\circ\sigma_n(n)}\right) \geq \mathcal{B}\left(u_{\sigma_1\circ\cdots\circ\sigma_n(n)}\right).
\end{aligned}$$

Fix $\epsilon > 0$ and $n_\epsilon \in \mathbb{N}$ such that $V(x_0) - \mathcal{B}(u_n) < \epsilon$ for $n \geq n_\epsilon$; since $\sigma_1 \circ \cdots \circ \sigma_m \geq Id$, we have

$$V(x_0) - \mathcal{B}(v_n) < \epsilon \quad \forall n \geq n_\epsilon.$$

Eventually we prove the admissibility of v. We have established that relation (2.31) holds for v: this implies that $v \in L^\infty_{loc}([0,+\infty))$ and $v > 0$ almost everywhere in $[0,+\infty)$. In order to prove that the discounted integral of v is finite, start by observing that, since $(v_n)_{n\in\mathbb{N}}$ is a maximizing sequence, relation (2.10) holds for the functions v_n with n sufficiently big, i.e.:

$$\int_0^{+\infty} e^{-\rho t} v_n(t)\mathrm{d}t \leq K_1(x_0) \quad \forall n \geq n_0.$$

By (2.30), we have, for every fixed $T > 0$:

$$\int_0^T e^{-\rho t} v(t)\mathrm{d}t = \lim_{n\to\infty} \int_0^T e^{-\rho t} v_n(t)\mathrm{d}t \leq \limsup_{n\to\infty} \int_0^{+\infty} e^{-\rho t} v_n(t)\mathrm{d}t \leq K_1(x_0).$$

Passing to the limit for $T \to +\infty$ we obtain:

$$\int_0^{+\infty} e^{-\rho t} v(t)\mathrm{d}t \leq K_1(x_0). \tag{2.39}$$

□

At this point we need to take into account the functions $\log(v_n)$ and extract a subsequence

$(v_{n,n})_{n\in\mathbb{N}}$ from $(v_n)_{n\in\mathbb{N}}$ - in order to ensure that the values $\mathcal{B}(x_0; v_{n,n})$ have the right asymptotic behavior. This is done thorugh a "standard" diagonalization.

Lemma 62. *Take $(v_n)_{n\in\mathbb{N}}$ and v as in Lemma 61. There exists a sequence $(v_{n,n})_{n\in\mathbb{N}}$, extracted from $(v_n)_{n\in\mathbb{N}}$, and a function $u_* \in \Lambda(x_0)$, satisfying:*

$$\text{for every } T \in \mathbb{N}:$$

$$\log v_{n,n} \rightharpoonup \log u_* \quad \text{in } L^1([0,T]) \tag{2.40}$$

$$\eta(x_0,T) \le u_* \le N(x_0,T) \qquad \text{a. e. in } [0,T]. \tag{2.41}$$

Moreover:

$$u_* \le v \quad \text{a. e. in } [0,+\infty). \tag{2.42}$$

Proof. Observe that the sequence $(\log v_n)_{n\in\mathbb{N}}$ is uniformly bounded in the $L^\infty_{[0,1]}$ norm, by (2.31). Precisely, the following relation holds almost everywhere in $[0,1]$:

$$\log \eta(x_0,1) \le \log v_n \le \log N(x_0,1) \quad \forall n \ge 1.$$

Hence by the Dunford-Pettis criterion there exists a function $f^1 \in L^1([0,1])$ and a sequence $(v_{n,1})_n$ extracted form (v_n) such that

$$\log v_{n,1} \rightharpoonup f^1 \quad \text{in } L^1([0,1]).$$

Again by (2.31), $(v_{n,1})_n$ satisfies, almost everywhere in $[0,2]$:

$$\log \eta(x_0,2) \le \log v_{n,1} \le \log N(x_0,2) \quad \forall n \ge 2;$$

therefore there exist $f^2 \in L^1([0,2])$ and $(v_{n,2})_n$ extracted from $(v_{n,1})_n$ such that

$$\log v_{n,2} \rightharpoonup f^2 \quad \text{in } L^1([0,2]),$$

and so on. This shows that, for every $T \in \mathbb{N}$:

$$\text{almost everywhere in [0,T]} : \forall n \ge T:$$
$$\log \eta(x_0,T) \le \log v_{n,n} \le \log N(x_0,T)$$

and there exists a function $f \in L^1_{loc}([0,+\infty))$ such that

$$\log v_{n,n} \rightharpoonup f \quad \text{in } L^1([0,T]) \quad \text{as } n \to \infty.$$

Define $u_* := e^f$; then relations (2.40) and (2.41) are easy consequences of this definition and of the properties of the weak convergence.

We now prove relation (2.42). Fix $0 < t_0 < t_1 < T$ and let t_0 be a Lebesgue point for both $\log u_*$ and v. By Jensen's inequality we have, for every $n \in \mathbb{N}$:

$$\frac{\int_{t_0}^{t_1} \log v_{n,n}(s)\, \mathrm{d}s}{t_1 - t_0} \leq \log\left(\frac{\int_{t_0}^{t_1} v_{n,n}(s)\, \mathrm{d}s}{t_1 - t_0}\right);$$

since $(v_{n,n})_n$ is a subsequence of $(v_n)_n$, passing to the limit for $n \to +\infty$ in the previous relation, we obtain by (2.30) and (2.40):

$$\frac{\int_{t_0}^{t_1} \log u_*(s)\, \mathrm{d}s}{t_1 - t_0} \leq \log\left(\frac{\int_{t_0}^{t_1} v(s)\, \mathrm{d}s}{t_1 - t_0}\right).$$

Passing now to the limit for $t_1 \to t_0$ yields to $\log u_*(t_0) \leq \log v(t_0)$. By the Lebesgue Point Theorem, t_0 is a generic element of a full measure subset of $[0, T]$. This implies (2.42). Hence, by (2.39):

$$\int_0^{+\infty} e^{-\rho t} u_*(t) \mathrm{d}t \leq \int_0^{+\infty} e^{-\rho t} v(t) \mathrm{d}t \leq K_1(x_0).$$

This relation, together with (2.41), proves that $u_* \in \Lambda(x_0)$. □

2.3.3 Step three: Functional convergence

In this last sub-section, we complete the proof of the existence of an optimal control for the Shallow Lake problem with monotone dynamics. First we establish that the states associated with the new maximizing sequence $(v_n)_n$ converge pointwise to the state associated with the control v obtained in Lemma 61. Then, relying *inter alia* on the relation between v and u_* established in (2.42), we will be able to prove the final step of the main result, i.e. the convergence $\mathcal{B}(x_0; v_{n,n}) \to \mathcal{B}(x_0; u_*)$.

Proposition 63. *Let $(v_n)_n$ and v be as in Lemma 61. Then*

$$x(\cdot; x_0, v_n) \to x(\cdot; x_0, v) \quad \textit{pointwise in } [0, +\infty).$$

Proof. Fix $T > 0$ and set $x_n := x(\cdot; x_0, v_n)$, $x := x(\cdot; x_0, v)$. By Remark 55 and by (2.31), the following uniform estimate holds:

$$|x - x_n| \leq x(\cdot; x_0, N(x_0, T)) \quad \text{in } [0, T], \forall n \geq T. \tag{2.43}$$

Now fix $t \in [0, T]$ and $n \geq T$. Subtracting the state equation for x from the state equation for

x_n, we obtain, for almost every $s \in [0,t]$:

$$\begin{aligned} \dot{x_n}(s) - \dot{x}(s) &= F(x_n(s)) - F(x(s)) + v_n(s) - v(s) \\ &= h_n(s)[x_n(s) - x(s)] + v_n(s) - v(s), \end{aligned}$$

where $h_n := h(x_n, x)$ is the function defined in Remark 54.

Integrating both sides of this equation between 0 and t, then taking absolute values yields to:

$$|x_n(t) - x(t)| \quad \le \int_0^t |h_n(s)| \, |x_n(s) - x(s)| \, \mathrm{d}s \quad + \left| \int_0^t [v_n(s) - v(s)] \, \mathrm{d}s \right|. \tag{2.44}$$

Observe that, for every $s \in [0,t]$:

$$|h_n(s)| \, |x_n(s) - x(s)| \quad \le \quad b_0 x(s; x_0, N(x_0, T)),$$

by Remark 54 and by (2.43).

Since the function on the right hand side of the latter relation obviously belongs to $L^1([0,t])$, passing to the limsup in (2.44) and remembering (2.30), we obtain by the Fatou–Lebesgue theorem:

$$\begin{aligned} \limsup_{n\to+\infty} |x_n(t) - x(t)| &\le \limsup_{n\to+\infty} \int_0^t |h_n(s)| \, |x_n(s) - x(s)| \, \mathrm{d}s \\ &\le \int_0^t \limsup_{n\to+\infty} |h_n(s)| \, |x_n(s) - x(s)| \, \mathrm{d}s \\ &\le b_0 \int_0^t \limsup_{n\to+\infty} |x_n(s) - x(s)| \, \mathrm{d}s. \end{aligned} \tag{2.45}$$

Hence by Gronwall's inequality:

$$\limsup_{n\to+\infty} |x_n(t) - x(t)| = 0,$$

for every $t \in [0,T]$. This is equivalent to

$$\lim_{n\to+\infty} x_n = x \quad \text{in } [0,T],$$

which proves the thesis, since $T > 0$ is generic. $\square$

For completeness sake, we give a proof of the reverse Fatous's Lemma.

Lemma 64. *Let (E, σ, μ) a measure space, f_n $(n \in \mathbb{N})$ and g μ-measurable functions in E,*

$F \subseteq E$ a full measure set such that:

$$\forall n \in \mathbb{N} : f_n \leq g \quad \textit{in } F$$
$$\int_E g \, d\mu < +\infty.$$

Then

$$\limsup_{n\to+\infty} \int_E f_n \, d\mu \leq \int_E \limsup_{n\to+\infty} f_n \, d\mu.$$

Proof. Case I. $\int_E g \mathrm{d}\mu = -\infty$. Then

$$\limsup_{n\to+\infty} \int_E f_n \mathrm{d}\mu = -\infty$$

and the thesis is trivially true.

Case II. $\int_E g \mathrm{d}\mu \in -(\infty, +\infty)$

The sequence

$$a_n := g - \sup_{k\geq n} f_k$$

satisfies

$$0 \leq a_n \uparrow g - \limsup_{m\to+\infty} f_m \quad \text{in } F.$$

Hence by Monotone convergence:

$$\int_E \left(g - \sup_{k\geq n} f_k \right) \mathrm{d}\mu = \int_E a_n \mathrm{d}\mu \uparrow \int_E \left(g - \limsup_{m\to+\infty} f_m \right) \mathrm{d}\mu. \tag{2.46}$$

Observe that the quantities

$$\int_E \left(-\sup_{k\geq n} f_k \right) \mathrm{d}\mu := \int_E \left(g - \sup_{k\geq n} f_k \right) \mathrm{d}\mu - \int_E g \mathrm{d}\mu$$
$$\int_E \left(-\limsup_{m\to+\infty} f_m \right) \mathrm{d}\mu := \int_E \left(g - \limsup_{m\to+\infty} f_m \right) \mathrm{d}\mu - \int_E g \mathrm{d}\mu$$

make sense and belong to $(-\infty, +\infty]$. It follows from (2.46) that:

$$\lim_{n\to+\infty} \int_E \left(-\sup_{k\geq n} f_k \right) \mathrm{d}\mu = \int_E \left(-\limsup_{m\to+\infty} f_m \right) \mathrm{d}\mu, \tag{2.47}$$

by the assumption on $\int_E g \mathrm{d}\mu$. Hence:

$$\inf_{n\in\mathbb{N}} \int_E \sup_{k\geq n} f_k \mathrm{d}\mu \quad = \quad \int_E \limsup_{m\to+\infty} f_m \mathrm{d}\mu.$$

Finally, it is a consequence of the definition of sup that

$$\limsup_{m\to+\infty}\int_E f_m \mathrm{d}\mu \le \inf_{n\in\mathbb{N}}\int_E \sup_{k\ge n} f_k \mathrm{d}\mu.$$

$\square$

Now with a simple integration by parts we obtain a useful decomposition of the objective functional:

$$\begin{aligned}
\forall \mathrm{u}\in\Lambda(x_0):\ \mathcal{B}(x_0;\mathrm{u}) &= \int_0^{+\infty} e^{-\rho t}\left(\log \mathrm{u}(t) - cx^2(t)\right)\mathrm{d}t\\
&= \int_0^{+\infty} e^{-\rho t}\log \mathrm{u}(t)\,\mathrm{d}t - c\int_0^{+\infty} e^{-\rho t}x^2(t)\,\mathrm{d}t\\
&= \lim_{T\to+\infty} e^{-\rho T}\int_0^T \log \mathrm{u}(s)\,\mathrm{d}s +\\
&\quad\ \rho\int_0^{+\infty} e^{-\rho t}\left(\int_0^t \log \mathrm{u}(s)\,\mathrm{d}s - \frac{c}{\rho}x^2(t)\right)\mathrm{d}t\\
&=: \lim_{T\to+\infty} e^{-\rho T}\int_0^T \log \mathrm{u}(t)\,\mathrm{d}t + \mathcal{B}_1(x_0;\mathrm{u})
\end{aligned}$$

where

$$\mathcal{B}_1(x_0;\mathrm{u}) := \rho\int_0^{+\infty} e^{-\rho t}\left(\int_0^t \log \mathrm{u}(s)\,\mathrm{d}s - \frac{c}{\rho}x^2(t;x_0,\mathrm{u})\right)\mathrm{d}t.$$

With this notation, we can complete the proof of the main result.

Proof of Theorem 58. Choose u_* and $(v_{n,n})_{n\in\mathbb{N}}$ as in Lemma 62. We are going to show that $\mathcal{B}(x_0;u_*)\ge V(x_0)$. We can assume that relation (2.12) holds for the functions $v_{n,n}$, for every $n\in\mathbb{N}$. Hence, by Jensen's inequality, we have for every $t>0$:

$$\begin{aligned}
e^{-\rho t}\int_0^t \log v_{n,n}(s)\,\mathrm{d}s &\le te^{-\rho t}\log\left(\frac{\int_0^t v_{n,n}(s)\,\mathrm{d}s}{t}\right)\\
&\le te^{-\rho t}\log\left(\frac{K(x_0)e^{\rho t}}{t}\right).
\end{aligned}\tag{2.48}$$

This implies that $\lim_{t\to+\infty} e^{-\rho t}\int_0^t \log v_{n,n}(s)\,\mathrm{d}s \le 0$ and consequently

$$\mathcal{B}(x_0;v_{n,n}) \le \mathcal{B}_1(x_0;v_{n,n}).\tag{2.49}$$

Moreover

$$\int_0^{+\infty} te^{-\rho t}\log\left(\frac{K(x_0)e^{\rho t}}{t}\right)\mathrm{d}t < +\infty.\tag{2.50}$$

Set $x_{n,n} := x\left(\cdot; x_0, v_{n,n}\right)$, $x := \left(\cdot; x_0, v\right)$ and $x_* := \left(\cdot; x_0, u_*\right)$. Relations (2.48) and (2.50) imply that the hypotheses of Lemma 64 are satisfied for the functions:

$$
\begin{aligned}
f_n(t) &= e^{-\rho t}\left(\int_0^t \log v_{n,n}\left(s\right) \mathrm{d}s - \frac{c}{\rho} x_{n,n}^2\left(t\right)\right) \\
g(t) &= t e^{-\rho t} \log\left(\frac{K(x_0) e^{\rho t}}{t}\right).
\end{aligned}
$$

Combining this result with relations (2.49), (2.40) and with Proposition 63 we obtain, since $\left(x_{n,n}\right)_n$ is extracted from $\left(x_n\right)_n$:

$$
\begin{aligned}
V\left(x_0\right) &= \lim_{n\to+\infty} \mathcal{B}\left(x_0; v_{n,n}\right) \leq \limsup_{n\to+\infty} \mathcal{B}_1\left(x_0; v_{n,n}\right) \\
&= \rho \limsup_{n\to+\infty} \int_0^{+\infty} e^{-\rho t}\left(\int_0^t \log v_{n,n}\left(s\right) \mathrm{d}s - \frac{c}{\rho} x_{n,n}^2\left(t\right)\right) \mathrm{d}t \\
&\leq \rho \int_0^{+\infty} e^{-\rho t} \limsup_{n\to+\infty}\left(\int_0^t \log v_{n,n}\left(s\right) \mathrm{d}s - \frac{c}{\rho} x_{n,n}^2\left(t\right)\right) \mathrm{d}t \\
&= \rho \int_0^{+\infty} e^{-\rho t}\left(\int_0^t \log u_*\left(s\right) \mathrm{d}s - \frac{c}{\rho} x^2\left(t\right)\right) \mathrm{d}t \\
&\leq \rho \int_0^{+\infty} e^{-\rho t}\left(\int_0^t \log u_*\left(s\right) \mathrm{d}s - \frac{c}{\rho} x_*^2\left(t\right)\right) \mathrm{d}t \\
&= \mathcal{B}_1\left(x_0; u_*\right).
\end{aligned}
$$

The last inequality is a consequence of (2.42) and of Remark 55.
Finally observe that by (2.41), for every $t \geq 0$:

$$
t e^{-\rho t} \log \eta\left(x_0, t+1\right) \leq e^{-\rho t} \int_0^t \log u_*\left(s\right) \mathrm{d}s \leq t e^{-\rho t} \log N\left(x_0, t+1\right),
$$

which implies that the estimated quantity vanishes for $t \to +\infty$, by (3.15) and (2.23).
Hence $\mathcal{B}_1\left(x_0; u_*\right) = \mathcal{B}\left(x_0; u_*\right)$, and this concludes the proof. $\square$

Chapter 3

Shallow Lake models with non-monotone dynamics

In this chapter we prove the existence of optimal solutions to Shallow Lake problems with non-monotone dynamics, under the assumption that the discount exponent in the objective functional is sufficiently big with respect to the derivative of the dynamics. From the methodological viewpoint, such goal requires a significant improvement of the technique introduced in the previous chapter: a new "discount reduction" procedure is implemented; the localization procedure is empowered and the interpolation cycle is adapted to the new context.

Again the convexity-concavity hypothesis for the dynamics contained in Assumption 3 is not essential in order to be able to carry out the proofs and could be replaced by a suitable comparative relation with affine functions (see the final remark in subsection 1.1.2 and the comment at the end of the introduction to Chapter 2).
This could be seen as a hint of a possible generalizability of the methods here developed to higher dimensional optimal control problems.

3.1 Definition of the problem and preliminary results.

The dynamics of the problem is described by the following evolution equation in the unknown $x(\cdot)$:

$$\begin{cases} \dot{x}\,(t) = F\,(x\,(t)) + u\,(t) & t \geq 0 \\ x\,(0) = x_0. \end{cases} \tag{3.1}$$

The state equation obeys to the following conditions.

Assumption 3. *The dynamics F has the following properties:*

i) $F \in \mathcal{C}^1\left([0,+\infty)\right)$

ii) $F(0) = 0$, $\lim_{x\to+\infty} F(x) = -\infty$

iii) $F'_+(0) = -s_0 < 0$, $\lim_{x\to+\infty} F'(x) = -s_\infty \in (-\infty, 0)$

iv) there exists $\bar{x} > 0$ such that F is convex in $[0,\bar{x}]$ and concave in $[\bar{x},+\infty)$

v) $a := F'(\bar{x}) = \max_{[0,+\infty)} F' > 0$.

Clearly, the behaviour of F for negative inputs is not relevant for modeling purposes. Nevertheless, a conventional assumption for such domain is technically needed.

Assumption 4. *Let $-b_0 := \min\{-s_0, -s_\infty\}$. We assume that $F(x) = -b_0 x$ for every $x < 0$.*

The last assumption implies that F' has discontinuity in 0 when $s_0 < s_\infty$. Anyway, this possibility is by no means harmful.

It is a consequence of these assumptions that

$$-b_0 \le F' \le a.$$

Finally, we assume that the discount factor is sufficiently big with respect to the dynamics of the problem.

Assumption 5.

$$\rho > a + b_0.$$

Remark 65. In the case of the archetypal Shallow Lake model, namely:

$$F(x) = -bx + \frac{x^2}{x^2+1} \quad \forall x \ge 0,$$

we have $b_0 = s_0 = s_\infty = b$, $a = -b + \frac{3\sqrt{3}}{8} > 0$ (the non-monotonicity assumption) and $\rho > \frac{3\sqrt{3}}{8}$. The case $b \ge \frac{3\sqrt{3}}{8}$ is covered by the previous analysis, without any restriction on the value of ρ.

Remark 66. i) The dynamics $F : \mathbb{R} \to \mathbb{R}$ is Liptschitz-continuous and satisfies:

$$\begin{aligned} F(x) &\geq -b_0 x \quad \forall x \in \mathbb{R} \\ F(x) &\geq -bx - M \quad \forall x \in \mathbb{R} \\ F(x) &\leq -bx + M \quad \forall x \geq 0, \end{aligned}$$

for some constants $b, M > 0$.

ii) The state initial value problem (3.1) - for fixed $x_0 \geq 0$ and $u \in L^1_{loc}([0, +\infty))$ - admits a unique solution, which we denote by $x(\cdot; x_0, u)$.

The set of the admissible controls *is:*

$$\Lambda(x_0) = \left\{ u \in L^1_{loc}([0, +\infty)) \,\middle|\, u > 0 \text{ a.e. in } [0, +\infty) \text{ and } \int_0^{+\infty} e^{-\rho t} u(t)\,\mathrm{d}t < +\infty \right\},$$

and the *objective functional*, defined for $u \in \Lambda(x_0)$, is:

$$\mathcal{B}(x_0; u) := \int_0^{+\infty} e^{-\rho t} \left[\log u(t) - c x^2(t; x_0, u) \right] \mathrm{d}t.$$

Remark 67. The objective functional is not identically equal to $-\infty$. As a trivial example, consider the control $u \equiv 1 \in \Lambda(x_0)$.

Proposition 68. *i) For every $x_0 \geq 0$, $u \in \Lambda(x_0)$ and $t \geq 0$:*

$$x(t; x_0, u) \geq e^{-b_0 t} \left(x_0 + \int_0^t e^{b_0 s} u(s)\, ds \right) \tag{3.2}$$

$$x(t; x_0, u) \geq e^{-bt} \left(x_0 + \int_0^t e^{bs} [u(s) - M]\, ds \right) \tag{3.3}$$

$$x(t; x_0, u) \leq e^{-bt} \left(x_0 + \int_0^t e^{bs} [u(s) + M]\, ds \right). \tag{3.4}$$

ii) For every $x_0 \geq 0$ and $u_1, u_2 \in \Lambda(x_0)$:

$$x(t; x_0, u_1) - x(t; x_0, u_2) = \int_0^t \exp\left(\int_s^t h(u_1, u_2)(\tau)\, d\tau \right) (u_1(s) - u_2(s))\, ds, \tag{3.5}$$

where $h(u_1, u_2) : [0, +\infty) \to \mathbb{R}$ is a continuous function satisfying:

$$-b_0 \leq h(u_1, u_2)(t) \leq a \quad \forall t \geq 0.$$

iii) Let $s_1, s_2 \geq 0$ and $u_1, u_2 \in \Lambda(x_0)$; then for every $t_0 \geq 0$ and every $t_1 \in (t_0, +\infty]$, if $u_1 \geq u_2$

almost everywhere in $[t_0, t_1]$ and $x(t_0; s_1, u_1) \geq x(t_0; s_2, u_2)$, then

$$x(t; s_1, u_1) \geq x(t; s_2, u_2) \quad \forall t \in [t_0, t_1]. \tag{3.6}$$

Proof. The proof of i) is a consequence of Remark 66, and is obtained essentially in the same way as in Remark 53 of the previous Chapter. The proof of point ii) is analogous to that of Remark 54, and point iii) follows from point ii). $\square$

Definition 69. The function $V : [0, +\infty) \to \mathbb{R}$ such that

$$V(x_0) = \sup_{u \in \Lambda(x_0)} \mathcal{B}(x_0; u) \quad \forall x_0 \geq 0$$

is called *value function*.
A sequence $(u_n)_{n\in\mathbb{N}} \subseteq \Lambda(x_0)$ is said to be *maximizing at* x_0 if

$$\lim_{n\to+\infty} \mathcal{B}(x_0; u_n) = V(x_0).$$

A control $u_* \in \Lambda(x_0)$ is *optimal* at x_0 if

$$\mathcal{B}(x_0; u_*) = V(x_0).$$

The proof of the boundedness of the value function is the same as in the monotone case. We state the result for completeness' sake.

Proposition 70. *i) The value function satisfies:*

$$V(x_0) \leq \frac{1}{\rho} \log\left(\frac{\rho + b_0}{\sqrt{2ec}}\right) \quad \forall x_0 \geq 0,$$

where e is the Napier's constant.

ii) For every $x_0 \geq 0$, there exists a constant $K(x_0) > 0$ such that, for every $u \in \Lambda(x_0)$ with $\mathcal{B}(x_0; u) \geq V(x_0) - 1$:

$$\int_0^{+\infty} e^{-\rho t} u(t)\, dt, \int_0^{+\infty} e^{-\rho t} x(t; x_0, u)\, dt \leq K(x_0).$$

3.2 Proof of the main result

In this section we give a proof of the existence of an optimal control for any Shallow Lake type problem with non monotone dynamics F. Namely, we will prove the following theorem.

Theorem 71. *For every $x_0 > 0$, the optimal control problem for the class $\Lambda(x_0)$ defined in Section 3.1 admits a solution $u_* \in L^\infty_{loc}([0, +\infty))$. Precisely, the optimum satisfies, for every $T > 0$:*

$$\eta(x_0, T) \leq u_* \leq N(x_0, T) \quad a.e. \text{ in } [0, T],$$

where $\eta(x_0, T)$ and $N(x_0, T)$ are strictly positive quantities not depending on u_*.

The proof will be divided into various lemmas forming two subsections, following the scheme outlined in the introduction.

3.2.1 Discount reduction and localization

In this subsection we will see how it is possible to improve a given admissible control (in the sense of the objective functional $\mathcal{B}$) in such way that the control obtained is bounded in a given time interval $[0, T]$, by a quantity that does not depend on the initial control. To perform this localization procedure, we need that the input control belongs to a special class of functions whose integral in $[0, +\infty)$ "bears" a bigger discount factor (i.e. a less negative discount exponent) than a general admissible control. The first lemma of this subsection ensures that such an input control can be chosen: we call this procedure *discount reduction.*

Both the localization procedure and the discount reduction procedure will preserve the values of the given control at $[0, T]$, as well as the property of being "sufficiently close" to optimality.

Definition 72. i) For every $x_0 > 0$, the quantities $T(x_0)$, $\gamma(x_0)$ are defined as follows:

If $\frac{2b_0+\rho}{2cx_0} \leq 1$, then

$$T(x_0) = 1$$
$$\gamma(x_0) = b_0.$$

If $\frac{2b_0+\rho}{2cx_0} > 1$, then $T(x_0)$ is such that $T(x_0) > 1$ and

$$\log\left(\frac{2b_0+\rho}{2cx_0}\right)\frac{1}{T(x_0)} + b_0 < \rho - a.$$

In this case:

$$\gamma(x_0) := \log\left(\frac{2b_0+\rho}{2cx_0}\right)\frac{1}{T(x_0)} + b_0.$$

ii) For every $x_0 > 0$, $K_1(x_0) = \frac{1}{\rho - a - \gamma(x_0)}$.

We introduce some additional notation. Herein, $K(x_0)$ is the positive constant in point ii) of Proposition 70, while $L^{1,+}_{loc}([0,+\infty))$ denotes the set of all the functions which are locally integrable and almost everywhere strictly positive in $[0,+\infty)$.

We set:

$$\begin{aligned}
S(x_0) &= \left\{ u \in \Lambda(x_0) \middle| \mathcal{B}(x_0;u) \geq V(x_0) - 1 \right\} \\
\Lambda_0(x_0) &= \left\{ u \in L^{1,+}_{loc}([0,+\infty)) \middle| \int_0^{+\infty} e^{-\rho t} u(t)\,\mathrm{d}t \leq K(x_0) \text{ and } \mathcal{B}(x_0;u) > -\infty \right\} \\
\forall T > 0 : \Lambda^T(x_0) &= \left\{ u \in L^{1,+}_{loc}([0,+\infty)) \middle| \int_0^{+\infty} e^{-(\rho-a)t} u(t)\,\mathrm{d}t \leq e^{aT} K(x_0) + K_1(x_0) \right\}.
\end{aligned}$$

Remark 73. By Proposition 70, ii) we have, for every $x_0 \geq 0$:

$$\begin{aligned}
&S(x_0) \subseteq \Lambda_0(x_0) \subseteq \Lambda(x_0) \\
&\Lambda^T(x_0) \subseteq \Lambda(x_0) \quad \forall T > 0.
\end{aligned}$$

Moreover, $S(x_0)$ is "closed under improvement", in the sense that:

$$u_1 \in S(x_0), u_2 \in \Lambda(x_0) \text{ and } \mathcal{B}(x_0;u_2) \geq \mathcal{B}(x_0;u_1) \implies u_2 \in S(x_0).$$

Lemma 74 (Undiscounting lemma)**.** *For every $x_0 > 0$, take $T(x_0)$ as in Definition 72. Then, for every $T \geq T(x_0)$ and every $u \in \Lambda_0(x_0)$, there exists a control $\mathcal{U}(T,u)$ satisfying:*

$$\begin{aligned}
&\mathcal{U}(T,u) \in \Lambda^T(x_0) \\
&\mathcal{U}(T,u) = u \quad \textit{in } [0,T] \\
&\mathcal{B}(x_0;\mathcal{U}(T,u)) \geq \mathcal{B}(x_0;u).
\end{aligned}$$

In particular,

$$u \in S(x_0) \implies \mathcal{U}(T,u) \in \Lambda^T(x_0) \cap S(x_0).$$

Proof. Fix $x_0 > 0$, take $\gamma(x_0)$, $T(x_0)$ as in Definition 72, and fix $T \geq T(x_0)$. We distinguish two cases.

If $\frac{2b_0+\rho}{2cx_0} \leq 1$ (which implies $\gamma(x_0) = b_0$ and $T(x_0) = 1$), set $\gamma_T := b_0$.

If $\frac{2b_0+\rho}{2cx_0} > 1$, set $\gamma_T := \log\left(\frac{2b_0+\rho}{2cx_0}\right)\frac{1}{T} + b_0$. Observe that in any case $\gamma_T \leq \gamma(x_0) < \rho - a$, remembering Definition 72.

Now define:

$$\mathcal{U}(T,u)(t) := u(t)\,\chi_{[0,T]}(t) + \left(u(t) \wedge e^{\gamma_T t}\right)\chi_{(T,+\infty)}(t).$$

We have:

$$
\begin{aligned}
\int_0^{+\infty} e^{-(\rho-a)t}\mathcal{U}(T,u)(t)\,\mathrm{d}t =& \int_0^T e^{at}e^{-\rho t}u(t)\,\mathrm{d}t + \int_T^{+\infty} e^{-(\rho-a)t}u(t)\,\mathrm{d}t \\
&\le e^{aT}K(x_0) + \int_0^{+\infty} e^{-(\rho-a-\gamma_T)t}\mathrm{d}t \\
&= e^{aT}K(x_0) + \frac{1}{\rho-a-\gamma_T} \\
&\le e^{aT}K(x_0) + \frac{1}{\rho-a-\gamma(x_0)} \\
&= e^{aT}K(x_0) + K_1(x_0).
\end{aligned} \tag{3.7}
$$

since $u \in \Lambda_0(x_0)$.

Hence, $\mathcal{U}(T,u) \in \Lambda^T(x_0)$ by definition of $\Lambda^T(x_0)$. Clearly, also the second required property is satisfied by $\mathcal{U}(T,u)$.

Now we prove that $\mathcal{B}(x_0;\mathcal{U}(T,u)) \ge \mathcal{B}(x_0;u)$. For simplicity of notation, call $\tilde{u} = \mathcal{U}(T,u)$ and $\tilde{x} = x(\cdot;x_0,\tilde{u})$.

Since $\mathcal{B}(x_0;u) > -\infty$, we can write:

$$
\mathcal{B}(x_0;\mathcal{U}(T,u)) - \mathcal{B}(x_0;u) = \int_0^{+\infty} \left\{e^{-\rho t}\left[\log\tilde{u}(t) - \log u(t)\right] + ce^{-\rho t}\left[x^2(t) - \tilde{x}^2(t)\right]\right\}\mathrm{d}t. \tag{3.8}
$$

Now observe that the non-positive function $t \to e^{-\rho t}\left[\log\tilde{u}(t) - \log u(t)\right]$ has finite integral (i.e., $> -\infty$) if and only if $\int_T^{+\infty} e^{-\rho t}\log(u(t)\vee 1)\,\mathrm{d}t < +\infty$.

Indeed, setting $g(t) := e^{-\rho t}\log u(t)\,\chi_{\left\{u > e^{\gamma_T(\cdot)}\right\}}(t)$, we have:

$$
\begin{aligned}
\int_T^{+\infty} e^{-\rho t}\log(u(t)\vee 1)\,\mathrm{d}t &= \int_T^{+\infty} e^{-\rho t}\log u(t)\,\chi_{\{u>1\}}(t)\,\mathrm{d}t \\
&= \int_T^{+\infty} \left\{e^{-\rho t}\log u(t)\,\chi_{\left\{1<u\le e^{\gamma_T(\cdot)}\right\}}(t) + g(t)\right\}\mathrm{d}t.
\end{aligned}
$$

The function g is non-negative, while the other function inside the integral satisfies:

$$
0 \le e^{-\rho t}\log u(t)\,\chi_{\left\{1<u\le e^{\gamma_T(\cdot)}\right\}}(t) \le \gamma_T t e^{-\rho t} \quad \text{for a.e. } t \ge 0.
$$

Hence:

$$\int_T^{+\infty} e^{-\rho t} \log\left(u\left(t\right) \vee 1\right) \mathrm{d}t = +\infty$$
$$\Longleftrightarrow$$
$$\int_T^{+\infty} g\left(t\right) = +\infty.$$

On the other hand:

$$\begin{aligned}\int_0^{+\infty} e^{-\rho t}\left[\log \tilde{u}\left(t\right) - \log u\left(t\right)\right] \mathrm{d}t &= \int_{(T,+\infty)\cap\left\{u>e^{\gamma_T(\cdot)}\right\}} e^{-\rho t}\left[\gamma_T t - \log u\left(t\right)\right] \mathrm{d}t \\ &= \int_T^{+\infty} \left\{\gamma_T t e^{-\rho t}\chi_{\left\{u>e^{\gamma_T(\cdot)}\right\}}\left(t\right) - g\left(t\right)\right\} \mathrm{d}t.\end{aligned}$$

Since

$$0 \leq \gamma_T t e^{-\rho t}\chi_{\left\{u>e^{\gamma_T(\cdot)}\right\}}\left(t\right) \leq \gamma_T t e^{-\rho t} \quad \text{for a.e. } t \geq 0,$$

we have:

$$\int_0^{+\infty} e^{-\rho t}\left[\log \tilde{u}\left(t\right) - \log u\left(t\right)\right] \mathrm{d}t = -\infty$$
$$\Longleftrightarrow$$
$$\int_T^{+\infty} g\left(t\right) = +\infty.$$

Since $u \in \Lambda\left(x_0\right)$, it is a fact that:

$$\begin{aligned}\int_T^{+\infty} e^{-\rho t} \log\left(u\left(t\right) \vee 1\right) \mathrm{d}t &\leq \int_T^{+\infty} e^{-\rho t}\left(u\left(t\right) \vee 1\right) \mathrm{d}t \\ &\leq \int_0^{+\infty} e^{-\rho t} u\left(t\right) \mathrm{d}t + \frac{1}{\rho} < +\infty,\end{aligned}$$

and consequently $\int_0^{+\infty} e^{-\rho t}\left[\log \tilde{u}\left(t\right) - \log u\left(t\right)\right] \mathrm{d}t > -\infty$. Hence, setting:

$$\begin{aligned}I_1 =& \int_0^{+\infty} e^{-\rho t}\left[\log \tilde{u}\left(t\right) - \log u\left(t\right)\right] \mathrm{d}t \\ I_2 =& c\int_0^{+\infty} e^{-\rho t}\left[x^2\left(t\right) - \tilde{x}^2\left(t\right)\right] \mathrm{d}t,\end{aligned}$$

we can write:

$$\int_0^{+\infty} \left\{e^{-\rho t}\left[\log \tilde{u}\left(t\right) - \log u\left(t\right)\right] + c e^{-\rho t}\left[x^2\left(t\right) - \tilde{x}^2\left(t\right)\right]\right\} \mathrm{d}t = I_1 + I_2. \tag{3.9}$$

Now we give estimates for I_1 and I_2. We have

$$\begin{aligned} I_1 &= \int_T^{+\infty} e^{-\rho t} \left[\log\left(u(t) \wedge e^{\gamma_T t}\right) - \log u(t)\right] \mathrm{d}t \\ &\geq \int_T^{+\infty} \frac{e^{-\rho t}}{u(t) \wedge e^{\gamma_T t}} \left[u(t) \wedge e^{\gamma_T t} - u(t)\right] \mathrm{d}t \\ &= \int_T^{+\infty} e^{-(\rho+\gamma_T)t} \left[u(t) \wedge e^{\gamma_T t} - u(t)\right] \mathrm{d}t, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} I_2 &= c\int_0^{+\infty} e^{-\rho t} \left[x^2(t) - \tilde{x}^2(t)\right] \mathrm{d}t \\ &\geq 2c\int_0^{+\infty} \tilde{x}(t)\, e^{-\rho t} \left[x(t) - \tilde{x}(t)\right] \mathrm{d}t \\ &= 2c\int_T^{+\infty} \tilde{x}(t)\, e^{-\rho t} \left[x(t) - \tilde{x}(t)\right] \mathrm{d}t. \end{aligned}$$

By Proposition 68, i) and ii), we have for every $t \geq T$:

$$\begin{aligned} \tilde{x}(t) &\geq e^{-b_0 t} x_0; \\ x(t) - \tilde{x}(t) &= \int_0^t e^{\int_s^t h(\tau)\mathrm{d}\tau} \left[u(s) - \tilde{u}(s)\right] \mathrm{d}s \\ &= \int_T^t e^{\int_s^t h(\tau)\mathrm{d}\tau} \left[u(s) - u(s) \wedge e^{\gamma_T s}\right] \mathrm{d}s \\ &\geq \int_T^t e^{-b_0(t-s)} \left[u(s) - u(s) \wedge e^{\gamma_T s}\right] \mathrm{d}s. \end{aligned}$$

Thus we obtain:

$$\begin{aligned} I_2 &\geq 2cx_0 \int_T^{+\infty} e^{-(b_0+\rho)t} \int_T^t e^{-b_0(t-s)} \left[u(s) - u(s) \wedge e^{\gamma_T s}\right] \mathrm{d}s\mathrm{d}t \\ &= 2cx_0 \int_T^{+\infty} e^{b_0 s} \left[u(s) - u(s) \wedge e^{\gamma_T s}\right] \int_s^{+\infty} e^{-(2b_0+\rho)t} \mathrm{d}t\mathrm{d}s \\ &= \frac{2cx_0}{2b_0+\rho} \int_T^{+\infty} e^{-(\rho+b_0)s} \left[u(s) - u(s) \wedge e^{\gamma_T s}\right] \mathrm{d}s. \end{aligned} \tag{3.11}$$

Focusing on the last integral in (3.10),we note that

$$\int_T^{+\infty} e^{-(\rho+\gamma_T)t} \left[u(t) \wedge e^{\gamma_T t} - u(t)\right] \mathrm{d}t > -\infty$$

by the admissibility of u.

Hence:

$$
\begin{aligned}
&\int_T^{+\infty}\left(\frac{2cx_0}{2b_0+\rho}e^{-b_0t}-e^{-\gamma_T t}\right)e^{-\rho t}\left[u\left(t\right)-u\left(t\right)\wedge e^{\gamma_T t}\right]\mathrm{d}t\\
=&\int_T^{+\infty}e^{-(\rho+\gamma_T)t}\left[u\left(t\right)\wedge e^{\gamma_T t}-u\left(t\right)\right]\mathrm{d}t\\
&+\frac{2cx_0}{2b_0+\rho}\int_T^{+\infty}e^{-(\rho+b_0)t}\left[u\left(t\right)-u\left(t\right)\wedge e^{\gamma_T t}\right]\mathrm{d}t\\
\leq& I_1+I_2.
\end{aligned}
\tag{3.12}
$$

Going back to the definition of γ_T, we wee that the function $t\rightarrow\left(\frac{2cx_0}{2b_0+\rho}e^{-b_0t}-e^{-\gamma_T t}\right)$ is everywhere-non-negative. Indeed, if $\frac{2cx_0}{2b_0+\rho}\geq 1$, then $\gamma_T=b_0$. If, on the contrary, $\frac{2cx_0}{2b_0+\rho}<1$, then $\gamma_T=\log\left(\frac{2b_0+\rho}{2cx_0}\right)\frac{1}{T}+b_0$. By a straightforward computation, this implies, for every $t\geq T$:

$$
\begin{aligned}
&\gamma_T\geq\log\left(\frac{2b_0+\rho}{2cx_0}\right)\frac{1}{t}+b_0\\
\iff&\frac{2cx_0}{2b_0+\rho}e^{-b_0t}-e^{-\gamma_T t}\geq 0.
\end{aligned}
$$

By (3.12) this implies $I_1+I_2\geq 0$; hence, by (3.8) and (3.9):

$$
\mathcal{B}\left(x_0;\mathcal{U}\left(T,u\right)\right)-\mathcal{B}\left(x_0;u\right)\geq 0.
$$

□

Remark 75. If $u\in\Lambda^T\left(x_0\right)$ then

$$
\int_0^{+\infty}e^{-(\rho-a)t}x\left(t\right)\mathrm{d}t\leq K_2\left(x_0\right)e^{aT}+K_3\left(x_0\right),
$$

for two suitable constants $K_2\left(x_0\right),K_3\left(x_0\right)>0$.
Indeed, since $x\left(t\right)\leq x_0+\frac{M}{b}+e^{-bt}\int_0^t e^{bs}u\left(s\right)\mathrm{d}s$ for every $t\geq 0$ by (3.4), we have:

$$
\begin{aligned}
\int_0^{+\infty}e^{-(\rho-a)t}x\left(t\right)\mathrm{d}t&\leq\left(x_0+\frac{M}{b}\right)\frac{1}{\rho-a}+\int_0^{+\infty}e^{-(\rho-a)t}e^{-bt}\int_0^t e^{bs}u\left(s\right)\mathrm{d}s\mathrm{d}t\\
&=\left(x_0+\frac{M}{b}\right)\frac{1}{\rho-a}+\frac{1}{\rho+b-a}\int_0^{+\infty}e^{-(\rho-a)t}u\left(t\right)\mathrm{d}t\\
&\leq\frac{K\left(x_0\right)}{\rho+b-a}e^{aT}+K_1\left(x_0\right)+\left(x_0+\frac{M}{b}\right)\frac{1}{\rho-a}\\
&=:K_2\left(x_0\right)e^{aT}+K_3\left(x_0\right).
\end{aligned}
$$

Lemma 76 (Lower localization lemma)**.** *There exists a function* $\eta : [0,+\infty)^2 \to (0,+\infty)$*, continuous and strictly decreasing in the second variable, with the following property:*

for every $x_0 \geq 0$*,* $T > 0$*, and every* $u \in \Lambda^T(x_0)$ *such that* $\mathcal{B}(x_0; u) > -\infty$*, there exists* $u_T \in \Lambda(x_0)$ *satisfying:*

$$
\begin{aligned}
&u_T = u \vee \eta(x_0, T) \quad \textit{a. e. in } [0,T] \\
&\mathcal{B}(x_0; u_T) \geq \mathcal{B}(x_0; u).
\end{aligned}
$$

In particular, $\log u_T$ *is bounded below by a quantity which does not depend on* u*, almost everywhere in* $[0,T]$*.*

Proof. First, take the constants $K_2(x_0)$, $K_3(x_0)$ as in Remark 75. Then set $K_4(x_0) := K_3(x_0) + \frac{2M}{b}\frac{1}{\rho - a}$, and define $\epsilon(x_0)$ and $L(x_0)$ in the following way:

$$
\begin{aligned}
\epsilon(x_0) &= \frac{1}{2c\left[K_2(x_0) + K_4(x_0)\right]} \\
L(x_0) &> \rho + a + 2c\epsilon(x_0) K_2(x_0) a + \frac{2c\epsilon(x_0)^2}{\rho - a}.
\end{aligned}
$$

Without loss of generality we can assume $\epsilon(x_0) < 1$. Now we define the function η and an auxiliary function ζ. For every, $T \geq 0$:

$$
\begin{aligned}
\eta(x_0, T) &:= \epsilon(x_0) e^{-L(x_0)T} \\
\zeta(x_0, T) &:= \frac{e^{-\rho T}}{2c\eta(x_0, T)} - \frac{T\eta(x_0, T)}{\rho - a} - K_2(x_0) e^{aT} - K_4(x_0).
\end{aligned}
$$

An easy but boring computation shows that

$$
\zeta(x_0, T) \geq 0 \quad \forall T \geq 0. \tag{3.13}
$$

Indeed, the choice of $\epsilon(x_0)$ is such that $\zeta(x_0, 0) = 0$, and the choice of $L(x_0)$ makes sure that $\frac{\mathrm{d}}{\mathrm{d}T}\zeta(x_0, T) > 0$ for $T \geq 0$.

Now fix $T > 0$, $x_0 \geq 0$ and take $u \in \Lambda^T(x_0)$. If $u \geq \eta(x_0, T)$ almost everywhere in $[0,T]$, then choose $u_T := u$ and there is nothing more to prove.

If, on the contrary, there exists a subset of $[0,T]$ of positive measure where $u < \eta(x_0, T)$, then define:

$$
\begin{aligned}
u_T &:= \left(u \vee \eta(x_0, T)\right)\chi_{[0,T]} + u\chi_{(T,+\infty)} \\
J &:= \int_0^T \left[u(s) \vee \eta(x_0, T) - u(s)\right] \mathrm{d}s.
\end{aligned}
$$

Observe that

$$0 < J = \int_{[0,T]\cap\{u<\eta\}} [u(s) \vee \eta(x_0,T) - u(s)]\,\mathrm{d}s$$
$$\leq T\eta(x_0,T). \tag{3.14}$$

Set for simplicity of notation $x_T = x(\cdot; x_0, u_T)$, $x = x(\cdot; x_0, u)$, $\eta = \eta(x_0, T)$.

We shall give two different estimates for the difference $x_T - x$, one "multiplicative", and one "additive" in J. By (3.5), we obtain for every $t \geq 0$:

$$x_T(t) - x(t) = \int_0^T e^{\int_s^t h(u_T,u)(\tau)\mathrm{d}\tau} [u_T(s) - u(s)]\,\mathrm{d}s$$
$$\leq Je^{at}.$$

By (3.3), (3.4) and (3.14) it follows that, for every $t \geq 0$:

$$x_T(t) - x(t) \leq \int_0^t e^{-b(t-s)} [u_T(s) - u(s) + 2M]\,\mathrm{d}s$$
$$\leq \frac{2M}{b} + J \leq \frac{2M}{b} + T\eta.$$

Moreover, since $u \in \Lambda^T(x_0)$, by Remark 75 we have:

$$\int_0^{+\infty} e^{-(\rho-a)t} x(t)\,\mathrm{d}t \leq K_2(x_0)\,e^{aT} + K_3(x_0).$$

The last three inequalities lead to the following estimate:

$$
\begin{aligned}
c\int_0^{+\infty} e^{-\rho t}\left[x_T^2(t)-x^2(t)\right]\mathrm{d}t &\leq 2c\int_0^{+\infty} e^{-\rho t}x_T(t)\left[x_T(t)-x(t)\right]\mathrm{d}t \\
&\leq 2cJ\int_0^{+\infty} e^{-(\rho-a)t}x_T(t)\,\mathrm{d}t \\
&= 2cJ\int_0^{+\infty} e^{-(\rho-a)t}\left[x_T(t)-x(t)\right]\mathrm{d}t \\
&\quad +2cJ\int_0^{+\infty} e^{-(\rho-a)t}x(t)\,\mathrm{d}t \\
&\leq 2cJ\left(\frac{2M}{b}+T\eta\right)\int_0^{+\infty} e^{-(\rho-a)t}\mathrm{d}t \\
&\quad +2cJ\left[K_2(x_0)\,e^{aT}+K_3(x_0)\right] \\
&= 2cJ\left[\left(\frac{2M}{b}+T\eta\right)\frac{1}{\rho-a}+K_2(x_0)\,e^{aT}+K_3(x_0)\right] \\
&= 2cJ\left[T\eta\frac{1}{\rho-a}+K_2(x_0)\,e^{aT}+K_3(x_0)+\frac{2M}{b}\frac{1}{\rho-a}\right] \\
&=: 2cJ\left[T\eta\frac{1}{\rho-a}+K_2(x_0)\,e^{aT}+K_4(x_0)\right],
\end{aligned}
$$

in which we used the convexity relation $x^2-y^2\leq 2x(x-y)$ at the beginning.

Additionally:

$$
\begin{aligned}
\int_0^{+\infty} e^{-\rho t}\left[\log u_T(t)-\log u(t)\right]\mathrm{d}t &= \int_0^{T} e^{-\rho t}\left[\log\left(u(t)\vee\eta\right)-\log u(t)\right]\mathrm{d}t \\
&\geq \int_0^{T} e^{-\rho t}\frac{u(t)\vee\eta-u(t)}{u(t)\vee\eta}\mathrm{d}t \\
&= \frac{1}{\eta}\int_0^{T} e^{-\rho t}\left[u(t)\vee\eta-u(t)\right]\mathrm{d}t \\
&\geq J\frac{e^{-\rho T}}{\eta}.
\end{aligned}
$$

Finally, since $\mathcal{B}(x_0;u) > -\infty$ by assumption, we deduce from the last two estimates that:

$$\begin{aligned}
\mathcal{B}(x_0;u_T) - \mathcal{B}(x_0;u) &= \int_0^{+\infty} e^{-\rho t}\left[\log u_T(t) - \log u(t)\right] \mathrm{d}t \\
&\quad -c\int_0^{+\infty} e^{-\rho t}\left[x_T^2(t) - x(t)\right] \mathrm{d}t \\
&\geq J\frac{e^{-\rho T}}{\eta} - 2cJ\left[T\eta\frac{1}{\rho-a} + K_2(x_0)e^{aT} + K_4(x_0)\right] \\
&= 2cJ\left[\frac{e^{-\rho T}}{2c\eta(x_0,T)} - \frac{T\eta(x_0,T)}{\rho-a} - K_2(x_0)e^{aT} - K_4(x_0)\right] \\
&= 2cJ\zeta(x_0,T) \\
&\geq 0.
\end{aligned}$$

where the last inequality holds by (3.13). □

The assumption $\rho > a + b_0$ will not be used in the following lemma.

Lemma 77 (Upper localization lemma)**.** *There exists a function $N : [0,+\infty) \times (0,+\infty) \to (0,+\infty)$, continuous and strictly increasing in the second variable, with the following properties:*

i) $N(x_0,T) > \eta(x_0,T)$ for every $x_0 \geq 0$, $T > 0$, where η is the function introduced in Lemma (76).

ii)

$$\lim_{T\to+\infty} Te^{-\rho T}\log N(x_0,T) = 0. \tag{3.15}$$

iii) for every $x_0 \geq 0$, $T > 0$ and for every $u \in \Lambda_0(x_0)$, there exists a control $u^T \in \Lambda(x_0)$ satisfying:

$$\begin{aligned}
&\mathcal{B}\left(x_0;u^T\right) \geq \mathcal{B}(x_0;u) \\
&u^T = u \wedge N(x_0,T) \quad \textit{a. e. in } [0,T].
\end{aligned}$$

In particular, u^T is bounded above by a quantity which does not depend on the original control u, almost everywhere in $[0,T]$.

Proof. For every fixed $T \geq 0$, the equation

$$\log\beta + (a+b_0)\beta = -Tb_0, \quad \beta > 0 \tag{3.16}$$

has a unique solution, which is strictly less than 1. Call this solution β_T, and define

$$N(x_0,T) := K_5(x_0)\frac{e^{2\rho(T+\beta_T)}}{\beta_T^2}, \tag{3.17}$$

where $K_5(x_0) = K(x_0) \vee 1$ and $K(x_0)$ is the constant introduced in Proposition 70, ii).

Set $f(x) := \log x + (a + b_0)x$ for $x > 0$; f is a strictly increasing, $\mathcal{C}^1$ function with $f'(x) > 0$ for every $x > 0$. By definition, $b_T = f^{-1}(-Tb_0)$ for every $T \geq 0$, hence $T \to \beta_T$ is a strictly decreasing, continuously differentiable function defined in $[0, +\infty)$ with values in $(0, 1)$.

Hence $N(x_0, \cdot) \in \mathcal{C}^1([0, +\infty))$ and $N(x_0, \cdot)$ is strictly increasing. Indeed, the same argument shows that $N(x_0, \cdot) \in \mathcal{C}^\infty([0, +\infty))$.

Also the function $T \to T + \beta_T$ is strictly increasing. Indeed:

$$\frac{\mathrm{d}}{\mathrm{d}T}(T + \beta_T) = 1 + \frac{\mathrm{d}}{\mathrm{d}T} f^{-1}(-Tb_0) = 1 - \frac{b_0}{f'(\beta_T)} = 1 - \frac{b_0}{\frac{1}{\beta_T} + a + b_0} > 0,$$

since $a > 0$.

The property $N > 1 > \eta$ is an immediate consequence of the definitions.

With regard to the property in (3.15), a simple computation - remembering relation (3.16) applied to β_T - leads to:

$$\begin{aligned}
\log N(x_0, T) &= \log K_1(x_0) \frac{e^{2\rho(T+\beta_T)}}{\beta_T^2} \\
&= \log K_1(x_0) + 2\rho(T + \beta_T) - 2\log \beta_T \\
&= \log K_1(x_0) + 2T(\rho + b_0) + 2\beta_T(\rho + a + b_0).
\end{aligned}$$

This shows that (3.15) holds, since $\beta_T \to 0$ as $T \to +\infty$.

Now fix $u \in \Lambda_0(x_0)$. If $u \leq N(x_0, T)$ almost everywhere in $[0, T]$, then set $u^T := u$, and the proof is over.

If there exists a non-negligible subset of $[0, T]$ in which $u > N(x_0, T)$ then define

$$\begin{aligned}
I &:= \int_0^T e^{-\rho t}\left[u(t) - u(t) \wedge N(x_0, T)\right] \mathrm{d}t \\
u^T &:= u \wedge N(x_0, T) \cdot \chi_{[0,T]} + (u + I) \cdot \chi_{(T, T+\beta_T]} + u \cdot \chi_{(T+\beta_T, +\infty)}.
\end{aligned}$$

Obviously $u^T \in \Lambda(x_0)$, since $u \in \Lambda(x_0)$ and $N(x_0, T) > 0$.

First we prove an ordering relation between the orbits, i.e.:

$$0 \leq x\left(\cdot; x_0, u^T\right) \leq x(\cdot; x_0, u) \quad \text{in } [0, +\infty). \tag{3.18}$$

Clearly $x\left(\cdot; x_0, u^T\right) \geq 0$, by the admissibility of u^T. For simplicity of notation we set $N = N(x_0, T)$, $x^T = x\left(\cdot; x_0, u^T\right)$ and $x = x(\cdot; x_0, u)$.

The desired relation holds in $[0,T]$ by point iii) in Proposition 68, since $u^T \leq u$ in $[0,T]$.

Fix $t \in (T, T+\beta_T]$, and set $h := h\left(u, u^T\right)$, like in point ii) in the same Proposition. Hence:

$$\begin{aligned} x(t) - x^T(t) &= \int_0^T e^{\int_s^t h \mathrm{d}\tau} \left(u(s) - u(s) \wedge N\right) \mathrm{d}s \\ &\quad -I \int_T^t e^{\int_s^t h \mathrm{d}\tau} \mathrm{d}s. \end{aligned}$$

The first term in the right hand side of the equality is estimated in the following way:

$$\begin{aligned} \int_0^T e^{\int_s^t h \mathrm{d}\tau} \left(u(s) - u(s) \wedge N\right) \mathrm{d}s &\geq \int_0^T e^{-b_0(t-s)} \left(u(s) \wedge N - u(s)\right) \mathrm{d}s \\ &\geq e^{-b_0(T+\beta_T)} \int_0^T e^{-\rho s} \left(u(s) \wedge N - u(s)\right) \mathrm{d}s \\ &= I e^{-b_0(T+\beta_T)} \end{aligned} \tag{3.19}$$

The second term satisfies:

$$\begin{aligned} -I \int_T^t e^{\int_s^t h \mathrm{d}\tau} \mathrm{d}s &\geq -I \int_T^t e^{a(t-s)} \mathrm{d}s \\ &= -\frac{I}{a} \left(e^{a(t-T)} - 1\right) \\ &\geq -\frac{I}{a} \left(e^{a\beta_T} - 1\right) \\ &\geq -I \beta_T e^{a\beta_T} \end{aligned}$$

Thus we obtain:

$$x(t) - x^T(t) \geq I \left(e^{-b_0(T+\beta_T)} - \beta_T e^{a\beta_T}\right), \tag{3.20}$$

and the last quantity is zero, by definition of β_T (relation (3.16)).

This implies that $x \geq x_T$ also in $(T+\beta_T, +\infty)$, again by Proposition 68, iii). Hence, relation (3.18) holds.

By the concavity of log, we have:

$$
\begin{aligned}
& \int_0^{+\infty} e^{-\rho t}\left[\log u^T(t) - \log u(t)\right] \mathrm{d}t \\
= & \int_0^T e^{-\rho t}\left[\log\left(u(t) \wedge N\right) - \log u(t)\right] \mathrm{d}t \\
& + \int_T^{T+\beta_T} e^{-\rho t}\left[\log\left(u(t) + I\right) - \log u(t)\right] \mathrm{d}t \\
\geq & \int_0^T e^{-\rho t} \frac{u(t) \wedge N - u(t)}{u(t) \wedge N} \mathrm{d}t \\
& + I \int_T^{T+\beta_T} \frac{e^{-\rho t}}{u(t) + I} \mathrm{d}t \\
= & \frac{1}{N} \int_0^T e^{-\rho t}\left[u(t) \wedge N - u(t)\right] \mathrm{d}t \\
& + I \int_T^{T+\beta_T} \frac{e^{-\rho t}}{u(t) + I} \mathrm{d}t \\
= & I\left(\int_T^{T+\beta_T} \frac{e^{-\rho t}}{u(t) + I} \mathrm{d}t - \frac{1}{N}\right).
\end{aligned}
\tag{3.21}
$$

Set $\mu(T) = \frac{e^{-\rho T}}{\rho}\left[1 - e^{-\rho\beta_T}\right]$, i.e. $\mu(T) = \int_T^{T+\beta_T} e^{-\rho t}\mathrm{d}t$. By Jensen's inequality we obtain:

$$
\begin{aligned}
\int_T^{T+\beta_T} \frac{e^{-\rho t}}{u(t) + I} \mathrm{d}t = & \mu(T) \int_T^{T+\beta_T} \frac{1}{u(t) + I} \frac{e^{-\rho t}}{\mu(T)} \mathrm{d}t \\
\geq & \frac{\mu(T)^2}{\int_T^{T+\beta_T} e^{-\rho t}\left[u(t) + I\right] \mathrm{d}t} \\
= & \frac{\mu(T)^2}{\int_T^{T+\beta_T} e^{-\rho t} u(t) \, \mathrm{d}t + \beta_T \int_0^T e^{-\rho t}\left[u(t) - u(t) \wedge N\right]} \\
\geq & \frac{\mu(T)^2}{\int_0^{T+\beta_T} e^{-\rho t} u(t) \, \mathrm{d}t} \\
\geq & \frac{e^{-2\rho T}}{\rho^2} \cdot \frac{\left[1 - e^{-\rho\beta_T}\right]^2}{K(x_0)} \\
\geq & \frac{e^{-2\rho(T+\beta_T)} \beta_T^2}{K_5(x_0)},
\end{aligned}
$$

where we have used $\beta_T < 1$ and $\int_0^{+\infty} e^{-\rho t} u(t) \, \mathrm{d}t \leq K(x_0)$.

The latter relation, together with (3.21), implies:

$$\begin{aligned}\int_0^{+\infty} e^{-\rho t}\left[\log u^T(t) - \log u(t)\right]\mathrm{d}t &\geq I\left(\int_T^{T+\beta_T} \frac{e^{-\rho t}}{u(t)+I}\mathrm{d}t - \frac{1}{N}\right)\\ &\geq I\left(\frac{e^{-2\rho(T+\beta_T)}\beta_T^2}{K_5(x_0)} - \frac{1}{N}\right)\\ &= 0,\end{aligned} \tag{3.22}$$

by definition of N (relation (3.17)). This implies, by (3.20):

$$\begin{aligned}\mathcal{B}\left(x_0; u^T\right) - \mathcal{B}(x_0; u) &= \int_0^{+\infty} e^{-\rho t}\left[\log u^T(t) - \log u(t)\right]\mathrm{d}t\\ &\quad + c\int_0^{+\infty} e^{-\rho t}\left[x^2(t; x_0, u) - x^2\left(t; x_0, u^T\right)\right]\mathrm{d}t\\ &\geq 0.\end{aligned}$$

This concludes the proof. □

We can resume the bounding properties of the two localization lemmas in the following

Corollary 78 (Localization Lemma)**.** *There exists two functions $\eta, N : [0,+\infty)^2 \to (0,+\infty)$ satisfying:*

i) for every $x_0 \geq 0$:

$$\lim_{T\to+\infty} Te^{-\rho T}\log\eta(x_0, T) = \lim_{T\to+\infty} Te^{-\rho T}\log N(x_0, T) = 0.$$

ii)

$$\eta < N \ \ in \ \ [0,+\infty)^2.$$

iii) For every $x_0 \geq 0$, the functions $\eta(x_0,\cdot)$, $N(x_0,\cdot)$ belong to $\mathcal{C}^1([0,+\infty))$ and are, respectively, strictly decreasing and strictly increasing.

iv) For every $x_0 \geq 0$, $T > 0$ and $u \in S(x_0) \cap \Lambda^T(x_0)$ there exists $\mathcal{L}(T,u) \in \Lambda(x_0)$ such that

$$\begin{aligned}&\mathcal{B}(x_0; \mathcal{L}(T,u)) \geq \mathcal{B}(x_0; u)\\ &\mathcal{L}(T,u) = u \vee \eta(x_0,T) \wedge N(x_0,T) \quad a.\ e.\ in\ [0,T].\end{aligned}$$

In particular $\mathcal{L}(T,u) \in S(x_0)$ and the norm $\|\log\mathcal{L}(T,u)\|_{L^\infty\infty[0,T]}$ is bounded above by a quantity which does not depend on u.

Proof. Points i) and ii) are proven taking η and N like in Lemmas (76) and (77). For point iii), fix x_0, T and u as in the hypothesis. Then we can apply Lemma (76) to u and obtain a function

$u_T \in S(x_0)$, by Remark 73. In particular $u_T \in \Lambda_0(x_0)$, thus it is possible to apply Lemma 77 to u_T in order to obtain a function $(u_T)^T \in S(x_0)$. Define $\mathcal{L}(T,u) := (u_T)^T$: the properties of $\mathcal{L}(T,u)$ follow from the statements of Lemmas (76) and (77). $\square$

3.2.2 Interpolation and convergence of the values

From this point on, the initial state $x_0 > 0$ is to be considered fixed. Consequently, we set $T_0 = T(x_0)$ for simplicity of notation, where $T(x_0)$ is the quantity introduced in Definition 72.

Lemma 79 (Interpolation Lemma)**.** *There exist two functions $v, u_* \in \Lambda(x_0)$ and a sequence $(w_n)_{n\in\mathbb{N}} \subseteq S(x_0)$ with the following properties.*

MAXIMIZING PROPERTY :

$$\lim_{n\to+\infty} \mathcal{B}(x_0; w_n) = V(x_0). \tag{3.23}$$

CONVERGENCE PROPERTIES :

$$w_n \rightharpoonup v \ \textit{in}\ L^1([0,T]) \quad \forall T>0, \tag{3.24}$$

$$\log w_n \rightharpoonup \log u_* \ \textit{in}\ L^1([0,T]) \quad \forall T>0. \tag{3.25}$$

BOUNDEDNESS PROPERTY :

$$\begin{aligned}
&\forall m\in\mathbb{N} : \textit{almost everywhere in}\,[0,T_0+m] : \\
&\forall n\geq m : \eta(x_0,T_0+m) \leq f_n \leq N(x_0,T_0+m) \\
&\textit{for } f_n = w_n,\ v \ \textit{or}\ u_*,
\end{aligned} \tag{3.26}$$

where η, N are the functions defined in Lemmas 76 and 77.

Moreover, the functions u_* and v are in the following relation:

$$u_* \leq v \quad a.e.\ \ in\ \ [0,+\infty). \tag{3.27}$$

Proof. We split the proof into two steps. We will omit the dependence of the functions η and N on x_0, for the sake of simplicity.

Step 1. We define the function v and an auxiliary sequence $(v_n)_{n\in\mathbb{N}} \subseteq \Lambda_0(x_0)$. We prove that

$v \in \Lambda(x_0)$ and v satisfies:

$$\begin{aligned}&\forall m \in \mathbb{N} : \text{almost everywhere in } [0, T_0 + m] : \\ &\eta(x_0, T_0 + m) \leq v \leq N(x_0, T_0 + m),\end{aligned} \tag{3.28}$$

while $(v_n)_{n \in \mathbb{N}}$ satisfies

$$\begin{aligned}&\forall m \in \mathbb{N} : \text{almost everywhere in } [0, T_0 + m] : \\ &\forall n \geq m : \eta(x_0, T_0 + m) \leq v_n \leq N(x_0, T_0 + m),\end{aligned} \tag{3.29}$$

$$\lim_{n \to +\infty} \mathcal{B}(x_0; v_n) = V(x_0). \tag{3.30}$$

Then we will prove that v and $(v_n)_n$ are tied by the relation:

$$v_n \rightharpoonup v \text{ in } L^1([0, T]) \quad \forall T > 0; \tag{3.31}$$

this will conclude the first step.

Consider a sequence $(u_n)_{n \in \mathbb{N}} \subseteq \Lambda(x_0)$ such that

$$\lim_{n \to +\infty} \mathcal{B}(x_0; u_n) = V(x_0).$$

We can assume that $(u_n)_{n \in \mathbb{N}} \subseteq S(x_0)$; thus in particular $(u_n)_{n \in \mathbb{N}} \subseteq \Lambda_0(x_0)$, by Remark 73.

We want to apply the Localization Lemma 78, starting from $(u_n)_{n \in \mathbb{N}}$, in order to gain some compactness property. Actually, such lemma requires that the input control belongs to some $\Lambda^T(x_0)$ - and this will lead us to use the Undiscounting Lemma 74. So the procedure will follow the scheme:

... $\hookrightarrow$ undiscounting $\hookrightarrow$ localization $\hookrightarrow$ extraction of a convergent subsequence $\hookrightarrow$ undiscounting...

where each new cycle refers to a bigger interval.

The key point is that both the undiscounting procedure and the localization procedure preserve the values of the function at smaller intervals, as well as the property of being in $S(x_0)$. To see this, fix $u \in S(x_0)$ and assume that, for a certain $m \in \mathbb{N}$:

$$\eta(T_0 + m) \leq u \leq N(T_0 + m) \quad \text{a.e. in } [0, T_0 + m]. \tag{3.32}$$

Using the notation of Lemmas 74 and 78, we see that the control $\mathcal{U}(T_0 + m + 1, u)$ is well-defined (since $u \in \Lambda_0(x_0)$) and belongs to $\Lambda^{T_0+m+1}(x_0) \cap S(x_0)$ (since $u \in S(x_0)$). Hence, also the control $\mathcal{L}(T_0+m+1, \mathcal{U}(T_0+m+1,u))$ is well-defined and belongs to $S(x_0)$. We have, almost everywhere in $[0, T_0+m]$:

$$\begin{aligned} & \mathcal{L}(T_0+m+1, \mathcal{U}(T_0+m+1,u)) \\ = \;& \mathcal{U}(T_0+m+1,u) \vee \eta(T_0+m+1) \wedge N(T_0+m+1) \\ = \;& u \vee \eta(T_0+m+1) \wedge N(T_0+m+1) \\ = \;& u. \end{aligned} \tag{3.33}$$

The first equality holds by definition of the operator $\mathcal{L}$, the second is a consequence of the definition of the operator $\mathcal{U}$, and the third follows from (3.32) and the monotonicity of η and N.

We define recursively: a family of sequences $\left\{(u_n^m)_{n\in\mathbb{N}} \mid m \in \mathbb{N}\right\}$, a family of functions $\{u^m \mid m \in \mathbb{N}\}$ such that $u^m \in L^1([0, T_0+m])$, and a family of functions $\{\sigma_m \mid m \in \mathbb{N}\}$ such that $\sigma_m : \mathbb{N} \to \mathbb{N}$, σ_m is strictly increasing. Recall that, by the Dunford-Pettis criterion, a sequence $(u_n^m)_{n\in\mathbb{N}}$ which is bounded in $L^\infty([A,B])$ (thus, in $L^1([A,B])$) admits a subsequence $\left(u^m_{\sigma(n)}\right)_{n\in\mathbb{N}}$ weakly converging in $L^1([A,B])$.

Call Σ the class of all the strictly increasing functions $\sigma : \mathbb{N} \to \mathbb{N}$ (thus such that $\sigma \geq Id$).

Define:

$$\begin{cases} u_n^0 = \mathcal{L}(T_0, \mathcal{U}(T_0, u_n)) \quad \forall n \in \mathbb{N} \\ \sigma_0 \in \Sigma, u^0 \in L^1([0,T_0]) \text{ s.t. } u^0_{\sigma_0(n)} \rightharpoonup u^0 \text{ in } L^1([0,T_0]) \\ \\ \forall m \in \mathbb{N}: \\ u_n^{m+1} = \mathcal{L}\left(T_0+m+1, \mathcal{U}\left(T_0+m+1, u^m_{\sigma_m(n)}\right)\right) \quad \forall n \in \mathbb{N} \\ \sigma_{m+1} \in \Sigma, u^{m+1} \in L^1([0,T_0+m+1]) \text{ s.t. } u^{m+1}_{\sigma_{m+1}(n)} \overset{n\to\infty}{\rightharpoonup} u^{m+1} \text{ in } L^1([0,T_0+m+1]) \end{cases}$$

We proceed to define v and $(v_n)_{n\in\mathbb{N}}$. For every $m, n \in \mathbb{N}$ we have, by construction:

$$u^m_{\sigma_m(n)} = \mathcal{L}\left(T_0+m, \mathcal{U}\left(T_0+m, u^{m-1}_{\sigma_{m-1}\circ\sigma_m(n)}\right)\right),$$

and this implies $u^m_{\sigma_m(n)} \in S(x_0)$ and $u^m_{\sigma_m(n)}(t) \in [\eta(T_0+m), N(T_0+m)]$ for almost every $t \in [0, T_0+m]$. Hence, by the implication (3.32) $\implies$ (3.33), we have:

$$\begin{aligned} u^{m+1}_{\sigma_{m+1}(n)} &= \mathcal{L}\left(T_0+m+1, \mathcal{U}\left(T_0+m+1, u^m_{\sigma_m\circ\sigma_{m+1}(n)}\right)\right) \\ &= u^m_{\sigma_m\circ\sigma_{m+1}(n)} \qquad \text{a.e. in } [0, T_0+m]. \end{aligned} \tag{3.34}$$

Consequently, for every fixed $m \in \mathbb{N}$ and every $g \in L^\infty([0, T_0 + m])$ we have:

$$\begin{aligned}
\int_0^{T_0+m} g(t)\, u^{m+1}(t)\, \mathrm{d}t &= \lim_{n\to\infty} \int_0^{T_0+m} g(t)\, u^{m+1}_{\sigma_{m+1}(n)} \mathrm{d}t \\
&= \lim_{n\to\infty} \int_0^{T_0+m} g(t)\, u^{m}_{\sigma_m \circ \sigma_{m+1}(n)} \mathrm{d}t \\
&= \lim_{n\to\infty} \int_0^{T_0+m} g(t)\, u^{m}_{\sigma_m(n)} \mathrm{d}t \\
&= \int_0^{T_0+m} g(t)\, u^m(t)\, \mathrm{d}t,
\end{aligned}$$

since $\sigma_{m+1} \in \Sigma$. This implies

$$u^{m+1} = u^m \quad \text{a.e. in } [0, T_0 + m].$$

Since this holds for every $m \in \mathbb{N}$, we can define without ambiguities the following function:

$$v(t) := \begin{cases} u^0(t) & \text{if } t \in [0, T_0] \\ u^m(t) & \text{if } t \in [0, T_0 + m] \text{ for some } m \in \mathbb{N} \end{cases} \qquad \forall t \geq 0.$$

This definition has two immediate consequences:

$$\begin{aligned}
& v \in L^1_{loc}([0, +\infty)), \\
& u^m_{\sigma_m(k)} \overset{k\to\infty}{\rightharpoonup} v \text{ in } L^1([0, T_0 + m]) \quad \forall m \in \mathbb{N}.
\end{aligned} \tag{3.35}$$

Hence, for every $m \in \mathbb{N}$:

$$\int_0^{T_0+m} e^{-\rho t} v(t)\, \mathrm{d}t = \lim_{k\to\infty} \int_0^{T_0+m} e^{-\rho t} u^m_{\sigma_m(k)}(t)\, \mathrm{d}t \leq \limsup_{k\to\infty} \int_0^{+\infty} e^{-\rho t} u^m_{\sigma_m(k)}(t)\, \mathrm{d}t \leq K(x_0),$$

since $u^m_{\sigma_m(k)} \in \Lambda_0(x_0)$. Thus

$$\int_0^{+\infty} e^{-\rho t} v(t)\, \mathrm{d}t \leq K(x_0). \tag{3.36}$$

Again by (3.35), we have

$$\liminf_{k\to\infty} u^m_{\sigma_m(k)} \leq v \leq \limsup_{k\to\infty} u^m_{\sigma_m(k)} \quad \text{a. e. in } [0, T_0 + m],$$

by a well known property of weak convergence. We have already observed that for every $m, k \in \mathbb{N}$, $u^m_{\sigma_m(k)} \in [\eta(T_0 + m), N(T_0 + m)]$ almost everywhere in $[0, T_0 + m]$; hence, remembering that a numerable intersection of full measure sets is a full measure set, we can exchange the quantifiers "$\forall k \in \mathbb{N}$" and "a.e. in $[0, T_0 + m]$" . Thus we obtain for every fixed $m \in \mathbb{N}$:

$$\begin{aligned}
&\text{almost everywhere in } [0, T_0 + m]: \\
&\liminf_{k\to\infty} u^m_{\sigma_m(k)} \geq \eta\,(T_0 + m) \\
&\limsup_{k\to\infty} u^m_{\sigma_m(k)} \leq N\,(T_0 + m)\,.
\end{aligned}$$

Combining the last three relations we obtain (3.28). As a consequence of the latter relation, $v \in L^\infty_{loc}([0, +\infty))$ and $v > 0$ almost everywhere in $[0, +\infty)$. Hence, remembering (3.36), we deduce that $v \in \Lambda\,(x_0)$.

We now define $(v_n)_{n\in\mathbb{N}}$ as the diagonal sequence:

$$\forall n \in \mathbb{N} : v_n := u^n_{\sigma_n(n)}.$$

Clearly $(v_n)_{n\in\mathbb{N}} \subseteq S\,(x_0)$, since every function in the range of the operator $\mathcal{L}$ has this property.

We prove (3.29). It follows from (3.34) that for every $m, n, j \in \mathbb{N}$:

$$u^{m+j}_{\sigma_{m+j}(n)} = u^m_{\sigma_m\circ\cdots\circ\sigma_{m+j}(n)} \quad \text{a.e. in } [0, T_0 + m]\,.$$

This is proven by an easy induction on j, and implies that, for every $n, m \in \mathbb{N}$, with $m \leq n$ we have (setting $j = n - m$):

$$\begin{aligned}
v_n = u^n_{\sigma_n(n)} &= u^{m+j}_{\sigma_{m+j}(m+j)} = u^m_{\sigma_m\circ\cdots\circ\sigma_{m+j}(m+j)} \\
&= u^m_{\sigma_m\circ\sigma_{m+1}\circ\cdots\circ\sigma_n(n)} \quad \text{a.e. in } [0, T_0 + m]\,.
\end{aligned} \tag{3.37}$$

Note that this makes sense also if $n = m$. By definition of $\left(u^m_{\sigma_m(k)}\right)_{k\in\mathbb{N}}$:

$$\begin{aligned}
&\forall m \in \mathbb{N} : \forall n \geq m : \text{almost everywhere in } [0, T_0 + m]: \\
&\eta\,(T_0 + m) \leq v_n \leq N\,(T_0 + m)\,,
\end{aligned}$$

which implies (3.29), by an exchange of quantifiers.

To prove (3.30), start by observing that, for every $n \in \mathbb{N}$ and every $j \in [0, n]$, we can prove by induction on j that:

$$\mathcal{B}\,(x_0; v_n) \geq \mathcal{B}\left(x_0; u^{n-j}_{\sigma_{n-j}\circ\cdots\circ\sigma_n(n)}\right),$$

by definition of $\left\{(u^m_n)_{n\in\mathbb{N}} \mid m \in \mathbb{N}\right\}$, because every application of the operators $\mathcal{U}$, $\mathcal{L}$ gives a bigger value of the functional $\mathcal{B}(x_0,\cdot)$ (see the statements of Lemmas 74 and 78). Hence:

$$\begin{aligned}\mathcal{B}(x_0;v_n) &\geq \mathcal{B}\left(x_0;u^0_{\sigma_0\circ\cdots\circ\sigma_n(n)}\right)\\ &\geq \mathcal{B}\left(x_0;u_{\sigma_0\circ\cdots\circ\sigma_n(n)}\right).\end{aligned}$$

Relation (3.30) follows from the fact that $\left(u_{\sigma_0\circ\cdots\circ\sigma_n(n)}\right)_{n\in\mathbb{N}}$ is extracted from $(u_n)_{n\in\mathbb{N}}$ - since $\sigma_0\circ\cdots\circ\sigma_n\in\Sigma$ - and the latter is a maximizing sequence by assumption.

Finally we prove relation (3.31).

By (3.37), we have for every $m\in\mathbb{N}$ and $n\geq m$:

$$v_n=u^m_{\sigma_m\circ\tau_m(n)}\quad\text{a.e. in }[0,T_0+m],$$

where $\tau_m:\mathbb{N}\to\mathbb{N}$ is defined as

$$\tau_m(n):=\begin{cases}n & \text{if } n<m+1\\ \sigma_{m+1}\circ\cdots\circ\sigma_n(n) & \text{if } n\geq m+1.\end{cases}$$

Actually $\tau_m\in\Sigma$ for every $m\in\mathbb{N}$: indeed, τ_m is the identity on $[0,m]$ and, for every $n\geq m+1$:

$$\begin{gathered}\tau_m(n)<\tau_m(n+1)\\ \Longleftrightarrow\\ \sigma_{m+1}\circ\cdots\circ\sigma_n(n)<\sigma_{m+1}\circ\cdots\circ\sigma_n\circ\sigma_{n+1}(n+1)\\ \Longleftrightarrow\\ n<\sigma_{n+1}(n+1),\end{gathered}$$

which is certainly true since $\sigma_{n+1}(n+1)\geq n+1$.

Exchanging again the quantifiers, we obtain that, for every fixed $m\in\mathbb{N}$, $(v_n)_{n\geq m}$ can be regarded as a sub-sequence of $\left(u^m_{\sigma_m(k)}\right)_{k\in\mathbb{N}}$ in $[0,T_0+m]$, and (3.31) follows from the definitions of v and $\left(u^m_{\sigma_m(k)}\right)_{k\in\mathbb{N}}$. This concludes the proof of the first step.
Up to now, we have repeatedly used the fact that every function in the range of the operators $\mathcal{L}(T,\cdot)$ is bounded *below* - almost everywhere - in $[0,T]$ by a positive quantity (depending only on x_0 and T). Nevertheless, we could have invoked the Dunford-Pettis criterion relying only on the fact that $\mathcal{L}(T,u)\in[0,N(x_0,T)]$ a.e. in $[0,T]$ - and for this only one localization lemma would have been necessary.
Step 2. We define $(w_n)_{n\in\mathbb{N}}$ as a sub-sequence of $(v_n)_{n\in\mathbb{N}}$. In order that the former can be chosen to satisfy (3.25), it will be crucial that $v_n\geq\eta(x_0,T)$ a.e. in $[0,T]$.

Set $v^0_n:=v_n$ for every $n\in\mathbb{N}$. By (3.29) , for every $T>0$, the sequence $\left(\log v^0_n\right)_{n\in\mathbb{N}}$ is uniformly bounded (with respect to n) in the $L^{\infty}_{[0,T]}$ norm. Hence, with a standard diagonalization and repeatedly using the Dunford-Pettis criterion, we can extract - recursively in m - $\left(v_n^{m+1}\right)_{n\in\mathbb{N}}$ from $\left(v_n^m\right)_{n\in\mathbb{N}}$ in such way that

$$\begin{aligned}&\forall m\in\mathbb{N}: \text{almost everywhere in } [0,T_0+m]:\\&\forall n\in\mathbb{N}: \log\eta\left(T_0+m\right)\leq\log v_n^m\leq\log N\left(T_0+m\right),\end{aligned}$$

and that

$$\begin{aligned}&\forall m\in\mathbb{N}:\exists f_m\in L^1\left([0,T_0+m]\right):\\&\log v_n^m \overset{n\to\infty}{\rightharpoonup} f_m \quad \text{in } L^1\left([0,T_0+m]\right).\end{aligned}$$

Define

$$u_* := \exp\left(f_0\chi_{[0,T_0)} + \sum_{m\geq 1} f_m\chi_{[T_0+m-1,T_0+m)}\right).$$

By the essential uniqueness of the weak limit, $f_{m+j}=f_m$ a.e. in $[0,T_0+m]$, for every $m,j\in\mathbb{N}$. Hence, for every fixed $m\in\mathbb{N}$, $f_m=\log u_*$ a.e. in $[0,T_0+m]$; furthermore, $\left(v_n^n\right)_{n\geq m}$ is a sub-sequence of $\left(v_n^m\right)_{n\geq 1}$. Thus

$$\log v_n^n \rightharpoonup \log u_* \quad \text{in } L^1\left([0,T_0+m]\right).$$

Define $w_n := v_n^n$, for every $n\in\mathbb{N}$. Then (3.25) holds, and, since $\left(w_n\right)_n$ is a sub-sequence of $\left(v_n\right)_n$, also (3.23) and (3.24) hold, by (3.30) and (3.31). For the same reason, relation (3.29) implies that (3.26) hols for w_n. Also the function u_* satisfies (3.26): with the same argument used for v, we obtain $\log\eta\left(T_0+m\right)\leq\log u_*\leq\log N\left(T_0+m\right)$ almost everywhere in $[0,T_0+m]$, for every $m\in\mathbb{N}$.

Finally we prove (3.27). Fix $0<t_0<t_1<T$ and let t_0 be a Lebesgue point for both $\log u_*$ and v. By Jensen's inequality we have, for every $n\in\mathbb{N}$:

$$\frac{\int_{t_0}^{t_1}\log w_n\left(s\right)\mathrm{d}s}{t_1-t_0}\leq\log\left(\frac{\int_{t_0}^{t_1} w_n\left(s\right)\mathrm{d}s}{t_1-t_0}\right).$$

Passing to the limit for $n\to+\infty$ in the previous relation, we obtain by (3.25) and (3.31):

$$\frac{\int_{t_0}^{t_1}\log u_*\left(s\right)\mathrm{d}s}{t_1-t_0}\leq\log\left(\frac{\int_{t_0}^{t_1} v\left(s\right)\mathrm{d}s}{t_1-t_0}\right).$$

Passing now to the limit for $t_1\to t_0$ yields to $\log u_*\left(t_0\right)\leq\log v\left(t_0\right)$. By the Lebesgue Point

Theorem, t_0 is a generic element of a full measure subset of $[0, T]$; hence (3.27) follows. In particular $\int_0^{+\infty} e^{-\rho t} u_* (t)\, \mathrm{d}t \le K(x_0)$; hence, by (3.26), $u_* \in \Lambda(x_0)$. $\square$

At this point we can complete the proof of Theorem 71 with the same steps as in the monotone dynamics case. Proposition 63 holds also in the present case with the same proof up to substituting b_0 with $b_0 + a$. Hence, we have the pointwise convergence of the maximizing sequence of states to the state controlled by the function v in Lemma 79. The final argument is the same as in the proof of Theorem 58, with the sequence $(w_n)_n$ defined in Lemma 79 considered in the place of the sequence $(v_{n,n})_n$.

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