A characterization of the Chern and Bernwald connections

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Abstract

We present a global characterization of the Chern and Bernwald connections induced by a Finsler metric, illustrating their similarities and differences with respect to the Cartan connection. Furthermore, using the symplectic structure canonically associated to a Finsler metric we describe a minimal compatibility condition between a vertical connection and a Finsler metric.

0. Introduction

In the study of the geometry of Finsler manifolds, one of the main tools is a connection generalizing to this case the classical Levi-Civita connection induced by a Riemannian metric. The first such generalization has been proposed by E. Cartan [C]; shortly later, S.S. Chern [Ch] suggested a different generalization. Both these connections can be used (see, e.g., [BC] and [AP]) to prove in the context of Finsler manifolds most of the standard results of Riemannian geometry.

The Cartan connection and the Chern connection were defined in quite different ways. The Cartan connection was an explicit generalization of the Levi-Civita connection, and it was defined using the language of vector fields and covariant differentiation. On the other hand, the Chern connection appeared as a solution of the local equivalence problem of Finsler metrics, and it was defined using the language of differential forms. A third connection useful in Finsler geometry is the Bernwald connection (see, e.g., [B, p. 44]), again defined in terms of covariant differentiation using local coordinates.

The aim of this note is to present parallel characterizations of these connections showing explicitly similarities and dissimilarities. The starting point is the theory of good vertical connections developed in the first chapter of [AP], theory that we shall briefly summarize in the first section of this paper. Let M be a smooth manifold, and $\pi\colon TM \to M$ the standard projection. Roughly speaking, a good vertical connection is a connection on the vertical bundle (the subbundle of $T(TM)$ given by the kernel of $d\pi$) that can be canonically extended to a connection on the whole of $T(TM)$. In particular, a good vertical connection has a well-defined torsion, which is a $T(TV)$-valued 2-form on TM. Furthermore, a Finsler metric on M yields a canonically defined Riemannian metric on the vertical bundle; therefore we can talk about metric compatibility of a vertical connection.

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Both the Cartan, the Chern and the Bernwald connections are (or can be thought of as) good vertical connections. We shall show that they are characterized by different degrees of metric compatibility and vanishing of the torsion. To be more specific, in [AP, Theorem 1.4.2] we characterized the Cartan connection as the unique good vertical connection which is fully metric compatible and has a minimal vanishing of the torsion. In this paper we shall show that the Chern connection can be characterized as the unique good vertical connection with an intermediate amount of metric compatibility and of vanishing of the torsion; so it is in a sense simpler than the Cartan connection, but it still retains enough metric compatibility to be used for the study of the geometry of the Finsler metric. Finally, the Bernwald connection can be characterized as the unique good vertical connection with a minimal amount of metric compatibility and the most vanishing of the torsion — a situation exactly opposite to the one encountered for the Cartan connection.

All these connections give rise to the same geodesics in M, which are exactly the geodesics in M of the given Finsler metric. We end this paper using the symplectic structure on TM naturally associated to a Finsler metric to give minimal conditions on a good vertical connection ensuring that its geodesics are exactly the geodesics of the Finsler metric. An example of such a connection, which in general does not agree with either the Cartan, Chern or Bernwald connections, is the one induced by a weakly Kähler complex Finsler metric (see [AP, Section 2.3]).

After the completion of this work I became aware of the paper [A], where it is independently proved a result [A, Proposition 3.1] which is essentially equivalent to Theorem 2.2 of this note, although expressed in different terms, and where it is also noticed the identity between the Chern and the Rund connections (see the remark after the proof of Theorem 2.2).

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1. Preliminaries

Let M be a (connected) smooth manifold of dimension n, and denote by $\pi\colon TM \to M$ the canonical projection of the tangent bundle TM onto M. By definition, the *vertical bundle* $\mathcal{V} \subset T(TM)$ is the kernel of the differential of π; it is a rank n vector bundle over TM. If $\{x^1, \ldots, x^n\}$ are local coordinates around a point $p \in M$, a tangent vector $u \in T_pM$ can be written as

$$u = u^i \left.\frac{\partial}{\partial x^i}\right|_p,$$

where we are using the Einstein convention. In particular, $\{x^1, \ldots, x^n, u^1, \ldots, u^n\}$ are local coordinates on TM. Setting $\partial_i = \partial/\partial x^i$ and $\dot\partial_a = \partial/\partial u^a$, it is easy to check that $\{\dot\partial_1, \ldots, \dot\partial_n\}$ is a local frame for $\mathcal{V}$.

Let π^*TM denote the pull-back bundle over TM; the fiber of π^*TM over $u \in TM$ is just $T_{\pi(u)}M$. Since several presentations of Finsler geometry are based on π^*TM whereas our presentation is based on the bundle $\mathcal{V}$, we record here the following easy fact:

Lemma 1.1: *The bundles π^*TM and $\mathcal{V}$ are canonically isomorphic.*

Proof: For any $p \in M$ let $j_p\colon T_pM \to TM$ be the inclusion, and for any $u \in T_pM$ let $k_u\colon T_pM \to T_u(T_pM)$ denote the usual identification. Then for any $u \in TM$ we can define the injective map $\iota_u \colon T_{\pi(u)}M \to T_u(TM)$ by

$$\iota_u = (dj_{\pi(u)})_u \circ k_u.$$

Since $\pi \circ j_{\pi(u)} \equiv \pi(u)$, the image of ι_u is contained in $\mathcal{V}_u$; being $\dim \mathcal{V}_u = n = \dim T_{\pi(u)}M$ it follows that ι_u is an isomorphism of $T_{\pi(u)}M$ with $\mathcal{V}_u$. Since, as previously remarked, the $T_{\pi(u)}M$ are exactly the fibers of π^*TM, we can patch together the isomorphisms ι_u to get a global bundle isomorphism $I \colon \pi^*TM \to \mathcal{V}$ by setting $I(v) = \iota_u(v)$ for any $v \in (\pi^*TM)_u$ and any $u \in TM$. □

As a consequence of this proof, the vertical bundle is equipped with a canonical section, the *radial vertical vector field* $\iota \colon TM \to \mathcal{V}$ given by $\iota(u) = \iota_u(u)$. In local coordinates,

$$\iota(u) = \iota\left(u^i \left.\frac{\partial}{\partial x^i}\right|_{\pi(u)} \right) = u^i \dot\partial_i|_u.$$

A *horizontal bundle* is a rank n subbundle $\mathcal{H}$ of $T(TM)$ such that $T(TM) = \mathcal{H} \oplus \mathcal{V}$. We recall (see [AP, Proposition 1.1.2]) that every horizontal bundle comes provided with a canonically defined bundle isomorphism $\Theta \colon \mathcal{V} \to \mathcal{H}$ such that $(d\pi \circ \Theta)_u = \iota_u^{-1}$ for any $u \in TM$. In particular, setting $\delta_j = \Theta(\dot\partial_j)$ for $j = 1, \ldots, n$, the set $\{\delta_1, \ldots, \delta_n\}$ is a local frame for $\mathcal{H}$. Since, by construction, $d\pi(\delta_i) = \partial/\partial x^i = d\pi(\partial_i)$, every δ_i must be of the form

$$\delta_i = \partial_i - \Gamma^a_{;i}\dot\partial_a$$

for suitable coefficients $\Gamma^a_{;i}$, called the *Christoffel symbols* of the horizontal bundle $\mathcal{H}$.

We now describe a standard procedure for the construction of horizontal bundles. A *vertical connection* is just a linear connection ∇ on the vertical bundle. If ∇ is a vertical connection, we can define a bundle map $\Lambda \colon T(TM) \to \mathcal{V}$ by

$$\Lambda(X) = \nabla_X \iota,$$

where ι is the radial vertical vector field previously introduced. We say that a vertical connection ∇ is *good* if $\Lambda|_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{V}$ is a bundle isomorphism. It turns out (see [AP, Lemma 1.2.1]) that ∇ is good iff $\operatorname{Ker}\Lambda$ is a horizontal bundle.

In local coordinates, we can write $\nabla\dot\partial_i = \omega^j_i \otimes \dot\partial_j$, where (ω^j_i) is a locally defined matrix of 1-forms:

$$\omega^j_i = \tilde\Gamma^j_{i;h}\,dx^h + \tilde\Gamma^j_{ia}\,du^a$$

for suitable coefficients $\tilde\Gamma^j_{i;h}$ and $\tilde\Gamma^j_{ia}$. Then $\Lambda(\dot\partial_i) = [\delta^j_i + \tilde\Gamma^j_{hi}u^h]\dot\partial_j$, where δ^j_i is the Kronecker delta, and so ∇ is good iff the matrix

$$L^j_i = \delta^j_i + \tilde\Gamma^j_{hi}u^h \tag{1.1}$$

is invertible.

If we write $X = X^i\partial_i + \dot X^a\dot\partial_a \in T(TM)$, then

$$\nabla_X\iota = [\dot X^a + u^h(\tilde\Gamma^a_{h;i}X^i + \tilde\Gamma^a_{hb}\dot X^b)]\dot\partial_a;$$

it follows that over the zero section of TM the kernel of Λ is always generated by $\partial_1, \ldots, \partial_n$. So this theory is interesting only over the complement of the zero section in TM; as we shall see in the next section, this fits nicely with the theory of Finsler metrics.

Now let ∇ be a good vertical connection and set $\mathcal{H} = \operatorname{Ker}\Lambda$. Using the associated bundle isomorphism $\Theta\colon \mathcal{V} \to \mathcal{H}$ we can extend ∇ to a connection on $T(TM)$ just by setting

$$\nabla H = \Theta \circ \nabla\bigl(\Theta^{-1}(H)\bigr)$$

for any horizontal vector field H, and then extending by linearity. If $\{\delta_1, \ldots, \delta_n, \dot\partial_1, \ldots, \dot\partial_n\}$ is the local frame of $T(TM)$ associated to the decomposition $T(TM) = \mathcal{H} \oplus \mathcal{V}$, let $\{dx^1, \ldots, dx^n, \psi^1, \ldots, \psi^n\}$ be the dual coframe, where

$$\psi^a = du^a + \Gamma^a_{;j}\, dx^j. \tag{1.2}$$

With respect to this coframe we can write the connection forms ω^j_i as

$$\omega^j_i = \Gamma^j_{i;h}\, dx^h + \Gamma^j_{ia}\, \psi^a, \tag{1.3}$$

where

$$\Gamma^j_{i;h} = \tilde\Gamma^j_{i;h} - \tilde\Gamma^j_{ib}\Gamma^b_{;h} \qquad \text{and} \qquad \Gamma^j_{ia} = \tilde\Gamma^j_{ia}. \tag{1.4}$$

Note that, by construction, $\nabla\delta_i = \omega^j_i \otimes \delta_j$. We also recall that (see [AP, Lemma 1.2.2])

$$\Gamma^j_{;h} = \Gamma^j_{i;h} u^i. \tag{1.5}$$

The tangent bundle TM is naturally equipped with the $T(TM)$-valued global 1-form

$$\eta = dx^i \otimes \partial_i + du^a \otimes \dot\partial_a.$$

It is easy to check that η can also be written as

$$\eta = dx^i \otimes \delta_i + \psi^a \otimes \dot\partial_a.$$

The *torsion* of the good vertical connection ∇ is the $T(TM)$-valued 2-form $\theta = D\eta$, where D is the exterior differential on $T(TM)$-valued forms induced by the (extension to $T(TM)$ of the) connection ∇. In local coordinates (see [AP, (1.3.3) and (1.3.4)]) the torsion is given by $\theta = \theta^i \otimes \delta_i + \dot\theta^a \otimes \dot\partial_a$, where

$$\begin{aligned} \theta^i &= \tfrac12 [\Gamma^i_{k;h} - \Gamma^i_{h;k}]\, dx^h \wedge dx^k + \Gamma^i_{jc}\psi^c \wedge dx^j, \\ \dot\theta^a &= \tfrac12 [\delta_j(\Gamma^a_{;i}) - \delta_i(\Gamma^a_{;j})]\, dx^j \wedge dx^i + [\dot\partial_b(\Gamma^a_{;i}) - \Gamma^a_{b;i}]\, \psi^b \wedge dx^i + \tfrac12 [\Gamma^a_{cb} - \Gamma^a_{bc}]\, \psi^b \wedge \psi^c. \end{aligned} \tag{1.6}$$

As shown in [AP, Proposition 1.3.1], the 2-form θ satisfies the usual definition of torsion:

$$\nabla_X Y - \nabla_Y X = [X, Y] + \theta(X, Y)$$

for any sections X, Y of $T(TM)$.

2. The connections

A *Finsler metric* on a smooth manifold M is a function $F\colon TM \to \mathbb{R}^+$ satisfying the following properties:

(a) $G = F^2$ is smooth on $\tilde{M}$, the complement of the zero section in TM;
(b) $F(u) > 0$ for all $u \in \tilde{M}$;
(c) $F(\lambda u) = |\lambda| F(u)$ for all $u \in TM$ and $\lambda \in \mathbb{R}$;
(d) the matrix $\big(G_{ab}(u)\big)$ is positive definite for all $u \in \tilde{M}$, where $G_{ab} = \partial^2 G/\partial u^a \partial u^b$.

The homogeneity property (b) of the Finsler metric implies the following identities, to be used several times in the sequel:

$$G_a u^a = 2G, \qquad G_{ab} u^b = G_a, \qquad G_{abc} u^c = 0, \tag{2.1}$$

where subscripts indicate derivatives as in property (d) above. We shall use a semicolon to denote derivatives with respect to the x^i variables; for instance,

$$G_{;i} = \frac{\partial G}{\partial x^i}, \qquad G_{ab;j} = \frac{\partial^3 G}{\partial x^j \partial u^a \partial u^b},$$

and so on.

Using the property (d) we can define a Riemannian metric on $\mathcal{V}$ (or on $\pi^* TM$, thanks to Lemma 1.1) over $\tilde{M}$ by setting

$$\forall V, W \in \mathcal{V}_u \qquad\qquad \langle V | W \rangle_u = \tfrac{1}{2} G_{ab}(u) V^a W^b,$$

where $V = V^a \dot{\partial}_a$ and $W = W^b \dot{\partial}_b$. It is easy to check that $\langle \cdot | \cdot \rangle$ is a well-defined Riemannian metric over $\mathcal{V}$. The factor $\frac{1}{2}$ is chosen so that the following equality holds:

$$\langle \iota | \iota \rangle \equiv G.$$

To study the geometry of Finsler metrics one would like to have a generalization of the Levi-Civita connection. At present, two such generalizations are mostly used: the Cartan connection [C] and the Chern connection [Ch, BC]. Another interesting connection related to Finsler geometry is the Bernwald connection (see, e.g., [B, p. 43]). In [AP, Theorem 1.4.2] we proved the following characterization of the Cartan connection:

Theorem 2.1: *Let $F\colon TM \to \mathbb{R}^+$ be a Finsler metric on a smooth manifold M, and let $\langle \cdot | \cdot \rangle$ be the Riemannian structure induced by F on the vertical bundle $\mathcal{V}$. Then the Cartan connection is the unique vertical connection ∇ such that:*

(i) *∇ is good;*
(ii) *for all $X \in T\tilde{M}$ and vertical vector fields V, W one has*

$$X\langle V | W \rangle = \langle \nabla_X V | W \rangle + \langle V | \nabla_X W \rangle;$$

(iii) *$\theta(V, W) = 0$ for all V, $W \in \mathcal{V}$, and $\theta(H, K) \in \mathcal{V}$ for all H, $K \in \mathcal{H}$, where θ is the torsion of the linear connection on $T\tilde{M}$ induced by ∇, and $\mathcal{H}$ is the horizontal bundle induced by ∇.*

The aim of this section is to prove analogous characterizations for the Chern and Bernwald connections. Looking to (1.6) we see that condition (iii) in Theorem 2.1 amounts to the vanishing of two components of the torsion: the $dx^h \wedge dx^k$ component in the horizontal part, and the $\psi^b \wedge \psi^c$ in the vertical part. As we shall see, asking for the vanishing of other components of the torsion and, at the same time, not requiring the full metric compatibility described in condition (ii) we shall recover both the Chern and the Bernwald connections.

But let us start with the Chern connection.

Theorem 2.2: *Let $F: TM \to \mathbb{R}^+$ be a Finsler metric on a smooth manifold M, and let $\langle\cdot|\cdot\rangle$ be the Riemannian structure induced by F on the vertical bundle $\mathcal{V}$. Then the Chern connection is the unique vertical connection ∇ such that:*

(i) *∇ is good;*

(ii) *for all $H \in \mathcal{H}$ and vertical vector fields V, W one has*

$$H\langle V|W\rangle = \langle\nabla_H V|W\rangle + \langle V|\nabla_H W\rangle,$$

where $\mathcal{H}$ is the horizontal bundle induced by ∇;

(iii) *$\theta(X,Y) \in \mathcal{V}$ for all X, $Y \in T\tilde{M}$, where θ is the torsion of the linear connection on $T\tilde{M}$ induced by ∇.*

As a consequence, $\theta(V,W) = 0$ for all V, $W \in \mathcal{V}$. Furthermore, both the Cartan connection and the Chern connection induce the same horizontal bundle.

Remark: Usually, the Chern connection (see, e.g., [BC]) is defined over $\mathbb{P}TM$, the projectivized tangent bundle; so, strictly speaking, the vertical connection ∇ of Theorem 2.2 is the pull-back of the Chern connection under the canonical projection $\tilde{M} \to \mathbb{P}TM$. Anyway, the only difference is the consideration of $u^1, \ldots, u^n$ as actual coordinates on $\tilde{M}$ instead of homogeneous coordinates on $\mathbb{P}TM$.

Proof of Theorem 2.2: We shall show that a vertical connection satisfying properties (i)-(iii) must necessarily be the Chern connection. To do so, we shall recover the connection forms ω_i^j by determining the coefficients $\Gamma_{i;h}^j$ and Γ_{ic}^j — see (1.3) and (1.4).

Recalling (1.6), we see that in local coordinates condition (iii) says that

$$\Gamma_{k;h}^j = \Gamma_{h;k}^j \qquad \text{and} \qquad \Gamma_{ic}^j \equiv 0; \tag{2.2}$$

in particular, recalling (1.1) and (1.4), we see that (i) is satisfied, and that $\theta(V,W) = 0$ for all V, $W \in \mathcal{V}$.

Now (ii) yields

$$\delta_h(G_{kr}) = 2\delta_h\langle\dot\partial_k|\dot\partial_r\rangle = 2\langle\nabla_{\delta_h}\dot\partial_k|\dot\partial_r\rangle + 2\langle\dot\partial_k|\nabla_{\delta_h}\dot\partial_r\rangle = G_{jr}\Gamma_{k;h}^j + G_{kj}\Gamma_{r;h}^j,$$
$$\delta_r(G_{hk}) = G_{jk}\Gamma_{h;r}^j + G_{hj}\Gamma_{k;r}^j, \qquad \delta_k(G_{rh}) = G_{jh}\Gamma_{r;k}^j + G_{rj}\Gamma_{h;k}^j;$$

and so recalling (2.2) we get

$$\begin{aligned}\Gamma_{h;k}^j &= \frac{1}{2}G^{jr}[\delta_k(G_{rh}) - \delta_r(G_{hk}) + \delta_h(G_{kr})]\\ &= \frac{1}{2}G^{jr}[G_{rh;k} - G_{hk;r} + G_{kr;h}] - \frac{1}{2}G^{jr}[G_{rhi}\Gamma_{;k}^i - G_{hki}\Gamma_{;r}^i + G_{kri}\Gamma_{;h}^i],\end{aligned}$$

where (G^{jr}) is the inverse matrix of (G_{jr}). Hence to recover $\Gamma^j_{h;k}$ it suffices to determine the Christoffel symbols $\Gamma^j_{;k}$. To this aim, we contract first with u^h and then with u^k. Recalling (1.5) we get

$$\Gamma^j_{;k} = \Gamma^j_{h;k}u^h = \frac{1}{2}G^{jr}[G_{r;k} - G_{k;r} + G_{kr;h}u^h] - \frac{1}{2}G^{jr}G_{kri}\Gamma^i_{;h}u^h,$$
$$\Gamma^j_{;k}u^k = G^{jr}[G_{r;k}u^k - G_{;r}],$$

where we used (2.1). Therefore

$$\Gamma^j_{;k} = \frac{1}{2}G^{jr}[G_{r;k} - G_{k;r} + G_{kr;h}u^h] - \frac{1}{2}G^{jr}G_{kri}G^{is}[G_{s;h}u^h - G_{;s}]; \tag{2.3}$$

hence the Christoffel symbols — and, *a fortiori,* the vertical connection — are uniquely determined by conditions (i)–(iii). We remark that (see [AP, (1.4.14)]) the $\Gamma^j_{;k}$ are the Christoffel symbols of the Cartan connection, and thus the induced horizontal bundle is the same as the one induced by the Cartan connection.

To end the proof we need to show that the connection we found is the Chern connection. To simplify our task, let us introduce the following symbols:

$$g_{rs} = \frac{1}{2}G_{rs}, \qquad \Gamma_{ihk} = g_{ij}\Gamma^j_{h;k}, \qquad M_{ihk} = -\frac{1}{2}G_{ihj}\Gamma^j_{;k},$$

so that

$$\Gamma_{ihk} = \frac{1}{2}\left(\frac{\partial g_{ih}}{\partial x^k} - \frac{\partial g_{hk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^h}\right) + \frac{1}{2}\big(M_{ihk} - M_{hki} + M_{kih}\big), \tag{2.4}$$

which is formally identical to [BC, (2.48b)]. Therefore to prove our claim it suffices to show that our M_{ihk} are given by the formula [BC, (2.48c)], because then our connection forms ω^j_i will coincide with the connection forms of the Chern connection, by [BC, (2.47a)]. Set

$$G^j = \frac{1}{2}G^{jr}(G_{r;h}u^h - G_{;r}); \tag{2.5}$$

an easy computation shows that this is the quantity defined in [BC, (2.25c)]. Then

$$\frac{\partial G^j}{\partial u^k} = -\frac{1}{2}G^{js}G_{sik}G^{ir}(G_{r;h}u^h - G_{;r}) + \frac{1}{2}G^{jr}(G_{kr;h}u^h + G_{r;k} - G_{k;r}) = \Gamma^j_{;k}, \tag{2.6}$$

and so

$$M_{ihk} = -\frac{\partial g_{ih}}{\partial u^j}\frac{\partial G^j}{\partial u^k},$$

as in [BC, (2.48c)], and we are done. □

Remark: The formulas (2.2), (2.6) and (2.4) show that the Chern connection coincides with the Rund connection introduced in [B, p. 43].

As shown in [AP, Chapter 1], to study the geometry of a Finsler metric the relevant information are contained in the horizontal bundle and the horizontal part of the connection; for this reason, the metric compatibility expressed in condition (ii) is enough to recover the variation formulas of the length functional and their usual consequences, as done in [BC].

We remark in passing that the Chern connection has some more metric compatibility besides the one displayed in condition (ii) of Theorem 2.2. Indeed (2.2) yields

$$U\langle V|W\rangle = \langle \nabla_U V|W\rangle + \langle V|\nabla_U W\rangle + \frac{1}{2}G_{abc}U^a V^b W^c$$

for any $U \in \mathcal{V}$ and vertical vector fields V, W. So (2.1) implies

$$\iota\langle V|W\rangle = \langle \nabla_\iota V|W\rangle + \langle V|\nabla_\iota W\rangle,$$

and

$$\forall X \in T\tilde{M} \qquad X\langle V|\iota\rangle = \langle \nabla_X V|\iota\rangle + \langle V|\nabla_X \iota\rangle. \tag{2.7}$$

Condition (iii) in Theorem 2.2 amounts to the vanishing of three components of the torsion: both components of the horizontal part, and the component $\psi^b \wedge \psi^c$ of the vertical part. If we require the vanishing of a fourth component (the $\psi^b \wedge dx^i$ component of the vertical part) and a minimal metric compatibility we recover the Bernwald connection:

Theorem 2.3: *Let $F\colon TM \to \mathbb{R}^+$ be a Finsler metric on a smooth manifold M, and let $\langle\cdot|\cdot\rangle$ be the Riemannian structure induced by F on the vertical bundle $\mathcal{V}$. Then the Bernwald connection is the unique vertical connection ∇ such that:*

(i) *∇ is good;*

(ii) *for all $H \in \mathcal{H}$ one has*

$$H(G) = 0,$$

where $\mathcal{H}$ is the horizontal bundle induced by ∇;

(iii) *$\theta(X,Y) \in \mathcal{V}$ for all X, $Y \in T\tilde{M}$, and $\theta(V,H) = 0$ for all $V \in \mathcal{V}$ and $H \in \mathcal{H}$, where θ is the torsion of the linear connection on $T\tilde{M}$ induced by ∇.*

Furthermore, both the Cartan connection and the Bernwald connection induce the same horizontal bundle.

Proof: Exactly as in the proof of Theorem 2.2, we assume that such a connection exists and we prove that it must be the Bernwald connection.

Condition (iii) now states that

$$\Gamma^j_{k;h} = \Gamma^j_{h;k}, \qquad \Gamma^j_{hc} \equiv 0 \qquad \text{and} \qquad \Gamma^j_{h;k} = \dot\partial_h(\Gamma^i_{;k}); \tag{2.8}$$

therefore (i) is satisfied, and it suffices to determine the Christoffel symbols $\Gamma^j_{;k}$. First of all, (2.8) and (1.5) imply

$$\Gamma^j_{;k} = \Gamma^j_{h;k}u^h = \Gamma^j_{k;h}u^h = \dot\partial_k(\Gamma^j_{;h})u^h = \dot\partial_k(\Gamma^j_{;h}u^h) - \Gamma^j_{;k};$$

therefore

$$\Gamma^j_{;k} = \frac{1}{2}\,\dot\partial_k(\Gamma^j_{;h}u^h), \tag{2.9}$$

and we are left to determine $\Gamma^j_{;h} u^h$.

Condition (ii) applied with $H = \delta_h$ yields $\delta_h(G) = 0$, that is

$$G_j \Gamma^j_{;h} = G_{;h}. \tag{2.10}$$

Applying $\dot\partial_k$ to (2.10) and recalling (2.8) we obtain

$$G_{k;h} - G_{jk}\Gamma^j_{;h} = G_j \Gamma^j_{k;h}. \tag{2.11}$$

Rewriting this with h and k interchanged and subtracting the result we get

$$G_{jh}\Gamma^j_{;k} = G_{jk}\Gamma^j_{;h} + G_{h;k} - G_{k;h}, \tag{2.12}$$

again by (2.8). Then

$$G_{jh}\Gamma^j_{;k} u^k = G_j \Gamma^j_{;h} + G_{h;k}u^k - 2G_{;h},$$

and (2.10) yields

$$\Gamma^j_{;k} u^k = G^{jh}(G_{h;k}u^h - G_{;h}).$$

Thus, by (2.9), (2.5) and (2.6) we have recovered the usual Christoffel symbols of the Cartan connection. Finally, recalling (2.8) and [B, pp. 39 and 44] we see that we have found the Bernwald connection, as claimed. Indeed, the ${F_i}^k{}_j$, ${C_i}^k{}_j$ and ${N^k}_i$ of [B] by definition correspond respectively to our $\Gamma^k_{i;j}$, Γ^k_{ij} and $\Gamma^k_{;i}$. □

Again, the Bernwald connection has slightly more metric compatibility than the one showed by condition (ii). Indeed, (2.11) says that

$$\delta_h \langle \dot\partial_k | \iota \rangle = \langle \nabla_{\delta_h} \dot\partial_k | \iota \rangle$$

for all h, $k = 1, \ldots, n$; therefore, recalling that $\nabla_H \iota = 0$ for any $H \in \mathcal{H}$ by definition, we find that (2.7) holds for the Bernwald connection too.

We end this section with a final remark on the remaining component of the torsion, the $dx^i \wedge dx^j$ component of the vertical part. As shown in (1.6), it depends only on the horizontal bundle, and therefore it is the same for both the Cartan, the Chern and the Bernwald connections. In particular, [AP, Proposition 1.4.4.(i)] shows that

$$\forall H, K \in \mathcal{H} \qquad\qquad \langle \theta(H,K) | \iota \rangle = 0$$

for each of these connections.

3. Geodesics

In this section we shall describe a minimal compatibility condition between a good vertical connection and a Finsler metric.

Let ∇ be a good vertical connection on a manifold M. We shall say that a curve $\sigma\colon [a,b] \to M$ is a *geodesic* for ∇ if the tangent curve $\dot\sigma$ (which is a curve in TM) is a geodesic for the connection induced by ∇ over TM.

To describe the geodesics of a good vertical connection we need the *radial horizontal vector field* $\chi = \Theta \circ \iota$:

Lemma 3.1: *Let ∇ be a good vertical connection over a smooth manifold M of dimension n. Then a curve $\sigma\colon[a,b]\to M$ is a geodesic for ∇ iff $\ddot\sigma$ is horizontal iff $\dot\sigma$ is an integral curve of χ iff*

$$\ddot\sigma^a+\Gamma^a_{;j}(\dot\sigma)\dot\sigma^j=0 \tag{3.1}$$

for $a=1,\dots,n$.

Proof: In local coordinates, $\dot\sigma=\dot\sigma^j\frac{\partial}{\partial x^j}|_\sigma$; therefore

$$\ddot\sigma=\dot\sigma^j\partial_j+\ddot\sigma^a\dot\partial_a=\dot\sigma^j\delta_j+(\ddot\sigma^a+\Gamma^a_{;j}(\dot\sigma)\dot\sigma^j)\dot\partial_a, \tag{3.2}$$

and $\ddot\sigma$ is horizontal iff (3.1) is satisfied. Since $\chi(\dot\sigma)=\dot\sigma^j\delta_j$, the tangent curve $\dot\sigma$ is an integral curve of χ iff (3.1) is satisfied.

Now set $\ddot\sigma^{\mathcal H}=\dot\sigma^j\delta_j$ and $\ddot\sigma^{\mathcal V}=(\ddot\sigma^a+\Gamma^a_{;j}(\dot\sigma)\dot\sigma^j)\dot\partial_a$. The curve $\dot\sigma$ in TM is a geodesic for ∇ iff $\nabla_{\ddot\sigma}\ddot\sigma=0$, that is iff $\nabla_{\ddot\sigma}\ddot\sigma^{\mathcal H}=0$ and $\nabla_{\ddot\sigma}\ddot\sigma^{\mathcal V}=0$. Now

$$\begin{aligned}\nabla_{\ddot\sigma}\ddot\sigma^{\mathcal H}&=[\ddot\sigma^h+\dot\sigma^j\tilde\Gamma^h_{c;j}\dot\sigma^c+\ddot\sigma^a\tilde\Gamma^h_{ca}\dot\sigma^c]\delta_h\\&=[\ddot\sigma^h+\Gamma^h_{;j}\dot\sigma^j+\Gamma^h_{cb}\Gamma^b_{;j}\dot\sigma^c\dot\sigma^j+\Gamma^h_{ca}\ddot\sigma^a\dot\sigma^c]\delta_h\\&=[\delta^h_b+\Gamma^h_{cb}\dot\sigma^c][\ddot\sigma^b+\Gamma^b_{;j}\dot\sigma^j]\delta_h,\end{aligned}$$

by (1.5), where all Γ's are evaluated at $\dot\sigma$. Being ∇ a good connection, by (1.1) the matrix $(\delta^h_b+\Gamma^h_{cb}\dot\sigma^c)$ is invertible; it follows that $\nabla_{\ddot\sigma}\ddot\sigma^{\mathcal H}=0$ iff (3.1) is satisfied — and hence $\ddot\sigma^{\mathcal V}\equiv0$. Therefore σ is a geodesic for ∇ iff (3.1) is satisfied, and we are done. □

On the other hand, a curve $\sigma\colon[a,b]\to M$ is a *geodesic* for a Finsler metric F on a manifold M if it is an extremal for the length functional. A standard computation in local coordinates shows that σ is a geodesic for F iff

$$\ddot\sigma^a+G^{ba}[G_{b;j}\dot\sigma^j-G_{;b}]=0 \tag{3.3}$$

for all $a=1,\dots,n$, where G^{ba}, $G_{b;j}$ and $G_{;b}$ are evaluated at $\dot\sigma$. Comparing (3.1) and (3.3) we see that the geodesics for a good vertical connection ∇ are exactly the geodesics for a Finsler metric F iff

$$\Gamma^a_{;j}u^j=G^{ba}[G_{b;j}u^j-G_{;b}]. \tag{3.4}$$

There is a more intrinsic way to express this result (which is the minimal compatibility one can ask for a good vertical connection and a Finsler metric). Using the Finsler metric F we can define a canonical 1-form α on $\tilde M$ by setting in local coordinates

$$\alpha=G_i\,dx^i$$

(the form $\alpha/2F$ was called *Hilbert form* in [BC]), and then introduce a canonical 2-form

$$\omega=d\alpha=G_{i;j}\,dx^j\wedge dx^i+G_{ia}\,du^a\wedge dx^i; \tag{3.5}$$

since $\omega^n=\pm(\det G_{ia})^2\,dx^1\wedge\cdots\wedge dx^n\wedge du^1\wedge\cdots\wedge du^n$, the 2-form ω is a symplectic form on $\tilde M$. Let X_G denote the hamiltonian vector field of G with respect to ω, that is the unique vector field on $\tilde M$ such that $dG=\omega(\cdot,X_G)$. Writing $X_G=X^i\partial_i+\dot X^a\dot\partial_a$, we see that X_G must satisfy

$$[(G_{i;j}-G_{j;i})X^i-G_{aj}\dot X^a]\,dx^j+G_{ia}X^i\,du^a=G_{;j}\,dx^j+G_a\,du^a;$$

therefore

$$X_G=u^i\partial_i-G^{aj}(G_{j;i}u^i-G_{;j})\dot\partial_a.$$

Comparing with (3.4) we have obtained

Proposition 3.2: *Let $F\colon TM \to \mathbb{R}^+$ be a Finsler metric on a smooth manifold M, and let ∇ be a good vertical connection over M. Then the geodesics for ∇ are exactly the geodesics for F iff the radial horizontal vector field χ is the hamiltonian vector field for G with respect to the symplectic form ω given by* (3.5).

In particular, Proposition 3.2 holds for the Cartan, Chern and Bernwald connections — and thus they all have the same geodesics.

We end this paper with yet another example of a good vertical connection inducing again the same geodesics as the Finsler metric F. We need the following

Proposition 3.3: *Let $F\colon TM \to \mathbb{R}^+$ be a Finsler metric on a smooth manifold M, and let $\langle\cdot|\cdot\rangle$ be the Riemannian structure induced by F on the vertical bundle $\mathcal{V}$. Let ∇ be a good vertical connection over M such that*

(i) *for all $H \in \mathcal{H}$ and vertical vector fields V one has*

$$H\langle V|\iota\rangle = \langle \nabla_H V|\iota\rangle,$$

where $\mathcal{H}$ is the horizontal bundle induced by ∇ (and we recall that $\nabla_H \iota = 0$ by definition);

(ii) *$\langle \theta(H,\chi)|\chi\rangle = 0$ for all $H \in \mathcal{H}$, where θ is the torsion of the linear connection on $\tilde{TM}$ induced by ∇.*

Then the geodesics for ∇ are exactly the geodesics for F.

Proof: Condition (i) applied with $H = \delta_h$ and $V = \dot\partial_k$ yields

$$G_{k;h} - G_{ik}\Gamma^i_{;h} = G_i\Gamma^i_{k;h}; \tag{3.6}$$

in particular, contracting by u^k and recalling (1.5) we get

$$G_{;h} = G_i\Gamma^i_{;h}. \tag{3.7}$$

Rewriting (3.6) with h and k exchanged and subtracting we obtain

$$G_{k;h} - G_{h;k} - G_{ik}\Gamma^i_{;h} + G_{ih}\Gamma^i_{;k} = G_i(\Gamma^i_{k;h} - \Gamma^i_{h;k}).$$

Contracting again by u^k and recalling (3.7) we get

$$G_{;h} - G_{h;k}u^k + G_{ih}\Gamma^i_{;k}u^k = G_i(\Gamma^i_{k;h} - \Gamma^i_{h;k})u^k.$$

Now (ii) in local coordinates means exactly $G_i(\Gamma^i_{k;h} - \Gamma^i_{h;k})u^k = 0$; therefore we obtain

$$\Gamma^i_{;k}u^k = G^{ih}(G_{h;k}u^k - G_{;h}),$$

that is (3.4), and we are done. □

Let M be a complex manifold, and $T^{1,0}M$ its holomorphic tangent bundle. A *complex Finsler metric* on M is a function $F\colon T^{1,0}M \to \mathbb{R}^+$ such that

(a) $G = F^2$ is smooth on the complement of the zero section in $T^{1,0}M$;
(b) $F(\lambda v) = |\lambda| F(v)$ for all $v \in T^{1,0}M$ and $\lambda \in \mathbb{C}$;
(c) the matrix $(\partial^2 G/\partial v^\alpha \partial \overline{v^\beta})$ is positive definite on the complement of the zero section in $T^{1,0}M$.

If moreover the Hessian (with respect to the real coordinates) of G is positive definite, we say that the complex Finsler metric is *strongly convex.*

Using the canonical isomorphism of $T^{1,0}M$ with the real tangent bundle $T_{\mathbb{R}}M$, a strongly convex complex Finsler metric induces a Finsler metric on the underlying smooth real manifold. As described in [AP, Section 2.3], to any complex Finsler metric is canonically associated a good vertical connection $\tilde{\nabla}$ (the *Chern-Finsler connection*); again, using the canonical isomorphisms we get a good vertical connection ∇ over $T_{\mathbb{R}}M$. A tedious but completely straightforward computation (very similar to the ones carried out in [AP, Section 2.6]) shows that ∇ satisfies condition (i) of Proposition 3.3 always, and condition (ii) iff $\tilde{\nabla}$ is weakly Kähler (which is exactly condition (ii) expressed in complex terms; see [AP, Section 2.3]). On the other hand, except in very special cases, the horizontal bundle associated to ∇ is different from the one associated to the Cartan (Chern, Bernwald) connection; see [AP, Theorem 2.6.4]. Therefore the real good vertical connection induced by a strictly convex complex weakly Kähler Finsler metric is an example of another good vertical connection still having the same geodesics as the given Finsler metric.

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