Risoluzione degli esercizi della pagina precedente

1(i) Autovalori di A:

$$\det\begin{pmatrix} 1-\lambda & 0 \\ 3 & 2-\lambda \end{pmatrix}=0 \iff (1-\lambda)(2-\lambda)=0 \iff \lambda_1=1,\ \lambda_2=2;$$

autovettori:

per λ_1, $\begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 3x+y=0 \iff \begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ -3 \end{pmatrix}c,\ c\in\mathbb{C};$

per λ_2, $\begin{pmatrix} -1 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x=0 \iff \begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}c,\ c\in\mathbb{C}.$

Dunque

$$\underline{W}(t)=\begin{pmatrix} e^t & 0 \\ -3e^t & e^{2t} \end{pmatrix},$$

$$V_0=\left\{\underline{W}(t)\underline{c},\ \underline{c}\in\mathbb{C}^2\right\}=\left\{c_1\begin{pmatrix} 1 \\ -3 \end{pmatrix}e^t+c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{2t},\ c_1,c_2\in\mathbb{C}\right\}.$$

1(ii) Autovalori: $\det\begin{pmatrix} -\lambda & -4 \\ 1 & -\lambda \end{pmatrix}=0 \iff \lambda^2+4=0 \iff \lambda=\pm 2i;$

autovettori:

per $\lambda=2i$, $\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x=2iy \iff \begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2i \\ 1 \end{pmatrix}c,\ c\in\mathbb{C},$

per $\lambda=-2i$, $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -2i \\ 1 \end{pmatrix}c,\ c\in\mathbb{C}$. Sostituendo $\begin{pmatrix} 2i \\ 1 \end{pmatrix}e^{2it}$ e $\begin{pmatrix} -2i \\ 1 \end{pmatrix}e^{-2it}$ con $\frac{1}{2}\begin{pmatrix} 2ie^{2it}-2ie^{-2it} \\ e^{2it}+e^{-2it} \end{pmatrix}$ e $\frac{1}{2i}\begin{pmatrix} 2ie^{2it}+2ie^{-2it} \\ e^{2it}-e^{-2it} \end{pmatrix}$, cioè con $\begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix}$ e $\begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}$,

si ottiene $\underline{W}(t)=\begin{pmatrix} -2\sin 2t & 2\cos 2t \\ \cos 2t & \sin 2t \end{pmatrix}$ e $V_0=\left\{\begin{pmatrix} -2c_1\sin 2t+2c_2\cos 2t \\ c_1\cos 2t+c_2\sin 2t \end{pmatrix},\ c_1,c_2\in\mathbb{C}\right\}.$

iii). Autovalori: $\det\begin{pmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} = (3-\lambda)(-1-\lambda)+4 = \lambda^2-2\lambda+1=0$ (42)

$\iff \lambda=1$ doppio

Autovettori:

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x=2y \iff \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$ (ad

esempio). Si trova un solo autovettore, con

$$\underline{u}_1(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t$$

Cerchiamo $\underline{u}_2(t)$ della forma $\underline{u}_2(t) = \begin{pmatrix} a+bt \\ c+dt \end{pmatrix} e^t$; sostituendo nel sistema

$$\underline{u}_2'(t) = \begin{pmatrix} b+a+bt \\ d+c+dt \end{pmatrix} e^t = A\underline{u}_2(t) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a+bt \\ c+dt \end{pmatrix} e^t,$$

da cui

$$\begin{cases} b+a = 3a-4c \\ b = 3b-4d \\ d+c = a-c \\ d = b-d \end{cases} \iff \begin{cases} a = d+2c \\ b = 2d \end{cases} \iff$$

(scegliendo ad esempio $c=0, d=1$): $\begin{cases} a=1 \\ b=2 \\ c=0 \\ d=1 \end{cases}$ Dunque $\underline{u}_2(t) = \begin{pmatrix} 1+t \\ t \end{pmatrix} e^t$

e

$$W(t) = \begin{pmatrix} 2 & 1+t \\ 1 & t \end{pmatrix} e^t, \quad V_0 = \left\{ e^t \begin{pmatrix} 2c_1 + (1+t)c_2 \\ c_1 + tc_2 \end{pmatrix}, c_1, c_2 \in \mathbb{R} \right\}.$$

2(i) $A = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$, autovalori: $\det\begin{pmatrix} 1-\lambda & 0 \\ 3 & -2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda)=0 \iff$

$\lambda_1=1$, $\lambda_2=-2$. Autovettori:

per λ_1, $\begin{pmatrix}0&0\\3&-3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\3x-3y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \iff y=x \iff \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}c,\ c\in\mathbb{C}$;

per λ_2, $\begin{pmatrix}-3&0\\-3&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-3x\\-3x\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \iff x=0 \iff \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}c,\ c\in\mathbb{C}$.

Dunque

$$\underline{W}(t)=\begin{pmatrix}e^t&0\\e^t&e^{-2t}\end{pmatrix},\quad V_0=\left\{\begin{pmatrix}c_1e^t\\c_1e^t+c_2e^{-2t}\end{pmatrix},\ c_1,c_2\in\mathbb{C}\right\}.$$

2(ii) È lo stesso di 1(ii).

2(iii) $A=\begin{pmatrix}3&-4\\1&-1\end{pmatrix}$, autovalori: $\det\begin{pmatrix}3-\lambda&-4\\1&-1-\lambda\end{pmatrix}=(3-\lambda)(-1-\lambda)+4=0$ $\iff \lambda^2+2\lambda+1=0 \iff \lambda=1$ (doppio). Autovettori: per $\lambda=1$,

$$\begin{pmatrix}2&-4\\1&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x-4y\\x-2y\end{pmatrix}=0 \iff x=2y \iff \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}c,\ c\in\mathbb{C}.$$

Si ha allora $\underline{u}_1(t)=e^t\begin{pmatrix}2\\1\end{pmatrix}$. Poniamo $\underline{u}_2(t)=e^t(\underline{a}+t\underline{b})$, con $\underline{a},\underline{b}\in\mathbb{C}^2$ da determinare imponendo che $\underline{u}_2\in V_0$. Si ha

$$\underline{u}_2'(t)=e^t(\underline{a}+t\underline{b})+e^t\underline{b},\quad \underline{A}\,\underline{u}_2(t)=e^t(\underline{A}\underline{a}+t\underline{A}\underline{b}),\ \text{da cui}$$

$$\underline{u}_2'(t)-\underline{A}\,\underline{u}_2(t)=e^t\left[(\underline{a}+\underline{b}-\underline{A}\underline{a})+t(\underline{b}-\underline{A}\underline{b})\right].$$

Scegliamo allora $\underline{b}$ autovettore, $\underline{b}=\begin{pmatrix}2\\1\end{pmatrix}$, e $\underline{a}-\underline{A}\underline{a}=\begin{pmatrix}2\\1\end{pmatrix}$, ossia

$$\begin{pmatrix}a_1\\a_2\end{pmatrix}-\begin{pmatrix}3a_1-4a_2\\a_1-a_2\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix},$$ che fornisce ad esempio $\underline{a}=\begin{pmatrix}1\\1\end{pmatrix}$.

Perciò $\underline{u}_2(t) = e^t \begin{pmatrix} 1+2t \\ 1+t \end{pmatrix}$ e

$$\underline{W}(t) = \begin{pmatrix} 2e^t & e^t(1+2t) \\ e^t & e^t(1+t) \end{pmatrix}, \quad V_0 = \left\{ e^t \begin{pmatrix} 2c_1 + c_2(1+2t) \\ c_1 + c_2(1+t) \end{pmatrix}, c_1, c_2 \in \mathbb{C} \right\}.$$

3(i) $A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$, autovalori: $\det \begin{pmatrix} 1-\lambda & 3 & -2 \\ 0 & 2-\lambda & 4 \\ 0 & 1 & 2-\lambda \end{pmatrix} = 0 \iff$

$\iff (1-\lambda)\left((2-\lambda)^2 - 4\right) = (1-\lambda)(\lambda^2 - 4\lambda) = 0 \iff \lambda_1 = 0, \ \lambda_2 = 1, \ \lambda_3 = 4.$

Autovettori: per $\lambda_1 = 0$, $\begin{cases} x+3y-2z=0 \\ 2y+4z=0 \\ y+2z=0 \end{cases} \iff \begin{cases} x = 8z \\ y = -2z \end{cases} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ 1 \end{pmatrix} c,$

$c \in \mathbb{C}$. Per $\lambda_2 = 1$, $\begin{cases} 3y-2z=0 \\ y+z=0 \\ y+z=0 \end{cases} \iff \begin{cases} y=0 \\ z=0 \end{cases} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} c, \ c \in \mathbb{C}.$

Per $\lambda_3 = 4$, $\begin{cases} -3x+3y-2z=0 \\ -2y+4z=0 \\ y-2z=0 \end{cases} \iff \begin{cases} x = \frac{4}{3} z \\ y = 2z \end{cases} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix} c, \ c \in \mathbb{C}.$

Dunque

$$\underline{W}(t) = \begin{pmatrix} 8 & e^t & 4e^{4t} \\ -2 & 0 & 6e^{4t} \\ 1 & 0 & 3e^{4t} \end{pmatrix},$$

$$V_0 = \left\{ c_1 \begin{pmatrix} 8 \\ -2 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}, c_1, c_2, c_3 \in \mathbb{C} \right\}.$$

3(ii) $A = \begin{pmatrix} 0 & -1 & 1 \\ -2 & -1 & -6 \\ 0 & -1 & 1 \end{pmatrix}$; autovalori: $\det \begin{pmatrix} -\lambda & -1 & 1 \\ -2 & -1-\lambda & -6 \\ 0 & -1 & 1-\lambda \end{pmatrix} = 0$

se e solo se $-\lambda\left[\lambda^2-1-6\right]+2\left[-1+\lambda+1\right]=\lambda(\lambda^2-9)=0, \iff$

$\iff \lambda_1=0,\ \lambda_2=3,\ \lambda_3=-3$. Autovettori:

per $\lambda_1=0$, $\underline{v}_1=\begin{pmatrix} 7 \\ -2 \\ -2 \end{pmatrix}$; per $\lambda_2=3$, $\underline{v}_2=\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$; per $\lambda_3=-3$, $\underline{v}_3=\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$.

Perciò

$$V_0=\left\{c_1\begin{pmatrix} 7 \\ -2 \\ -2 \end{pmatrix}+c_2 e^{3t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}+c_3 e^{-3t}\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix},\ c_1, c_2, c_3\right\}.$$

4(i) Autovalori: $\det\begin{pmatrix} 4-\lambda & -3 \\ 8 & -6-\lambda \end{pmatrix}=(4-\lambda)(-6-\lambda)+24=\lambda^2+2\lambda=0 \iff$

$\lambda_1=0, \lambda_2=-2$. Autovalori; per $\lambda_1=0$, $\underline{v}_1=\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, per $\lambda_2=-2$, $\underline{v}_2=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Perciò $V_0=\left\{c_1\begin{pmatrix} 3 \\ 4 \end{pmatrix}+c_2 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$. La soluzione che per $t=0$ vale $\underline{x}=\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ è quella per cui $\begin{cases} 3c_1+c_2=0 \\ 4c_1+2c_2=2 \end{cases}$, ossia $c_1=-1, c_2=3$.

Dunque

$$\underline{u}(t)=\begin{pmatrix} -3 \\ -4 \end{pmatrix}+e^{-2t}\begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

4(ii) Autovalori:

$$\det\begin{pmatrix} -\lambda & -1 & -1 \\ 1 & 1-\lambda & -4 \\ -1 & -1 & 4-\lambda \end{pmatrix}=-\lambda(\lambda^2-5\lambda+4-4)-\lambda+\lambda=-\lambda^2(\lambda-5)=0$$

$\iff \lambda_1=\lambda_2=0$ (doppio) e $\lambda_3=5$. Autovettori: per $\lambda=0$, si ha $\underline{v}_1=\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$;

ci occorre un altro elemento $\underline{v} \in V_0$ della forma $\underline{v}(t) = \underline{a} + \underline{b}t$. Poiché $\underline{v}'(t) - A\underline{v}(t) = \underline{b} - A\underline{a} - A\underline{b}t$, dovrà essere $A\underline{b} = \underline{0}$ e $A\underline{a} = \underline{b}$. Possiamo scegliere $\underline{b} = \underline{v}_1 = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$, e l'equazione $A\underline{a} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$ diventa $\begin{cases} -y-z=5 \\ x+y-4z=-1 \\ -x-y+4z=1 \end{cases}$ e ha per soluzioni $\begin{cases} x = 5z+4 \\ y = -z-5 \end{cases}$; scelto $z=0$ si ha $\underline{a} = \begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix}$. Si ha dunque $\underline{u}_2(t) = \begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$. Infine per $\lambda_3 = 5$ si ha $\underline{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. Dunque

$$V_0 = \left\{ c_1 \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 4+5t \\ -5-t \\ 1 \end{pmatrix} + c_3 e^{5t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right.$$

. Si ha $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ per $t=0$ quando $\underline{c} = \begin{pmatrix} 8/5 \\ -5/4 \\ -\frac{73}{20} \end{pmatrix}$.

5(i) $\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) + 1 = 0 \iff$

$\lambda^2 - 5\lambda + 7 = 0 \iff \lambda = \frac{5 \pm i\sqrt{3}}{2}$. Autovettori per

$\lambda_1 = \frac{5+i\sqrt{3}}{2}$: $\begin{cases} 3x+y = \frac{5+i\sqrt{3}}{2}x \\ -x+2y = \frac{5+i\sqrt{3}}{2}y \end{cases} \iff y = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x$

$\iff \underline{v}_1 = \begin{pmatrix} 1 \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{pmatrix}$; per $\lambda_2 = \frac{5-i\sqrt{3}}{2}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix}$.

Quindi $\underline{u}_1(t) = \underline{v}_1 e^{\left(\frac{5}{2} + i\frac{\sqrt{3}}{2}\right)t}$, $\underline{u}_2(t) = \underline{v}_2 e^{\left(\frac{5}{2} - i\frac{\sqrt{3}}{2}\right)t}$, oppure

$$\underline{w}_1(t) = \frac{1}{2}\left[\underline{u}_1(t) + \underline{u}_2(t)\right] = \frac{e^{5/2\,t}}{2} \begin{pmatrix} 2\cos\frac{\sqrt{3}}{2}t \\ -\cos\frac{\sqrt{3}}{2}t - \sqrt{3}\sin\frac{\sqrt{3}}{2}t \end{pmatrix},$$

$$\underline{w}_2(t) = \frac{1}{2i}\left[\underline{u}_1(t) - \underline{u}_2(t)\right] = \frac{e^{5/2\,t}}{2i} \begin{pmatrix} 2i\sin\frac{\sqrt{3}}{2}t \\ i\left(\sqrt{3}\cos\frac{\sqrt{3}}{2}t - \sin\frac{\sqrt{3}}{2}t\right) \end{pmatrix}$$

e $V_0 = \left\{ c_1 \underline{u}_1(t) + c_2 \underline{u}_2(t),\ c_1, c_2 \in \mathbb{C} \right\} =$

$$= \left\{ e^{\frac{5}{2}t} \begin{pmatrix} \cos\frac{\sqrt{3}}{2}t & \sin\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t & +\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \ c_1, c_2 \in \mathbb{R} \right\}.$$

Si ha

$$W(t)^{-1} = \frac{2}{\sqrt{3}} e^{-5t} \begin{pmatrix} +\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t & -\sin\frac{\sqrt{3}}{2}t \\ +\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t & \cos\frac{\sqrt{3}}{2}t \end{pmatrix}$$

$$W(t)^{-1}\underline{f}(t) = \frac{2}{\sqrt{3}} e^{-5t} \begin{pmatrix} e^t\left[\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t\right] - e^{2t}\sin\frac{\sqrt{3}}{2}t \\ e^t\left[\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}t\sin\frac{\sqrt{3}}{2}t\right] + e^{2t}\cos\frac{\sqrt{3}}{2}t \end{pmatrix}$$

e omettiamo il calcolo finale, comunque fattibile, di

$$W(t)\int_0^t W(s)^{-1}\underline{f}(s)\,ds.$$

5(ii) $\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{pmatrix} = (3-\lambda)(1-\lambda)^2 = 0 \iff$

$\lambda_1 = 3$, $\lambda = 1$ (doppio). Autovettori: per λ_1

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} y = 0 \\ y - 2z = 0 \end{cases} \iff \begin{cases} x \text{ arbitrario} \\ y = 0 \\ z = 0 \end{cases}$$

$\Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Per λ: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x = 0 \\ y = 0. \end{cases}$

$\Rightarrow \underline{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Si hanno $\underline{u}_1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{3t}$, $\underline{u}_2(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t$, e

cerchiamo $\underline{u}_3(t) = \begin{pmatrix} a+bt \\ c+dt \\ e+ft \end{pmatrix} e^t$; sostituendo in $\underline{u}' = A\underline{u}$ si trova

$$= \left\{ e^{\frac{5}{2}t} \begin{pmatrix} \cos\frac{\sqrt{3}}{2}t & \sin\frac{\sqrt{3}}{2}t \\ \frac{1}{2}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t & +\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \ c_1, c_2 \in \mathbb{R} \right\}.$$

Si ha

$$W(t)^{-1} = \frac{2}{\sqrt{3}} e^{-5t} \begin{pmatrix} +\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t & -\sin\frac{\sqrt{3}}{2}t \\ +\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t & \cos\frac{\sqrt{3}}{2}t \end{pmatrix}$$

$$W(t)^{-1}\underline{f}(t) = \frac{2}{\sqrt{3}} e^{-5t} \begin{pmatrix} e^t\left[\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t\right] - e^{2t}\sin\frac{\sqrt{3}}{2}t \\ e^t\left[\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t\right] + e^{2t}\cos\frac{\sqrt{3}}{2}t \end{pmatrix}$$

e omettiamo il calcolo finale, comunque fattibile, di

$$\int_0^t W(s)^{-1}\underline{f}(s)\,ds.$$

5(ii) $\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{pmatrix} = (3-\lambda)(1-\lambda)^2 = 0 \iff$

$\lambda_1 = 3$, $\lambda = 1$ (doppio). Autovettori: per λ_1

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} y = 0 \\ y - 2z = 0 \end{cases} \iff \begin{cases} x \text{ arbitrario} \\ y = 0 \\ z = 0 \end{cases}$$

$\Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Per λ: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x = 0 \\ y = 0. \end{cases}$

$\Rightarrow \underline{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Si hanno $\underline{u}_1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{3t}$, $\underline{u}_2(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t}$, e

cerchiamo $\underline{u}_3(t) = \begin{pmatrix} a+bt \\ c+dt \\ e+ft \end{pmatrix} e^t$; sostituendo in $\underline{u}' = A\underline{u}$ si trova

$$\begin{cases} b+a+bt = 3a+3bt \\ d+c+dt = c+dt \\ f+e+ft = c+dt+e+ft \end{cases} \iff \begin{cases} b+a=3a \\ b=3b \\ d+c=c \\ d=d \\ f+e=c+e \\ f=f+d \end{cases} \iff \begin{cases} a=b=d=0. \\ f=c \text{ arbitr.} \\ e \text{ arbitrario} \end{cases}$$

$$\Longrightarrow \underline{u}_3(t) = \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix} e^t. \text{ Dunque}$$

$$W(t) = \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & 0 & e^t \\ 0 & e^t & te^t \end{pmatrix},$$

$$W(t)^{-1} = -e^{-5t} \begin{pmatrix} -e^{2t} & 0 & 0 \\ 0 & te^{4t} & -e^{4t} \\ 0 & -e^{4t} & 0 \end{pmatrix}$$

$$W(t)^{-1}\underline{f}(t) = -e^{-5t} \begin{pmatrix} -e^{2t} & 0 & 0 \\ 0 & te^{4t} & -e^{4t} \\ 0 & -e^{4t} & 0 \end{pmatrix} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} +te^{-3t} \\ -te^{-t} \\ +e^{-t} \end{pmatrix}$$

Quindi

$$\int_0^t W(s)^{-1}\underline{f}(s)\,ds = \begin{pmatrix} -\frac{t}{3}e^{-3t} - \frac{1}{9}e^{-3t} + \frac{1}{9} \\ te^{-t} + e^{-t} - 1 \\ -e^{-t} + 1 \end{pmatrix}$$

e infine

$$V_f = \left\{ \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & 0 & e^t \\ 0 & e^t & te^t \end{pmatrix} \begin{pmatrix} c_1 - \frac{t}{3}e^{-3t} - \frac{1}{9}e^{-3t} \\ c_2 + te^{-t} + e^{-t} \\ c_3 - e^{-t} \end{pmatrix}, \ c_1, c_2, c_3 \in \mathbb{R} \right\}.$$

6(i) $A=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, autovalori: $\det\begin{pmatrix}-\lambda&-1\\1&-\lambda\end{pmatrix}=\lambda^2+1=0 \iff$ 49

$\iff \lambda=\pm i$. Autovettori: per $\lambda=i$, $\begin{cases}-ix-y=0\\x-iy=0\end{cases} \iff \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}i\\1\end{pmatrix}c$,

con $c\in\mathbb{C}$; per $\lambda=-i$ si avrà $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-i\\1\end{pmatrix}c$, $c\in\mathbb{C}$. Abbiamo così

$\underline{u}_1(t)=\begin{pmatrix}i\\1\end{pmatrix}e^{it}$ e $\underline{u}_2(t)=\begin{pmatrix}-i\\1\end{pmatrix}e^{-it}$, oppure

$\underline{u}_1(t)=\begin{pmatrix}-\sin t\\\cos t\end{pmatrix}$, $\underline{u}_2(t)=\begin{pmatrix}\cos t\\\sin t\end{pmatrix}$. Allora $W(t)=\begin{pmatrix}-\sin t&\cos t\\\cos t&\sin t\end{pmatrix}$,

e $V_0=\{\underline{W}(t)\underline{c},\ \underline{c}\in\mathbb{C}^2\}$. Poiché $W(t)^{-1}=-\begin{pmatrix}\sin t&-\cos t\\-\cos t&-\sin t\end{pmatrix}$,

avremo una soluzione particolare del sistema non omogeneo ponendo $\underline{v}(t)=$
$=W(t)\int_0^t W(s)^{-1}\underline{f}(s)ds$, con $\int_0^t W(s)^{-1}\underline{f}(s)ds=\int_0^t\begin{pmatrix}-\sin s&\cos s\\\cos s&\sin s\end{pmatrix}\begin{pmatrix}e^{-s}\\\sin s\end{pmatrix}ds=$

$$=\begin{pmatrix}\int_0^t[-\sin s\, e^{-s}+\cos s\sin s]\,ds\\\int_0^t[\cos s\, e^{-s}+\sin^2 s]\,ds\end{pmatrix}.$$

Essendo

$$-\int_0^t \sin s\, e^{-s}ds=\frac12\left[(\cos s+\sin s)e^{-s}\right]_0^t=\frac12\left[(\cos t+\sin t)e^{-t}-1\right],$$

$$\int_0^t \cos s\sin s\, ds=\left[\frac{\sin^2 s}{2}\right]_0^t=\frac12\sin^2 t,$$

$$\int_0^t \cos s\, e^{-s}ds=\frac12\left[(-\cos s+\sin s)e^{-s}\right]_0^t=\frac12\left[(\sin t-\cos t)e^{-t}+1\right],$$

$$\int_0^t \sin^2 s\, ds=\left[\frac{s-\sin s\cos s}{2}\right]_0^t=\frac12\left[t-\sin t\cos t\right],$$

6(i) $A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, autovalori: $\det\begin{pmatrix}-\lambda & -1\\ 1 & -\lambda\end{pmatrix}=\lambda^2+1=0 \Leftrightarrow$

$\Leftrightarrow \lambda=\pm i$. Autovettori: per $\lambda=i$, $\begin{cases}-ix-y=0\\ x-iy=0\end{cases} \Leftrightarrow \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}i\\1\end{pmatrix}c$,

con $c\in\mathbb{C}$; per $\lambda=-i$ si avrà $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-i\\1\end{pmatrix}c$, $c\in\mathbb{C}$. Abbiamo così

$\underline{u}_1(t)=\begin{pmatrix}i\\1\end{pmatrix}e^{it}$ e $\underline{u}_2(t)=\begin{pmatrix}-i\\1\end{pmatrix}e^{-it}$, oppure

$\underline{u}_1(t)=\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}$, $\underline{u}_2(t)=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$. Allora $W(t)=\begin{pmatrix}-\sin t & \cos t\\ \cos t & \sin t\end{pmatrix}$,

e $V_0=\{\underline{W}(t)\underline{c},\ \underline{c}\in\mathbb{C}^2\}$. Poiché $W(t)^{-1}=-\begin{pmatrix}\sin t & -\cos t\\ -\cos t & -\sin t\end{pmatrix}$,

avremo una soluzione particolare del sistema non omogeneo ponendo

$$\underline{v}(t)=\int_0^t W(s)^{-1}\underline{f}(s)\,ds=\int_0^t\begin{pmatrix}-\sin s & \cos s\\ \cos s & \sin s\end{pmatrix}\begin{pmatrix}e^{-s}\\ \sin s\end{pmatrix}ds=$$

$$=\begin{pmatrix}\int_0^t[-\sin s\, e^{-s}+\cos s\sin s]\,ds\\ \int_0^t[\cos s\, e^{-s}+\sin^2 s]\,ds\end{pmatrix}.$$

Essendo

$$-\int_0^t \sin s\, e^{-s}ds=\frac{1}{2}\left[(\cos s+\sin s)e^{-s}\right]_0^t=\frac{1}{2}\left[(\cos t+\sin t)e^{-t}-1\right],$$

$$\int_0^t \cos s\sin s\,ds=\left[\frac{\sin^2 s}{2}\right]_0^t=\frac{1}{2}\sin^2 t,$$

$$\int_0^t \cos s\, e^{-s}ds=\frac{1}{2}\left[(-\cos s+\sin s)e^{-s}\right]_0^t=\frac{1}{2}\left[(\sin t-\cos t)e^{-t}+1\right],$$

$$\int_0^t \sin^2 s\,ds=\left[\frac{s-\sin s\cos s}{2}\right]_0^t=\frac{1}{2}\left[t-\sin t\cos t\right],$$

si trova

$$V_f = \left\{ W(t)\underline{c} + v(t),\ \underline{c}\in\mathbb{C}^2 \right\} =$$

$$= \left\{ \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} c_1 + \frac{1}{2}\left[(\cos t+\sin t)e^{-t} - 1 + \sin^2 t\right] \\ c_2 + \frac{1}{2}\left[(\sin t - \cos t)e^{-t} + 1 + t - \sin t\cos t\right] \end{pmatrix},\ \underline{c}\in\mathbb{C}^2 \right\}.$$

La soluzione che vale $\begin{pmatrix}1\\0\end{pmatrix}$ in $t=0$ è quella con $c_1=0, c_2=1$.

6(ii) $A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}$, $\det(A-\lambda I) = -(3+\lambda)(\lambda-2)^2 = 0$

se e solo se $\lambda_1=-3$, $\lambda_2=\lambda_3=2$ (doppio). Per $\lambda_1=-3$ si ha $\underline{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$; per $\lambda=2$ si ha $\underline{v}_2=\begin{pmatrix}0\\1\\1\end{pmatrix}$ e ci occorre $\underline{v}(t) = e^{2t}(\underline{a}+\underline{b}t) \in V_0$. Poiché

$$\underline{v}'(t) - A\underline{v}(t) = e^{2t}\left[2(\underline{a}+\underline{b}t) + \underline{b} - A\underline{a} - A\underline{b}t\right],$$

deve essere $A\underline{b} = 2\underline{b}$, cioè ad esempio $\underline{b}=\begin{pmatrix}0\\1\\1\end{pmatrix}$, e $A\underline{a}-2\underline{a}=\underline{b}$.

Perciò, se $\underline{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ si deve avere $\begin{cases} -5a_1=0 \\ a_2-a_3=1 \\ a_2-a_3=1 \end{cases}$, e scelto $\underline{a}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ si ha

$$W(t) = \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t} & e^{2t}(1+t) \\ 0 & e^{2t} & te^{2t} \end{pmatrix},\quad W(t)^{-1} = \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & -te^{-2t} & e^{-2t}(1+t) \\ 0 & e^{-2t} & -e^{-2t} \end{pmatrix},$$

si trova

$$V_f = \left\{ W(t)\underline{c} + v(t),\ \underline{c} \in \mathbb{C}^2 \right\} =$$

$$= \left\{ \begin{pmatrix} -c_1 \sin t + c_2 \cos t + \frac{1}{2}\left[(\cos t + \sin t)e^{-t} - 1 + \sin^2 t \right] \\ c_1 \cos t + c_2 \sin t + \frac{1}{2}\left[(\sin t - \cos t)e^{-t} + 1 + t - \sin t \cos t \right] \end{pmatrix}, \underline{c} \in \mathbb{C}^2 \right\}.$$

La soluzione che vale $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $t=0$ è quella con $c_1=0, c_2=1$.

6(ii) $A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}$, $\det(A-\lambda I) = -(3+\lambda)(\lambda-2)^2 = 0$

se e solo se $\lambda_1 = -3$, $\lambda_2 = \lambda_3 = 2$ (doppio). Per $\lambda_1 = -3$ si ha $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; per $\lambda = 2$ si ha $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ e ci occorre

$\underline{v}(t) = e^{2t}(\underline{a} + \underline{b}t) \in V_0$. Poiché

$$\underline{v}'(t) - A\underline{v}(t) = e^{2t}\left[2(\underline{a} + \underline{b}t) + \underline{b} - A\underline{a} - A\underline{b}t \right],$$

deve essere $A\underline{b} = 2\underline{b}$, cioè ad esempio $\underline{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, e $A\underline{a} - 2\underline{a} = \underline{b}$.

Perciò, se $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ si deve avere $\begin{cases} -5a_1 = 0 \\ a_2 - a_3 = 1 \\ a_2 - a_3 = 1 \end{cases}$, e scelto $\underline{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ si ha

$$W(t) = \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t} & e^{2t}(1+t) \\ 0 & e^{2t} & te^{2t} \end{pmatrix}, \quad W(t)^{-1} = \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & -te^{-2t} & e^{-2t}(1+t) \\ 0 & e^{-2t} & -e^{-2t} \end{pmatrix},$$

Cerchiamo un elemento $\underline{v}(t) \in V_f$ della forma $\underline{v}(t) = W(t)\int_0^t W(s)^{-1}\begin{pmatrix} s^2 \\ s \\ 0 \end{pmatrix} ds$.

Si ha
$$\int_0^t \underline{W}(s)^{-1}\begin{pmatrix} s^2 \\ s \\ 0 \end{pmatrix} ds = \int_0^t \begin{pmatrix} e^{3s} s^2 \\ -e^{-2s} s^2 \\ e^{-2s} s \end{pmatrix} ds.$$

Essendo
$$\int_0^t s^2 e^{3s} ds = \left[\frac{1}{3}s^2 e^{3s}\right]_0^t - \int_0^t \frac{2}{3} s e^{3s} ds = \left[\left(\frac{1}{3}s^2 - \frac{2}{9}s\right)e^{3s}\right]_0^t + \int_0^t \frac{2}{9}e^{3s} ds =$$
$$= \left[\left(\frac{1}{3}s^2 - \frac{2}{9}s + \frac{2}{27}\right)e^{3s}\right]_0^t = \left(\frac{1}{3}t^2 - \frac{2}{9}t\right)e^{3t} + \frac{2}{27}\left(e^{t}-1\right),$$

e, analogamente,
$$-\int_0^t s^2 e^{-2s} ds = \left[\frac{1}{2}s^2 e^{-2s}\right]_0^t - \int_0^t s e^{-2s} ds = \left[\left(\frac{1}{2}s^2 + \frac{1}{2}s + \frac{1}{4}\right)e^{-2s}\right]_0^t =$$
$$= \left[\frac{1}{2}t^2 + \frac{1}{2}t\right]e^{-2t} + \frac{1}{4}\left(e^{-2t}-1\right),$$

si ricava in definitiva

$$V_f = \left\{ \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & e^{2t}(1+t) \\ 0 & e^{2t} & te^{2t} \end{pmatrix} \begin{pmatrix} c_1 + \left(\frac{1}{3}t^2 - \frac{2}{9}t\right)e^{3t} + \frac{2}{27}\left(e^{3t}-1\right) \\ c_2 + \left(\frac{1}{2}t^2 + \frac{1}{2}t\right)e^{-2t} + \frac{1}{4}\left(e^{-2t}-1\right) \\ c_3 - \frac{1}{2}se^{-2t} + \frac{1}{4}\left(e^{-2t}-1\right) \end{pmatrix}, c_1, c_2, c_3 \in \mathbb{C} \right\}.$$

La soluzione $\underline{y}$ per la quale $\underline{y}(0) = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ si ottiene scegliendo $c_1 = -1$, $c_2 = 0$, $c_3 = -1$.

Cerchiamo un elemento $\underline{v}(t) \in V_p$ della forma

$$\underline{v}(t) = \int_0^t \underline{W}(s)^{-1} \begin{pmatrix} s^2 \\ s \\ 0 \end{pmatrix} ds = \int_0^t \begin{pmatrix} e^{3s} s^2 \\ -e^{-2s} s^2 \\ e^{-2s} s \end{pmatrix} ds .$$

Essendo

$$\int_0^t s^2 e^{3s}\, ds = \left[\frac{1}{3} s^2 e^{3s}\right]_0^t - \int_0^t \frac{2}{3} s e^{3s}\, ds = \left[\left(\frac{1}{3} s^2 - \frac{2}{9} s\right) e^{3s}\right]_0^t + \int_0^t \frac{2}{9} e^{3s}\, ds =$$

$$= \left[\left(\frac{1}{3} s^2 - \frac{2}{9} s + \frac{2}{27}\right) e^{3s}\right]_0^t = \left(\frac{1}{3} t^2 - \frac{2}{9} t\right) e^{3t} + \frac{2}{27}\left(e^{t} - 1\right),$$

e, analogamente,

$$-\int_0^t s^2 e^{-2s}\, ds = \left[\frac{1}{2} s^2 e^{-2s}\right]_0^t - \int_0^t s e^{-2s}\, ds = \left[\left(\frac{1}{2} s^2 + \frac{1}{2} s + \frac{1}{4}\right) e^{-2s}\right]_0^t =$$

$$= \left[\frac{1}{2} t^2 + \frac{1}{2} t\right] e^{-2t} + \frac{1}{4}\left(e^{-2t} - 1\right),$$

si ricava in definitiva

$$V_p = \left\{ \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & e^{2t}(1+t) \\ 0 & e^{2t} & t e^{2t} \end{pmatrix} \begin{pmatrix} c_1 + \left(\frac{1}{3} t^2 - \frac{2}{9} t\right) e^{3t} + \frac{2}{27}\left(e^{3t} - 1\right) \\ c_2 + \left(\frac{1}{2} t^2 + \frac{1}{2} t\right) e^{-2t} + \frac{1}{4}\left(e^{-2t} - 1\right) \\ c_3 - \frac{1}{2} s e^{-2t} + \frac{1}{4}\left(e^{-2t} - 1\right) \end{pmatrix}, \; c_1, c_2, c_3 \in \mathbb{C} \right\}.$$

La soluzione $\underline{y}$ per la quale $\underline{y}(0) = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ si ottiene scegliendo

$c_1 = -1$, $c_2 = 0$, $c_3 = -1$.